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Numerical Solution of Two-Dimensional Telegraph Equations Using Haar Wavelets

I. Singh
Dr.B.R.Ambedkar National Institute of Technology

S. Kumar*
Dr.B.R.Ambedkar National Institute of Technology

Abstract. We present, numerical solution of well known partial differential equation such as two-dimensional telegraph equations with the aid of three- dimensional Haar wavelet. Numerical observations illustrate the accuracy and efficiency of the proposed Haar wavelet method. High accuracy of the results has been observed even in the case of a small number of collocation points.

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1. Introduction

Several real life problems are formulated in the form of mathematical models. Partial differential equations are arising in many areas of physical and mathematical sciences and also in engineering. In the last few decades, wavelet analysis is a recently developed mathematical tool for solving linear and nonlinear differential and integral equations. Greater

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*Corresponding author

attempts have been done to find wavelets based solutions of differential equations. The basic methodology behind wavelet methods is working on the procedure, which reduces the differential equations in the system of linear and nonlinear equations, which are solved by any one of the significant methods such as Gauss-elimination method, Gauss-Jordan method, Matrix-inversion method, Newton method for solving nonlinear system of equations, etc. In the last few years, wavelet related approaches are gaining more popularity in the field of numerical analysis. Several types of wavelets and approximating functions have been developed for solving differential and integral equations. Wavelets have been used in different areas of science, engineering and other areas in which numerical approximations are required. In wavelet analysis, a function or signal can be expressed or represented in terms of a set of orthonormal basis functions known as wavelets, which are localized both in time and scale. The wavelet family is obtained from a continuous function $\psi(x)$, called mother wavelet, by translation and dilation of $\psi(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, where j and k are non-negative integers. In 1998, Ingrid Daubechie [10] introduced some wavelets which are differentiable and have small sized support; these wavelets are frequently applied for solving differential and integral equations. The main disadvantage of Daubechies wavelets is that they do not have an explicit expression, so analytical differentiation and integration is not possible.

In the last few decades, it is big and difficult challenge to obtain the numerical solutions of higher degree differential and integral equations. Many researcher are accepting the challenge and significant formulations or algorithms have been developed for solving $2D$ and $3D$ problems. There are many approaches such as finite difference method, finite element method, finite volume method, wavelet methods etc. for solving $2D$ and $3D$ partial differential equations. Wavelet methods have various advantages such as simple and fast algorithms, small computational cost, easy to handle, more efficient and more accurate. In the past literature, different wavelet techniques have been used for solving one dimensional problems. Among all wavelet families, Haar wavelet [13], being discontinuous and non-differentiable, is simplest orthonormal wavelet and has compact support. It is also known as Daubechies wavelet of order 1. Direct applications of Haar wavelets are not applied for solving partial differential equations. Two possibilities have been obtained for solving such equations: One possibility is to regularize the piecewise constant Haar function by interpolation splines as presented in [3] and [4]. But, this technique complicates the solution procedure. The other possibility is to expand the highest derivative of the differential equation into a Haar series as presented in [25]. Due to second possibility, simplicity of Haar wavelet is unaltered.

Wavelet collocation method for solving elliptic boundary value problems has been presented in [1]. Haar wavelet method has been presented for solving generalized Burger–Huxley equations in [5]. Approximation of Haar wavelet has been used for solving magnetohydrodynamic flow equations in [6]. Simple procedure has been developed, for the integration of Haar wavelet matrices for solving several differential equations in [7]. Operational matrices of integrations based on Haar wavelets have been introduced for analyzing the lumped and distributed-parameter dynamical system in [9]. Haar wavelet methods for solving Fisher and FitzHugh-Nagumo equations have been presented in [14] and [15] respectively. Chebyshev wavelet has been used for solving partial differential equations with boundary conditions of the telegraph type, in [16]. Some of the recurrence relations and procedures developed in [8], [17] and [18] may be useful for the solutions of two-and three-dimensional problems arising in sciences and engineering. In [20], [21] and [22] direct methods based on Haar wavelets have been established for solving linear and nonlinear higher order differential and integral equations. Haar wavelet based methods have been used for solving parabolic and elliptic partial differential equations in [23],[25],[28], [29] and [30]. Analytic studies on Burgers, Fisher, Huxley equations and combined forms of these equations have been presented in [31]. Numerical solutions of two-and three-dimensional Poisson equations and bi-harmonic equations using Haar wavelet have been presented in [27].

Consider the second-order linear two-space dimensional hyperbolic telegraph equation in the region $\{(x,y) : a \leqslant x \leqslant b,\ a \leqslant y \leqslant b\}$,

$$\frac{\partial^2 u}{\partial t^2}(x,y,t)+2\Gamma\frac{\partial u}{\partial t}(x,y,t)+\Delta^2 u(x,y,t) = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\digamma_1(x,y,t), \quad (1)$$

with initial conditions

$$u(x,y,0)=\mathbf{f}(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0)=\mathbf{f_{11}}(x,y), \tag{2}$$

and boundary conditions

$$\begin{aligned} &u(x,a,t)=\mathbf{g_{11}}(x,t), \quad u(x,b,t)=\mathbf{g_{12}}(x,t),\\ &u(a,y,t)=\mathbf{g_{21}}(y,t), \quad u(b,y,t)=\mathbf{g_{22}}(y,t), \qquad 0\leqslant t\leqslant T, \end{aligned} \tag{3}$$

where Γ, Δ are constant and $\mathbf{f}$, $\mathbf{f_{11}}$, $\mathbf{g_{11}}$, $\mathbf{g_{12}}$, $\mathbf{g_{21}}$, $\mathbf{g_{22}}$ are known functions. The hyperbolic partial differential equations have significant role in formulating fundamental equations in atomic physics and are also very useful in understanding various phenomena in applied sciences like engineering industry aerospace as well as in chemistry and biology too. In the literature, we found that many attempts have been taken for solving one-and two-dimensional telegraph equations. Several numerical schemes were developed for solving telegraph equation such as Taylor matrix method [2], dual reciprocity boundary integral method [11], unconditionally stable finite difference scheme [12]. Cubic and modified B-spline collocation method [24, 25], Chebyshev tau method [26], interpolating scaling function method [19] etc.

