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f-Grouplikes

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Abstract. A grouplike, which has been introduced earlier, is an algebraic structure between semigroups and groups and its axioms are generalization of the four group axioms. We observe that every grouplike is a homogroup (a semigroup containing an ideal subgroup) with a unique central idempotent. On the other hand, decomposer and associative functions on groups, semigroups and even magmas have been introduced in 2007. In this paper, we introduce special type of grouplikes (namely f-grouplike) that is motivated from the both topics. We prove that f-grouplikes is a proper subclass of Class United Grouplikes, and we study some of their properties.

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1. Introduction

The term "homogroup" was introduced by G. Thierrin on 1961, but earlier it was studied by Clifford, and Miller in [1]. A semigroup is called homogroup (see [1, 6, 7]) if it contains an ideal subgroup. Also, "grouplikes" have been introduced and studied by Hooshmand [2]. The grouplike axioms are generalization of the four group axioms and there is

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a close relation between grouplikes and homogroups. It is shown that a semigroup Γ is grouplike if and only if it contains a unique central idempotent e that is solvable (i.e. for every $x \in \Gamma$ there exists $y \in \Gamma$ in which $xy = yx = e$). The first ideas of the grouplikes are motivated from b-parts of real numbers, introduced by the first author and also are studied in [5]. Recall that an epigroup is a semigroup in which every element has a power that belongs to a subgroup (see [8]). Every "class united grouplike" (i.e. a grouplike with the property $exy = xy$ for all x, y, see [2]) is a unipotent epigroup and homogroup. The class united grouplikes are completely characterized in [2], and also regular grouplikes are studied in [4].

On the other hand, decomposer and associative functions on groups, semigroups (and even magmas) were introduced by the first author and their general form were characterized on arbitrary groups (see [3]). If $(G, \cdot)$ is a group and $f : G \to G$ is an associative function, then the f-multiplication "$\cdot_f$" is an associative binary operation and $(G, \cdot_f)$ is a semigroup with some additional properties.

2. Grouplikes

Recall that a magma (or groupoid) is a basic kind of algebraic structures which consists of a set X equipped with a single binary operation $X \times X \to X$. For every magma X, we consider $Z(X)$ and $It(X) = E(X)$ as the center and the set of all idempotents of X, respectively (it may be empty) and put $Zt(X) := Z(X) \cap It(X)$ that is the set of all central idempotents. Now, we give a summary about grouplikes introduced and studied in [2].

Definition 2.1. *We call a semigroup* $(\Gamma, \cdot)$ grouplike *if it satisfies the following axioms:*
(a) There exists $\varepsilon \in \Gamma$ *such that*

$$\varepsilon x = \varepsilon^2 x = x\varepsilon^2 = x\varepsilon \ : \ \forall x \in \Gamma,$$

(b) For every ε *satisfying (a) and every* $x \in \Gamma$*, there exists* $y \in \Gamma$ *such*

that

$$xy = yx = \varepsilon^2.$$

We call every $\varepsilon \in \Gamma$ *satisfying the axioms (a) and (b) an* identity-like.

Note that axiom (a) is equivalent to $Zt(\Gamma) \neq \emptyset$ and (b) says ε^2 is solvable for every ε satisfying (a). If $(\Gamma, \cdot)$ is a grouplike but not a group, then we call it *proper grouplike*. If a semigroup $(S, \cdot)$ satisfies axiom (a), then we call it *monoidlike*. By unipotent monoidlike [resp. grouplike], we mean a monoidlike [resp. grouplike] with only one idempotent (equivalently, a semigroup with a central idempotent and no other idempotents).

Lemma 2.2.([2]) *Every grouplike contains a unique idempotent identity-like element (denoted by e).*
Let Γ *be a grouplike and let* e *be the unique idempotent identity-like element of* Γ*. Then, we call* e standard identity-like *and use the notation* $(\Gamma, \cdot, e)$.
Note that every identity-like ε *satisfies* $\varepsilon^2 = e = e^2$*, by Lemma 2.2. Every* y *that is corresponded to* x *in axiom (b) is called* inverse-like of x *and is denoted by* x'_e *or* x'*, and the set of all inverse-likes* x' *is denoted by* $Inv_e(x) = Inv(x)$*. So* y *is an inverse-like of* x *(for a given identity-like* ε*) if and only if* $xy = yx = e$.
Therefore, we get the following axioms for grouplikes that is very similar to the four groups axioms:
(i) *Closure,*
(ii) *Associativity,*
(iii) *There exists a* $\underline{\text{unique}}$ *element* $e \in \Gamma$ *such that* $ex = xe$, $e^2 = e$ *for all* $x \in X$ *(i.e. e is its unique central idempotent),*
(iv) *For every* $x \in \Gamma$*, there exists* $y \in \Gamma$ *(not necessarily unique) such that* $xy = yx = e$.
(Of course, we can minimize these axioms and give several equivalent conditions for a semigroup to be grouplike, see [2]).