Kronecker product of two matrices:
For saving calculation time, we use the concept of Kronecker product of matrix A with matrix B of orders $p\times q$ respectively and is defined as:

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B\\ a_{21}B & a_{22}B & \cdots & a_{2q}B\\ \vdots & \vdots & \vdots & \vdots\\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{pmatrix}. \tag{4}$$

The first documented work on Kronecker products was written by Johann Georg Zehfuss between 1858 and 1868. In MATLAB, the Kronecker product of two matrices A and B is directly calculated with the command $kron(A,B)$.

Kronecker product of three matrices:
The Kronecker product of three A, B and C matrices each of orders $p \times q$ can be calculated as:

$$A \otimes B \otimes C = \begin{pmatrix} a_{11}E & a_{12}E & \cdots & a_{1q}E \\ a_{21}E & a_{22}E & \cdots & a_{2q}E \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1}E & a_{p2}E & \cdots & a_{pq}E \end{pmatrix}, \tag{5}$$

where E is of the form:

$$E = B \otimes C = \begin{pmatrix} b_{11}C & b_{12}C & \cdots & b_{1q}C \\ b_{21}C & b_{22}C & \cdots & b_{2q}C \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1}C & b_{p2}C & \cdots & b_{pq}C \end{pmatrix}. \tag{6}$$

In Section 2, we briefly describe Haar wavelet and establish some operational matrices of integrations. Description of three-dimensional Haar wavelet method for solving two-dimensional telegraph equations has been presented in Section 3. Error analysis has been derived for two-dimensional telegraph equations in Section 4 and in Section 5, numerical examples have been solved using the present method to illustrate the efficiency and accuracy of present wavelet method.

2. Haar Wavelet

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. These functions are defined as

$$\mathcal{H}_i(x) = \begin{cases} 1, & \alpha \leqslant x < \beta, \\ -1, & \beta \leqslant x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{7}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$ and $\gamma = \frac{k+1}{m}$. Integer $m = 2^j$, $(j = 0, 1, 2, ..., J)$ indicates the level of the wavelet,and $k = 0, 1, 2, ..., m-1$ is the translation parameter. Maximal level of resolution is J. The index $i = m+k+1$.

In case of minimal values, $m = 1, k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$, where $M = 2^J$. It is assumed that for value $i = 1$, the corresponding scaling function in $[0, 1]$ is as

$$\mathcal{H}_1(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{8}$$

Define the collocation points $x_l = \frac{(l-0.5)}{2M}$, where $l = 1, 2, 3, ..., 2M$. The operational matrix of integration, which is a $2M \times 2M$ square matrix, is defined by the relations

$$\mathcal{P}_{1,i}(x) = \int_0^x \mathcal{H}_i(t)dt, \tag{9}$$

and

$$\mathcal{P}_{n+1,i}(x) = \int_0^x \mathcal{P}_{n,i}(t)dt, \tag{10}$$

where $n = 1, 2, 3, 4, ...$ These integrals can be evaluated using (7) and first two of them are given below:

$$\mathcal{P}_{1,i}(x) = \begin{cases} x - \alpha, & x \in [\alpha, \beta), \\ \gamma - x, & x \in [\beta, \gamma), \\ 0, & \text{elsewhere;} \end{cases} \tag{11}$$

$$\mathcal{P}_{2,i}(x) = \begin{cases} 0, & x < \alpha, \\ \frac{1}{2}(x - \alpha)^2, & x \in [\alpha, \beta], \\ \frac{1}{2}[(x - \alpha)^2 - 2(x - \beta)^2], & x \in [\beta, \gamma], \\ \frac{1}{2}[(x - \alpha)^2 - 2(x - \beta)^2 + (x - \gamma)^2], & x > \gamma. \end{cases} \tag{12}$$

In general,

$$\mathcal{P}_{n,i}(x) = \begin{cases} 0, & x < \alpha, \\ \frac{1}{n!}(x - \alpha)^n, & x \in [\alpha, \beta], \\ \frac{1}{n!}[(x - \alpha)^n - 2(x - \beta)^n], & x \in [\beta, \gamma], \\ \frac{1}{n!}[(x - \alpha)^n - 2(x - \beta)^n + (x - \gamma)^n], & x > \gamma. \end{cases} \tag{13}$$

Haar wavelet functions are orthogonal to each other as given below

$$\int_0^1 \mathcal{H}_i(x)\mathcal{H}_l(x)dx = \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l. \end{cases} \tag{14}$$

Any square integrable function $f(x, y, t)$ in the interval $[0, 1) \times [0, 1) \times [0, 1)$ can be expanded by a Haar series of infinite terms

$$f(x, y, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)\mathcal{H}_k(t), \qquad x, y, t \in [0, 1], \tag{15}$$

where $\mathcal{C}_{ijk}$ are constants of the triple summation series, known as wavelet coefficient. For numerical approximation the above series is truncated up to finite terms say $2M$, that is

$$f(x, y, t) = \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)\mathcal{H}_k(t), \quad x, y, t \in [0, 1]. \tag{16}$$

3. Three-Dimensional Haar Wavelet Method for Solving Two-Dimensional Telegraph Equations

Consider the approximate wavelet solution of the form

$$\frac{\partial^6 u}{\partial x^2 \partial y^2 \partial t^2}(x, y, t) = \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)\mathcal{H}_k(t). \tag{17}$$

Integrating (17), twice with respect to x, from 0 to x, we obtain

$$\begin{aligned} \frac{\partial^4 u}{\partial y^2 \partial t^2}(x, y, t) = \frac{\partial^4 u}{\partial y^2 \partial t^2}(0, y, t) + x\frac{\partial^5 u}{\partial x \partial y^2 \partial t^2}(0, y, t) \\ + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(x)\mathcal{H}_j(y)\mathcal{H}_k(t). \end{aligned} \tag{18}$$

Putting $x = 1$ in (18), we obtain

$$\frac{\partial^5 u}{\partial x \partial y^2 \partial t^2}(0, y, t) = \frac{\partial^4 u}{\partial y^2 \partial t^2}(1, y, t) - \frac{\partial^4 u}{\partial y^2 \partial t^2}(0, y, t) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(1)\mathcal{H}_j(y)\mathcal{H}_k(t). \quad (19)$$

From (18) and (19), we obtain

$$\frac{\partial^4 u}{\partial y^2 \partial t^2}(x, y, t) = \frac{\partial^4 u}{\partial y^2 \partial t^2}(0, y, t) + x[\frac{\partial^4 u}{\partial y^2 \partial t^2}(1, y, t) - \frac{\partial^4 u}{\partial y^2 \partial t^2}(0, y, t)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))\mathcal{H}_j(y)\mathcal{H}_k(t). \quad (20)$$

Again, integrating (20), twice with respect to y, from 0 to y, we obtain

$$\frac{\partial^2 u}{\partial t^2}(x, y, t) = \psi_{11}(x, y, t) + y\psi_{12}(x, t) + x\psi_{13}(y, t) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))\mathcal{P}_{2,j}(y)\mathcal{H}_k(t). \quad (21)$$

where

$$\psi_{11}(x, y, t) = \frac{\partial^2 u}{\partial t^2}(x, 0, t) + \frac{\partial^2 u}{\partial t^2}(0, y, t) - \frac{\partial^2 u}{\partial t^2}(0, 0, t), \quad (22)$$

$$\psi_{12}(x, t) = \frac{\partial^3 u}{\partial y \partial t^2}(x, 0, t) - \frac{\partial^3 u}{\partial y \partial t^2}(0, 0, t) + x[\frac{\partial^3 u}{\partial y \partial t^2}(0, 0, t) - \frac{\partial^3 u}{\partial y \partial t^2}(1, 0, t)], \quad (23)$$

$$\psi_{13}(y, t) = (\frac{\partial^2 u}{\partial t^2}(1, y, t) - \frac{\partial^2 u}{\partial t^2}(1, 0, t) - \frac{\partial^2 u}{\partial t^2}(0, y, t) + \frac{\partial^2 u}{\partial t^2}(0, 0, t)). \quad (24)$$

Putting $y = 1$ in (21), we obtain

$$\psi_{12}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,1,t) - \psi_{11}(x,1,t) - x\psi_{13}(1,t) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))\mathcal{P}_{2,j}(1)\mathcal{H}_k(t). \quad (25)$$

From (21), using (25), we obtain

$$\frac{\partial^2 u}{\partial t^2}(x,y,t) = \psi_{11}(x,y,t) + x\psi_{13}(y,t) + y[\frac{\partial^2 u}{\partial t^2}(x,1,t) - \psi_{11}(x,1,t) - x\psi_{13}(1,t)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))\mathcal{H}_k(t). \quad (26)$$

Integrating (17) twice with respect to t from 0 to t, we obtain

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,t) = \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,0) + t\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t}(x,y,0) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)\mathcal{P}_{2,k}(t). \quad (27)$$

Putting $t = 1$ in (27), we obtain

$$\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t}(x,y,0) = \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,1) - \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,0) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)\mathcal{P}_{2,k}(1). \quad (28)$$

From (27), using (28), we obtain

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,t) = \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,0) + t[\frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,1) - \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y,0)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{H}_j(y)(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (29)$$

Again, integrating (29) twice with respect to x, from 0 to x, we obtain

$$\frac{\partial^2 u}{\partial y^2}(x,y,t) = \psi_{21}(x,y,t) + x\psi_{22}(y,t) + t\psi_{23}(x,y) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(x)\mathcal{H}_j(y)(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)), \quad (30)$$

where

$$\psi_{21}(x,y,t) = \frac{\partial^2 u}{\partial y^2}(0,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,0) - \frac{\partial^2 u}{\partial y^2}(0,y,0), \quad (31)$$

$$\psi_{22}(y,t) = \frac{\partial^3 u}{\partial x \partial y^2}(0,y,t) - \frac{\partial^3 u}{\partial x \partial y^2}(0,y,0) + t[\frac{\partial^3 u}{\partial x \partial y^2}(0,y,0) - \frac{\partial^3 u}{\partial x \partial y^2}(0,y,1)], \quad (32)$$

$$\psi_{23}(x,y) = (\frac{\partial^2 u}{\partial y^2}(x,y,1) - \frac{\partial^2 u}{\partial y^2}(0,y,1) - \frac{\partial^2 u}{\partial y^2}(x,y,0) + \frac{\partial^2 u}{\partial y^2}(0,y,0)). \quad (33)$$

Putting $x = 1$ in (30), we obtain:

$$\psi_{22}(y,t) = \frac{\partial^2 u}{\partial y^2}(1,y,t) - \psi_{21}(1,y,t) - t\psi_{23}(1,y) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(1)\mathcal{H}_j(y)(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (34)$$

Substituting (34) in (30), we obtain

$$\frac{\partial^2 u}{\partial y^2}(x,y,t) = \psi_{21}(x,y,t) + t\psi_{23}(x,y) + x[\frac{\partial^2 u}{\partial y^2}(1,y,t) - \psi_{21}(1,y,t) - t\psi_{23}(1,y)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))\mathcal{H}_j(y)(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (35)$$