Example 2.3. Consider the additive group $\mathbb{R}$ and fix $b \in \mathbb{R} \setminus \{0\}$. For each real number a denote by $[a]$ the largest integer not exceeding a and put $(a) = a - [a]$ (the decimal or fractional part of a). Now, set

$$[a]_b = b[\frac{a}{b}] \ , \ (a)_b = b(\frac{a}{b}).$$

We call $[a]_b$ b-integer part of a and $(a)_b$ b-decimal part of a. Also, $[\ ]_b$, $(\ )_b$ are called b-decimal part function and b-integer part function, respectively. By $+_b$ we mean the b-addition $x +_b y := (x + y)_b$ for all real numbers x, y. It is shown that

$$(x +_b y) +_b z = (x + y + z)_b = x +_b (y +_b z) \quad : \quad \forall x, y, z \in \mathbb{R},$$

$(\mathbb{R}, +_b, 0)$ is a proper grouplike (namely real b-grouplike, and specially real grouplike if $b = 1$). Moreover, the set of all its identity-likes is $b\mathbb{Z}$ that is also the set of all inverse-likes of 0.

Lemma 2.4.([2]) *For every grouplike* $(\Gamma, \cdot, e)$ *we have*
(i) $e\Gamma$ *is the least ideal which is also a maximal subgroup of* Γ *and* $Zt(\Gamma)$ *is singleton. Therefore, every grouplike is a homogroup with a unique central idempotent (and visa versa).*
(ii) Γ *is unipotent grouplike if and only if* $It(\Gamma) \subseteq Z(\Gamma)$ *(equivalently* $It(\Gamma) = Zt(\Gamma)$*), so every commutative grouplike is unipotent grouplike.*
(iii) $It(\Gamma) \subseteq Inv(e)$.
(v) In each of the following descriptions, e is the unique element of Γ*:*
- There exists a unique central idempotent element in Γ.
- There exists a unique idempotent identity-like in Γ.
- There exists the least idempotent in Γ.
- There exists a unique solvable identity-like in Γ.

Corollary 2.5. *For every semigroup* S*, the following statements are equivalent.*
(i) S *is a grouplike [resp. unipotent grouplike],*
(ii) S *has an ideal subgroup containing all its central idempotents [resp. idempotents],*
(iii) S *contains a minimum ideal which is also its maximal [resp. maximum] subgroup and* $Zt(S)$ *is singleton.*

For every $x, y \in \Gamma$ we use the notation $x \sim_e y$ if and only if $ex = ey$. The relation $\sim_e$ is a semigroup congruence and $\Gamma/\sim_e = \overline{\Gamma}$ (the set of all equivalent classes $\overline{x}$ that are gotten from $\sim_e$) is its quotient semigroup with the binary operation $\circ$ defined by $\overline{x} \circ \overline{y} = \overline{xy}$.

Theorem 2.6.([2]) *The quotient semigroup* $(\Gamma/\sim_e, \circ)$ *is a group and*

$\Gamma/\sim_e \cong e\Gamma$.
An important class of grouplikes is class united grouplikes that are characterized in [2]. It is shown that a grouplike is class united if and only if it satisfies the identity-like hypothesis ($\mathbf{H_1}$) (i.e. $exy = xy$ for all $x, y \in \Gamma$, which means e is a left bi-identity of Γ). Therefore, $(\Gamma, \cdot)$ is class united grouplike if and only it satisfies the four axioms (i)-(iv) and also (H_1). It is shown that every class united grouplike is unipotent epigroup ($x^2 = ex^2 \in e\Gamma$ for all x).
We say Γ_1 is isomorphic to Γ_2 and denote by $\Gamma_1 \cong \Gamma_2$ if there exists a grouplike isomorphism (equivalently semigroup isomorphism) between them.