Integrating (17) twice with respect to y, from 0 to y, we obtain:

$$\frac{\partial^4 u}{\partial x^2 \partial t^2}(x,y,t) = \frac{\partial^4 u}{\partial x^2 \partial t^2}(x,0,t) + y\frac{\partial^5 u}{\partial x^2 \partial y \partial t^2}(x,0,t) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{P}_{2,j}(1)\mathcal{H}_k(t). \quad (36)$$

Putting $y = 1$ in (36), we obtain

$$\frac{\partial^5 u}{\partial x^2 \partial y \partial t^2}(x,0,t) = \frac{\partial^4 u}{\partial x^2 \partial t^2}(x,1,t) - \frac{\partial^4 u}{\partial x^2 \partial t^2}(x,0,t) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)\mathcal{P}_{2,j}(y)\mathcal{H}_k(t). \quad (37)$$

From (36) and (37), we obtain

$$\frac{\partial^4 u}{\partial x^2 \partial t^2}(x,y,t) = \frac{\partial^4 u}{\partial x^2 \partial t^2}(x,0,t) + y[\frac{\partial^4 u}{\partial x^2 \partial t^2}(x,1,t) - \frac{\partial^4 u}{\partial x^2 \partial t^2}(x,0,t)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))\mathcal{H}_k(t). \quad (38)$$

Again, integrating (38), twice with respect t, from 0 to t, we obtain

$$\frac{\partial^2 u}{\partial x^2}(x,y,t) = \psi_{31}(x,y,t) + t\psi_{32}(x,y) + y\psi_{33}(x,t) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))\mathcal{P}_{2,k}(t), \quad (39)$$

where

$$\psi_{31}(x,y,t) = \frac{\partial^2 u}{\partial x^2}(x,y,0) + \frac{\partial^2 u}{\partial x^2}(x,0,t) - \frac{\partial^2 u}{\partial x^2}(x,0,0), \quad (40)$$

$$\psi_{32}(x,y) = \frac{\partial^3 u}{\partial x^2 \partial t}(x,y,0) - \frac{\partial^4 u}{\partial x^2 \partial t}(x,0,0) + y[\frac{\partial^3 u}{\partial x^2 \partial t}(x,0,0) - \frac{\partial^3 u}{\partial x^2 \partial t}(x,1,0)], \quad (41)$$

$$\psi_{33}(x,t) = (\frac{\partial^2 u}{\partial x^2}(x,1,t) - \frac{\partial^2 u}{\partial x^2}(x,1,0) - \frac{\partial^2 u}{\partial x^2}(x,0,t) + \frac{\partial^2 u}{\partial x^2}(x,0,0)). \quad (42)$$

Putting $t = 1$ in (39), we obtain

$$\psi_{32}(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y,1) - \psi_{31}(x,y,1) - y\psi_{33}(x,1) - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))\mathcal{P}_{2,k}(1). \quad (43)$$

From (39) and (43)), we obtain

$$\frac{\partial^2 u}{\partial x^2}(x,y,t) = \psi_{31}(x,y,t) + y\psi_{33}(x,t) + t[\frac{\partial^2 u}{\partial x^2}(x,y,1) - \psi_{31}(x,y,1) - y\psi_{33}(x,1)] + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (44)$$

Again, integrating (44), twice with respect to x, from 0 to x, we obtain

$$u(x,y,t) = \varphi_0(x,y,t) + x\varphi_1(y,t) + t\varphi_2(x,y) - yt\varphi_3(x) + y\varphi_4(x,t) + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(x)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)), \quad (45)$$

where

$$\varphi_0(x,y,t) = u(0,y,t) + u(x,y,0) - u(0,y,0) + u(x,0,t) - u(0,0,t) - u(x,0,0) + u(0,0,0), \quad (46)$$

$$\begin{aligned}\varphi_1(y,t) = &(\frac{\partial u}{\partial x}(0,y,t) - \frac{\partial u}{\partial x}(0,y,0) - \frac{\partial u}{\partial x}(0,0,t) + \frac{\partial u}{\partial x}(0,0,0)) \\ &+ t(\frac{\partial u}{\partial x}(0,y,0) - \frac{\partial u}{\partial x}(0,y,1) + \frac{\partial u}{\partial x}(0,0,1) - \frac{\partial u}{\partial x}(0,0,0)) \\ &- yt(\frac{\partial u}{\partial x}(0,1,0) - \frac{\partial u}{\partial x}(0,1,1) + \frac{\partial u}{\partial x}(0,0,1) - \frac{\partial u}{\partial x}(0,0,0)) \\ &+ y(\frac{\partial u}{\partial x}(0,1,0) - \frac{\partial u}{\partial x}(0,1,t) + \frac{\partial u}{\partial x}(0,0,t) - \frac{\partial u}{\partial x}(0,0,0)),\end{aligned} \quad (47)$$

$$\varphi_2(x,y) = u(x,y,1) - u(0,y,1) - u(x,y,0) + u(0,y,0) - u(x,0,1) \\ + u(0,0,1) + u(x,0,0) - u(0,0,0), \quad (48)$$

$$\varphi_3(x) = u(x,1,1) - u(0,1,1) - u(x,1,0) + u(0,1,0) - u(x,0,1) \\ + u(0,0,1) + u(x,0,0) - u(0,0,0), \quad (49)$$

$$\varphi_4(x,t) = u(x,1,t) - u(0,1,t) - u(x,1,0) + u(0,1,0) - u(x,0,t) \\ + u(0,0,t) + u(x,0,0) - u(0,0,0). \quad (50)$$

Putting $x = 1$ in (45), we obtain

$$\varphi_1(y,t) = u(1,y,t) - \varphi_0(1,y,t) - t\varphi_2(1,y) + yt\varphi_3(1) - y\varphi_4(1,t) \\ - \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}\mathcal{P}_{2,i}(1)(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (51)$$

From (45) and (51), we obtain

$$u(x,y,t) = \varphi_0(x,y,t) + xu(1,y,t) - x\varphi_0(1,y,t) - xt\varphi_2(1,y) \\ + xyt\varphi_3(1) - xy\varphi_4(1,t) + t\varphi_2(x,y) - yt\varphi_3(x) + y\varphi_4(x,t) \\ + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t) - t\mathcal{P}_{2,k}(1)). \quad (52)$$

Differentiating (52) with respect to t, we obtain:

$$\frac{\partial u}{\partial t}(x,y,t) = \frac{\partial \varphi_0}{\partial t}(x,y,t) + x\frac{\partial u}{\partial t}(1,y,t) - x\frac{\partial \varphi_0}{\partial t}(1,y,t) - x\varphi_2(1,y) \\ + xy\varphi_3(1) - xy\frac{\partial \varphi_4}{\partial t}(1,t) + \varphi_2(x,y) - y\varphi_3(x) + y\frac{\partial \varphi_4}{\partial t}(x,t) \\ + \sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M} \mathcal{C}_{ijk}(\mathcal{P}_{2,i}(x) - x\mathcal{P}_{2,i}(1))(\mathcal{P}_{2,j}(y) - y\mathcal{P}_{2,j}(1))(\mathcal{P}_{1,k}(t) - \mathcal{P}_{2,k}(1)). \quad (53)$$

Substituting the values from (26), (35), (44), (52) and (53) in (1), we obtain

$$\sum_{i=1}^{2M}\sum_{j=1}^{2M}\sum_{k=1}^{2M}\mathcal{C}_{ijk}[R_1+2\Gamma S_1+\Delta^2T_1-U_{11}-V_{11}]=\Omega(x,y,t), \qquad (54)$$

where

$$R_1=\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y)-y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t)-t\mathcal{P}_{2,k}(1)), \qquad (55)$$

$$S_1=(P_{2,i}(x)-xP_{2,i}(1))(P_{2,j}(y)-yP_{2,j}(1))(P_{1,k}(t)-P_{2,k}(1)), \quad (56)$$

$$T_1=(\mathcal{P}_{2,i}(x)-x\mathcal{P}_{2,i}(1))(\mathcal{P}_{2,j}(y)-y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t)-t\mathcal{P}_{2,k}(1)), \quad (57)$$

$$U_{11}=\mathcal{H}_i(x)(\mathcal{P}_{2,j}(y)-y\mathcal{P}_{2,j}(1))(\mathcal{P}_{2,k}(t)-t\mathcal{P}_{2,k}(1)), \qquad (58)$$

$$V_{11}=(\mathcal{P}_{2,i}(x)-x\mathcal{P}_{2,i}(1))\mathcal{H}_j(y)(\mathcal{P}_{2,k}(t)-t\mathcal{P}_{2,k}(1)), \qquad (59)$$

and

$$\Omega(x,y,t)=F_1(x,y,t)-EXTRA-TERMS. \qquad (60)$$

where EXTRA-TERMS termed as remaining terms from (26), (35), (44), (52) and (53). Discretisation by using $x\to x_l$, $y\to y_l$ and $t\to t_l$. From (54), we obtained Haar wavelet coefficients and the numerical solution can be obtained by substituting wavelet coefficients into (52). For finding R_1, S_1, T_1, U_{11}, V_{11} and Ω, we have use the concept of Kronecker product.

4. Error Analysis for Two-Dimensional Telegraph Equation

In this section we present the error analysis of two-dimensional partial differential equations such as two-dimensional telegraph equations. In order to analyze the convergence of our proposed method, we state and prove the following convergence theorem

Theorem 4.1. *Suppose that $u(x,y,t)$ satisfies a Lipschitz condition on $D=[0,1)\times[0,1)\times[0,1)$, that is there exist a positive constant L_1, L_2, L_3 and L_4, such that for all (x_1,y,t), (x_2,y,t), (x_3,y,t), (x_4,y,t), (x_5,y,t), (x_6,y,t), (x_7,y,t) and (x_8,y,t) in D, we have*

$$\begin{cases} | u(x_2, y, t) - u(x_1, y, t) |= L_1| x_2 - x_1 |, \\ | u(x_4, y, t) - u(x_3, y, t) |= L_2| x_4 - x_3 |, \\ | u(x_6, y, t) - u(x_5, y, t) |= L_3| x_6 - x_5 |, \\ | u(x_8, y, t) - u(x_7, y, t) |= L_4| x_8 - x_7 |. \end{cases} \tag{61}$$

Then, the error bound $\| E_m \|_2$ *obtained from above is*

$$\| E_m \|_2 \approx O(\frac{1}{m})^4. \tag{62}$$

Here, the order of convergence is of the order 4.

Proof. Consider $L_1 = L_2 = L_3 = L_4 = L$. Let $u_{exact}(x, y, t)$ and $u_{approximate}(x, y, t)$ be the exact and approximate solutions of the partial differential equation. The error at the Jth level of resolution is defined as:

$$\begin{aligned} E_m &= u_{exact}(x, y, t) - u_{approximate}(x, y, t) \\ &= \sum_{i_1=2M+1}^{\infty} \sum_{i_2=2M+1}^{\infty} \sum_{i_3=2M+1}^{\infty} \mathcal{C}_{i_1 i_2 i_3} \mathcal{H}_{i_1}(x) \mathcal{H}_{i_2}(y) \mathcal{H}_{i_3}(t) \\ &= \sum_{i_1, i_2, i_3=2M+1}^{\infty} \mathcal{C}_{i_1 i_2 i_3} \mathcal{H}_{i_1}(x) \mathcal{H}_{i_2}(y) \mathcal{H}_{i_3}(t), \end{aligned} \tag{63}$$

where

$$u_{approximate} = \sum_{i_1=1}^{2M} \sum_{i_2=1}^{2M} \sum_{i_3=1}^{2M} \mathcal{C}_{i_1 i_2 i_3} \mathcal{H}_{i_1}(x) \mathcal{H}_{i_2}(y) \mathcal{H}_{i_3}(t), \tag{64}$$

and the wavelet coefficients are calculated as follows:

$$\begin{aligned} \mathcal{C}_{i_1 i_2 i_3} &= \int_0^1 \int_0^1 \int_0^1 u(x, y, t) \mathcal{H}_{i_1}(x) \mathcal{H}_{i_2}(y) \mathcal{H}_{i_3}(t) dxdydt \\ &= \Big\langle \mathcal{H}_{i_1}(x), \Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x, y, t), \mathcal{H}_{i_3}(t) \Big\rangle \Big\rangle \Big\rangle. \end{aligned} \tag{65}$$