3. Decomposer, associative and canceler functions on groups

Let $(G, .)$ be a group with the identity element e. If f, g are functions from G to G, then define the functions $f.g$ and f^- by

$$f.g(x) = f(x)g(x) \quad , \quad f^-(x) = f(x)^{-1} \ : \qquad \forall x \in G.$$

We denote the identity function on G by ι_G and put $f^* = \iota_G.f^-$, $f_* = f^-.\iota_G$ and call f^* [resp. f_*] *left* $*$*-conjugate of* f [resp. *right* $*$*-conjugate of* f]. They are also called $*$*-conjugates of* f. Note that $f^*(e) = f_*(e) = f^-(e) = f(e)^{-1}$.
If $(G, +)$ is additive group, then the notations e, f^-, $f.g$, $f.g^-$ are replaced by 0, $-f$, $f+g$, $f-g$ and we have $f^* = f_* = \iota_G - f$.

Example 3.1. Fix $b \in \mathbb{R} \setminus \{0\}$. Then, $(\ )_b^* = [\ ]_b$ and $[\ ]_b^* = (\ )_b$, both are idempotent, so their compositions are zero and $(\mathbb{R})_b = \mathbb{R}_b = b[0,1) = \{bd | 0 \leqslant d < 1\}$ and $[\mathbb{R}]_b = b\mathbb{Z} = \langle b \rangle$.
If f is an arbitrary function from G to G and $f(x) = f(y)$, then $x = f^*(x)f(y) = f(y)f_*(x)$. The converse is valid if f is decomposer and we have the following definition (see [3]).

Definition 3.2. Let f be a function from G to G. We call f:

(a) *right [resp. left] decomposer* if

$$f(f^*(x)f(y)) = f(y) \quad [\text{resp. } f(f(x)f_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(b) *right [resp. left] strong decomposer* if

$$f(f^*(x)y) = f(y) \quad [\text{resp. } f(xf_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(c) *right canceler* [resp. *left canceler*] if

$$f(xf(y)) = f(xy) \ [\text{resp. } f(f(x)y) = f(xy)] \ : \quad \forall x, y \in G.$$

(d) *associative* if

$$f(xf(yz)) = f(f(xy)z) \ : \quad \forall x, y, z \in G.$$

(e) *strongly associative* if

$$f(xf(yz)) = f(f(xy)z) = f(xyz) \ : \quad \forall x, y, z \in G.$$

Note: We call f *decomposer* or *two-sided decomposer* [resp. canceler] if it is left and right decomposer [resp. canceler]. In each parts of the above and other definitions if $\underline{f(e) = e}$, then we will add the word *standard* to the titles.

Example 3.3. The b-decimal part function $f = (\ )_b$ is standard strong decomposer, canceler and strongly associative. But the b-integer part function $f^* = [\ ]_b$ is only standard decomposer.
Consider $G = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\} \cong D_4$ ($a^4 = b^2 = 1, bab = a^{-1} = a^3$). Put $\Omega = \{1, ba, ba^2, ba^3\}$ and

$$g(x) = \begin{cases} x & x \in \Omega \\ bx & x \notin \Omega \end{cases}$$

Considering the relation $x \notin \Omega \Leftrightarrow bx \in \Omega$, it can be seen that g is (standard) right strong decomposer.

Theorem 3.4. If $f : G \to G$, then all statements of **(A)**, **(B)** and **(C)** are equivalent

(A)
(i) f is associative,
(ii) There exists a constant $c \in G$ ($c = f(e)$) and a standard strongly associative function g such that $f = c \cdot g$ and $g(cxy) = g(xcy) = g(xyc)$ for all x, y, z (i.e $c \in C_c(f) = C_c(g)$, see [3]).
(ii) There exists a constant $d \in G$ and a standard strongly associative function h such that $f = h \cdot d$ and $h(dxy) = h(xdy) = h(xyd)$ for all x, y, z.
(B)
(i) f is strongly associative,
(ii) f is associative and idempotent,
(iii) f is associative and $f(e)$-periodic (i.e. $f(f(e)t) = f(tf(e)) = f(t)$, for all t),
(iv) f is strong decomposer,
(v) f is decomposer and $f^*(G) = f_*(G) \trianglelefteq G$,
(vi) f is canceler,
(ii) f is associative and $f(f(e)) = f(e)$.
(C)
(i) f is standard associative,
(ii) f is standard strongly associative,
(iii) f is standard case of all (one) items of (B).
By using the above theorem, it can be shown that if f is associative, then for every $x, y \in G$

$$f(f(x)y) = f(xf(y)) = f(f(e)xy) = f(xf(e)y) = f(xyf(e)) = f^2(xy),$$

$$f(f^*(x)y) = f(f^*(e)y) = f(yf^*(e)),$$

$$f(xf_*(y)) = f(xf_*(e)) = f(f_*(e)x).$$

Also, we observe that $f^*(G) = f_*(G)$ and it is a normal subgroup if f is strongly associative (see part (B) of the above theorem).