Here $\Big\langle . \Big\rangle$ shows the inner product. Define $\| \, . \, \|_2$ as:

$$\| E_m \|_2^2 = \int_0^1 \int_0^1 \int_0^1 \Big[u_{exact}(x, y, t) - u_{approximate}(x, y, t) \Big]^2 dxdydt. \tag{66}$$

From (63) and (66), we obtain

$$\| E_m \|_2^2 = \int_0^1 \int_0^1 \int_0^1 \Big[\sum_{i_1, i_2, i_3 = 2M+1}^{\infty} \mathcal{C}_{i_1 i_2 i_3} \mathcal{H}_{i_1}(x) \mathcal{H}_{i_2}(y) \mathcal{H}_{i_3}(t) \Big]^2 dxdydt. \tag{67}$$

Using definition of inner product, from (67), we obtain

$$\| E_m \|_2^2 = \sum_{i_1, i_2, i_3 = 2M+1}^{\infty} \sum_{p, q, r = 2M+1}^{\infty} \mathcal{C}_{i_1 i_2 i_3} \mathcal{C}_{pqr} \Big[\mathcal{H}(i_1, p) \Big] \Big[\mathcal{H}(i_2, q) \Big] \Big[\mathcal{H}(i_3, r) \Big], \tag{68}$$

where

$$\mathcal{H}(i_1, p) = \Big(\int_0^1 \mathcal{H}_{i_1}(x) \mathcal{H}_p(x) dx \Big), \tag{69}$$

$$\mathcal{H}(i_2, q) = \Big(\int_0^1 \mathcal{H}_{i_2}(y) \mathcal{H}_q(y) dy \Big), \tag{70}$$

$$\mathcal{H}(i_3, r) = \Big(\int_0^1 \mathcal{H}_{i_3}(t) \mathcal{H}_r(t) dt \Big). \tag{71}$$

Using orthogonality conditions, from (68), we obtain

$$\| E_m \|_2^2 = \frac{1}{m^3} \sum_{i_1, i_2, i_3 = 2M+1}^{\infty} \mathcal{C}_{i_1 i_2 i_3}^2. \tag{72}$$

Using definition of inner product space and (7), we can write

$$\begin{aligned} \Big\langle u(x, y, t), \mathcal{H}_{i_3}(t) \Big\rangle &= \int_0^1 u(x, y, t) \mathcal{H}_{i_3}(t) dt \\ &= \int_{k/m}^{k+0.5/m} u(x, y, t) dt - \int_{k+0.5/m}^{k+1/m} u(x, y, t) dt. \end{aligned} \tag{73}$$

Applying mean value theorem, that is there exist $t_1 \in [\frac{k}{m}, \frac{k+0.5}{m}]$ and $t_2 \in [\frac{k+0.5}{m}, \frac{k+1}{m}]$, such that

$$\begin{aligned}
&\Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle \\
&\quad = \Big[\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x,y,t_1) - \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x,y,t_2)\Big] \\
&\qquad = \frac{1}{2m}\Big[u(x,y,t_1) - u(x,y,t_2)\Big]. \qquad (74)
\end{aligned}$$

Again,

$$\Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle\Big\rangle = \Big\langle \mathcal{H}_{i_2}(y),\ \frac{1}{2m}\Big(u(x,y,t_1) - u(x,y,t_2)\Big)\Big\rangle, \qquad (75)$$

From (75), using the definition of inner product, we obtain

$$\Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle\Big\rangle = \int_0^1 \frac{1}{2m}\Big[u(x,y,t_1) - u(x,y,t_2)\Big]\mathcal{H}_{i_2}(y)dy, \qquad (76)$$

Using (7), from (76), we obtain

$$\begin{aligned}
&\Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle\Big\rangle \\
&\quad = \frac{1}{2m}\Big[\int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y,t_1)dy - \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y,t_1)dy\Big] \\
&\qquad + \frac{1}{2m}\Big[\int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y,t_2)dy - \int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y,t_2)dy\Big]. \qquad (77)
\end{aligned}$$

Applying mean value theorem, we obtain:

$$\begin{aligned}
&\Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle\Big\rangle \\
&= \frac{1}{2m}\Big[\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x,y_1,t_1) - \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x,y_2,t_1)\Big] \\
&+ \frac{1}{2m}\Big[\Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x,y_4,t_2) - \Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x,y_3,t_2)\Big]. \qquad (78)
\end{aligned}$$

After simplifications, from (78), we obtain

$$\Big\langle \mathcal{H}_{i_2}(y), \Big\langle u(x,y,t),\ \mathcal{H}_{i_3}(t)\Big\rangle\Big\rangle = \frac{1}{2^{j+2}m}\Big[u(x,y_1,t_1) - u(x,y_2,t_1) - u(x,y_3,t_2) + u(x,y_4,t_2)\Big]. \quad (79)$$

Hence,

$$\mathcal{C}_{i_1i_2i_3} = \Big\langle \mathcal{H}_{i_1}(x),\ \frac{1}{2^{j+2}m}\Big[u(x,y_1,t_1) - u(x,y_2,t_1) - u(x,y_3,t_2) + u(x,y_4,t_2)\Big]\Big\rangle$$
$$= \frac{1}{2^{j+2}m}\int_0^1 \Big[u(x,y_1,t_1) - u(x,y_2,t_1) - u(x,y_3,t_2) + u(x,y_4,t_2)\Big]\mathcal{H}_i(x)dx. \quad (80)$$

From (80), using (7), we obtain

$$\begin{aligned}\mathcal{C}_{i_1i_2i_3} = &\frac{1}{2^{j+2}m}\Big[\int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y_1,t_1)dx - \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y_1,t_1)dx\Big]\\ &- \frac{1}{2^{j+2}m}\Big[\int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y_2,t_1)dx + \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y_2,t_1)dx\Big]\\ &+ \frac{1}{2^{j+2}m}\Big[\int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y_4,t_2)dx - \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y_4,t_2)dx\Big]\\ &- \frac{1}{2^{j+2}m}\Big[\int_{\frac{k}{m}}^{\frac{k+0.5}{m}} u(x,y_3,t_2)dx - \int_{\frac{k+0.5}{m}}^{\frac{k+1}{m}} u(x,y_3,t_2)dx\Big]. \quad (81)\end{aligned}$$

Applying mean value theorem, from (81), we obtain

$$\begin{aligned}\mathcal{C}_{i_1i_2i_3} = &\frac{1}{2^{j+2}m}\Big[\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x_1,y_1,t_1) - \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x_2,y_1,t_1)\Big]\\ &+ \frac{1}{2^{j+2}m}\Big[-\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x_3,y_2,t_1) + \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x_4,y_2,t_1)\Big]\\ &+ \frac{1}{2^{j+2}m}\Big[\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x_5,y_4,t_2) - \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x_6,y_4,t_2)\Big]\\ &+ \frac{1}{2^{j+2}m}\Big[-\Big(\frac{k+0.5}{m} - \frac{k}{m}\Big)u(x_7,y_3,t_2) + \Big(\frac{k+1}{m} - \frac{k+0.5}{m}\Big)u(x_8,y_3,t_2)\Big]. \quad (82)\end{aligned}$$

After simplifications, from (82), we obtain

$$\begin{aligned}&|\mathcal{C}_{i_1i_2i_3}|\\&\leqslant \frac{1}{2^{2j+3}m}\Big[|u(x_1,y_1,t_1)-u(x_2,y_1,t_1)+u(x_4,y_2,t_1)-u(x_3,y_2,t_1)|\Big]\\&+\frac{1}{2^{2j+3}m}\Big[|u(x_5,y_4,t_2)-u(x_6,y_4,t_2)+u(x_8,y_3,t_2)-u(x_7,y_3,t_2)|\Big].\end{aligned} \tag{83}$$

Using (61), from (83), we obtain

$$|\mathcal{C}_{i_1i_2i_3}|\leqslant \frac{1}{2^{2j+3}m}\frac{4L}{2m}, \tag{84}$$

where $L = max\{L_1, L_2, L_3, L_4\}$. After simplifications, from (84), we obtain

$$|\mathcal{C}_{i_1i_2i_3}|\leqslant \frac{4L}{2^{2j+4}m^2}\leqslant \frac{L}{2^{2j+2}}\frac{1}{m^2}. \tag{85}$$

Squaring both sides, from (85), we obtain

$$\mathcal{C}^2_{i_1i_2i_3}\leqslant \frac{L^2}{2^{4j+4}}\frac{1}{m^4}. \tag{86}$$

By substituting (86) in (72), we obtain

$$\parallel E_m \parallel_2^2\leqslant \frac{1}{m^3}\sum_{i_1,i_2,i_3=2M+1}^{\infty}\frac{L^2}{2^{4j+4}m^4}. \tag{87}$$

After simplifications, from (87), we obtain

$$\parallel E_m \parallel_2^2\leqslant \frac{L^2}{m^7}\frac{1}{2^4}\sum_{i_1,i_2,i_3=2M+1}^{\infty}\frac{1}{2^{4j}}. \tag{88}$$

Expanding (88), we obtain

$$\parallel E_m \parallel_2^2\leqslant \frac{L^2}{m^7}\frac{1}{2^4}\sum_{j=J+1}^{\infty}\Big(\sum_{i_1=0}^{2^j-1}\sum_{i_2=0}^{2^j-1}\sum_{i_3=0}^{2^j-1}\frac{1}{2^{4j}}\Big). \tag{89}$$

From (89), after simplification, we obtain

$$\| E_m \|_2^2 \leqslant \frac{L^2}{m^7} \frac{1}{2^4} \sum_{j=J+1}^{\infty} \Big(\frac{1}{2^j} \Big). \tag{90}$$

From (90), after series summation, we obtain

$$\| E_m \|_2^2 \leqslant \frac{L^2}{m^8}. \tag{91}$$

After taking square root, we obtain

$$\| E_m \|_2 \approx O(\frac{1}{m^4}). \tag{92}$$

This shows that the convergence is of the order 4.

5. Numerical Experiments and Discussion

We illustrate here, the efficiency and accuracy of the present method by solving few numerical examples of two-dimensional telegraph equations.

Example 5.1. Consider the two-dimensional telegraph equation (1) with $\Gamma = 0$, $\Delta = 1$ and $F_1(x, y, t) = (\pi^2 + 1)\sin \pi x.\sin \pi y.\sin \pi t$. The exact solution of the problem is

$$u(x, y, t) = \sin \pi t.\sin \pi x.\sin \pi y. \tag{93}$$

The maximum absolute errors of Example 5.1 are $2.4155e{-}002$, $1.7278e{-}002$, $5.4222e - 003$ and $1.4323e - 003$ for $J = 0, 1, 2$ and $J = 3$ respectively.