4. f-Grouplikes

We call $\mathcal{G}$ a *class group* if $\mathcal{G}$ is a group for which all its elements are nonempty disjoint sets. For example, every quotient group is a class

group. Now, in view of [2] one can see that:
• Every grouplike $(\Gamma, \cdot, e)$ gives us the class group $\mathcal{G} = \overline{\Gamma} = \Gamma / \sim_e$, and conversely every class group $\mathcal{G}$ gives us the grouplike $\Gamma = \cup\mathcal{G}$.
• Every grouplike gives us the group $G = e\Gamma$, now as the converse we want to construct a class of grouplikes by using a given group G (instead of class groups). For the order we need to consider the f-multiplications. Consider a magma $(X, \cdot)$ and a function f from X to X. We get another binary operation in X by defining $x \cdot_f y = f(xy)$. In fact $\cdot_f = fo\cdot$, hence we call it *f-multiplication of "$\cdot$"*. Clearly $(X, \cdot_f)$ is a semigroup if and only if f is associative (e.g. if f is constant function). Also it is seen that $Z(X, \cdot) \subseteq Z(X, \cdot_f)$. But in general $Z(X, \cdot_f) \nsubseteq Z(X, \cdot)$, for if $X = G$ is a group with the center $\{e\}$ and f is a non-standard associative function on X, then ($e \neq f(e)$ and) $\{e, f(e)\} \subseteq Z(X, \cdot_f)$. By the following main theorem, we can introduce f-grouplikes in several ways.

Theorem 4.1. *If $f : G \to G$, then the following statements are equivalent:*
(i) $(G, \cdot_f)$ is a grouplike,
(ii) $(G, \cdot_f)$ is a monoidlike,
(iii) $(G, \cdot_f)$ is a semigroup,
(iv) f is associative.

Proof. Clearly $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. Now let f be associative. There exists a standard strongly associative function g such that $f = c{\cdot}g$ and $c = f(e)$. Putting $\Delta_g := g^*(G) = g_*(G)$, $\Delta_f := f^*(G) = f_*(G)$ and considering $\delta \in G$, we have $\Delta_f = c^{-1}\Delta_g = \Delta_g c^{-1}$, $\Delta_g \trianglelefteq G$ and

$$\delta \cdot_f x = \delta \cdot_f \delta \cdot_f x \Leftrightarrow f(\delta x) = f(f(\delta^2)x) \Leftrightarrow g(\delta x) = g(c\delta^2 x)$$
$$\Leftrightarrow c\delta^2 x \in \Delta_g \delta x \Leftrightarrow \delta \in c^{-1}\Delta_g = \Delta_f.$$

On the other hand if $\delta \in \Delta_f$, then $\delta = f^*(x_0)$ for some $x_0 \in G$ and

$$\delta \cdot_f x = f(f^*(x_0)x) = cg(g^*(x_0)c^{-1}x) = cg(c^{-1}x)$$
$$= cg(c^{-1}xg^*(x_0)) = cg(xg^*(x_0)c^{-1}) = f(xf^*(x_0)) = x \cdot_f \delta.$$

Now if $x \in G$ and $\delta \in \Delta_f$, then putting $y = f^-(e)\delta x^{-1}$ we have

$$x \cdot_f y = f(xy) = f(xf^-(e)\delta x^{-1}) = f(f^-(e)f(xf^-(e)\delta)x^{-1})$$

$$= f(f^-(e)f(xf^-(e)f^-(e))x^{-1}) = f(xf(e)^{-2}x^{-1}) = f(f(e)^{-2}xx^{-1}) = \delta \cdot_f \delta.$$

Similarly we have $f(yx) = f(f(e)^{-2}) = \delta \cdot_f \delta$. Therefore $(G, \cdot_f)$ is a grouplike and the proof is complete. $\square$

Therefore, if $f : (G.\cdot) \to (G, \cdot)$ is associative, then $(G, \cdot_f)$ is a grouplike and its standard identity-like is $e_f := f(f(e)^{-2}) = f(f^*(e)f_*(e))$.

Corollary 4.2. *If $f : G \to G$, then the following statements are equivalent:*
(i) $(G, \cdot_f, f(e))$ is a grouplike,
(ii) $(G, \cdot_f)$ is a grouplike and $\{f(e), f(e)^-3\} \subseteq f^*(G)$,
(iii) $(G, \cdot_f)$ is a grouplike and $f(e)^{-3} \in f^*(G)$,
(iv) f is associative and $f(f^*(e)f_*(e)) = f(e)$.