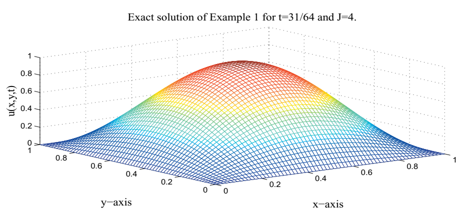

Figure 1: Exact solutions of Example 1 for $t = 31/64$ and $J = 4$.

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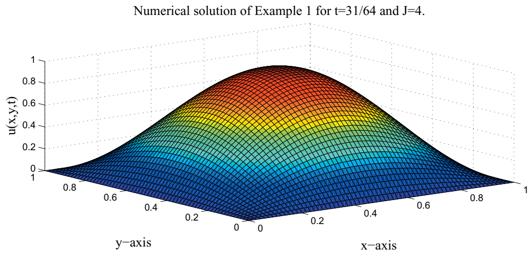

Figure 2: Numerical solutions of Example 1 for $t = 31/64$ and $J = 4$.

Example 5.2. Consider the two-dimensional telegraph equation (1) with $\Gamma = 0$, $\Delta = 1$ and $\digamma_1(x, y, t) = (6t(1-2t) + t^3(1-t) + 2\pi^2 t^3(1-t))$. The exact solution of the problem is

$$u(x, y, t) = t^3.(1-t).\sin \pi x.\sin \pi y. \tag{94}$$

The maximum absolute errors of Example 5.2 are $1.3225e{-}002$, $8.3354e{-}003$, $2.6554e-003$ and $6.9725e-004$ for $J = 0, 1, 2$ and $J = 3$ respectively.

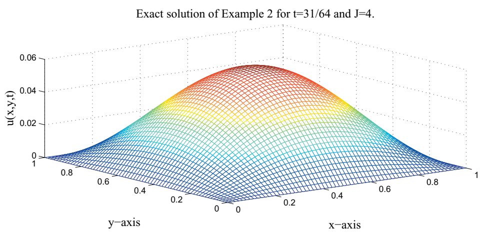

Figure 3: Exact solutions of Example 2 for $t = 31/64$ and $J = 4$.

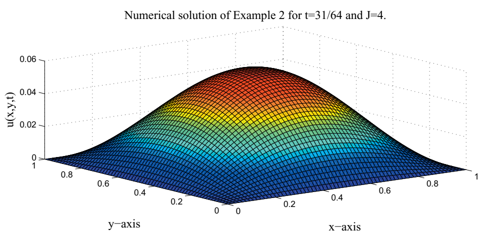

Figure 4: Numerical solutions of Example 2 for $t = 31/64$ and $J = 4$.

Example 5.3. Consider the two-dimensional telegraph equation (1) with $\Gamma = 0$, $\Delta = 0$ and

$$F_1(x,y,t) = 6t(1-2t)x^3(1-x)y^3(1-y) - 6x(1-2x)y^3(1-y)t^3(1-t) - 6y(1-2y)x^3(1-x)t^3(1-t). \quad (95)$$

The exact solution of the problem is

$$u(x,y,t) = t^3.(1-t).x^3.(1-x).y^3.(1-y). \quad (96)$$

The maximum absolute errors of Example 5.3 are $1.5095e{-}004$, $6.1064e{-}005$, $1.7972e-005$ and $4.6994e-006$ for $J = 0, 1, 2$ and $J = 3$ respectively.

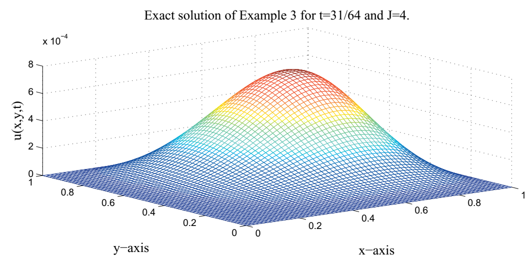

Figure 5: Exact solutions of Example 3 for $t = 31/64$ and $J = 4$.

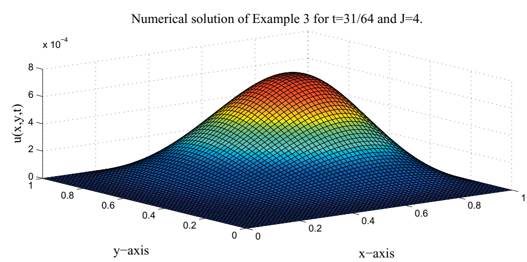

Figure 6: Numerical solutions of Example 3 for $t = 31/64$ and $J = 4$.

Example 5.4. In Example 5.1, if $\Gamma = 0$, $\Delta = 0$, then $\digamma_1(x, y, t) = \pi^2. \sin \pi x. \sin \pi y. \sin \pi t$. The exact solution is (93). The maximum absolute errors of Example 5.4 are $2.6419e-002$, $1.8987e-002$, $5.9681e-003$ and $1.5772e-003$ for $J = 0, 1, 2$ and $J = 3$ respectively.

6. Conclusion

We conclude from the above that, three-dimensional Haar wavelet based methods are also working as a powerful tool for solving two-dimensional telegraph equations. We also conclude here from the above, that the high dimensional Haar wavelet methods are more accurate, simple, fast and computationally efficient for solving partial differential equations. For getting the necessary accuracy the number of calculation points may be increased.

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Inderdeep Singh
Research Scholar of Mathematics
Department of Mathematics
Dr. B.R.Ambedkar National Institute of Technology
Jalandhar, Punjab-144011, India
E-mail: inderdeeps.ma.12@nitj.ac.in

Sheo Kumar
Professor of Mathematics
Department of Mathematics
Dr. B.R.Ambedkar National Institute of Technology
Jalandhar, Punjab-144011, India
E-mail: sheoks53@gmail.com