Proof. If $(G, \cdot_f, f(e))$ is a grouplike, then $f(e) = e_f = f(f(e)^{-2}) \in f^*(G)$ and so $f(e)^{-3} = f^*(f(e)^{-2}) \in f^*(G)$. Conversely, if $f(e)^{-3} \in f^*(G)$, then $f(e)^{-2} = f^*(x_0)f(e)$, for some $x_0 \in G$, and

$$f(f(e)^{-2}) = f(f^*(x_0)f(e)) = f(f^*(e)f(e)) = f(e).$$

Now, one can obtain these results by Theorem 4.1. $\square$

Corollary 4.3. *If $f : G \to G$, then the following statements are equivalent:*
(i) $(G, \cdot_f, f(e))$ is a grouplike and e is an identity-like element,
(ii) $(G, \cdot_f)$ is a grouplike and e is an identity-like,
(iii) $(G, \cdot_f)$ is a grouplike and $e \cdot_f f(e) = f(e)$,
(iv) f is associative and $f^2(e) = f(e)$,
(v) f is strongly associative.

Proof. Note that if e is an identity-like element, then $e \in f^*(G)$ and so $e = f^*(x_0)$ for some $x_0 \in G$. Thus

$$f(e) = f(f^*(x_0)e) = f(f^*(e)e) = f(f^*(x_0)f^*(e))$$

$$= f(f^*(e)f^*(e)) = e_f,$$

and

$$f(f(e)) = f(f(f(e)^{-2})) = f(f(e)^{-2}f(e)) = f(f(e)^{-1})$$

$$= f(f^*(x_0)f(e)^{-1}) = f(f^*(e)f(e)^{-1}) = e_f = f(e).$$

Therefore, one can drive them. $\square$

Corollary 4.4. *If $f : G \to G$, then the following statements are equivalent:*
(i) $(G, \cdot_f, e)$ is a grouplike,
(ii) $(G, \cdot_f)$ is a grouplike and e is idempotent,
(iii) f is standard associative,
(iv) f is standard strongly associative.

Proof. If $(G, \cdot_f, e)$ is a grouplike, then $e = f(f(e)^{-2}) \in f^*(G)$ and so

$$f(e) = f(f^*(x_0)e) = f(f^*(e)e) = f(f^*(x_0)f^*(e)) = f(f^*(e)f^*(e)) = e,$$

(i.e. $e \cdot_f e = f(e) = e$). Now, we arrive at the results by Theorem 4.1. $\square$

Note. The assumption "$(G, \cdot_f, f(e))$ is a grouplike" alone, does not imply "f is strongly associative". For instance consider $(\mathbb{R}, +_f, \frac{1}{2})$, where $f(x) = \frac{1}{2} + (x)$. Hence, we have

$$f \text{ is associative} \Leftrightarrow (G, \cdot_f, f(f(e)^{-2})) \text{ is a grouplike.}$$

$$f \text{ is standard (strongly) associative} \Leftrightarrow (G, \cdot_f, e) \text{ is a grouplike.}$$

$$f \text{ is strongly associative} \Rightarrow (G, \cdot_f, f(e)) \text{ is a grouplike.}$$

Now, we are ready to introduce f-grouplikes.

Definition 4.5. *We say a magma $(\Gamma, \centerdot)$ is f-grouplike if there exists a binary operation $\cdot$ in Γ such that $(\Gamma, \cdot)$ is group and there exists an associative function $f : (\Gamma, \cdot) \to (\Gamma, \cdot)$ such that $\centerdot = \cdot_f$.*
Therefore, every grouplike $(\Gamma, \centerdot)$ is f-grouplike if and only there exists a binary operation $\cdot$ in Γ such that $(\Gamma, \cdot)$ is group and $\centerdot$ is an f-multiplication of "$\cdot$". Note that, in this case $\centerdot = \cdot_f$ implies $f : (\Gamma, \cdot) \to (\Gamma, \cdot)$ is associative.

Example 4.6. If G is equal to the inner direct product of subgroups H and K (i.e., $G = H \dot{\times} K$), and f is one of the projections in H or K, then $(G, \cdot_f)$ is an f-grouplike. Also, every group, the real b-grouplike, and the Klein four-grouplike are some other examples of f-grouplikes.

Lemma 4.7. *A grouplike $(\Gamma, \centerdot, e)$ is f-grouplike if and only if there exists an f-grouplike $(G, \cdot_f)$ (described in Theorem 4.1) such that $(\Gamma, \centerdot) \cong (G, \cdot_f)$ (as the sense of isomorphic semigroups).*

Proof. Let $\mu : (G, \cdot_f) \to (\Gamma, \centerdot)$ be an isomorphism. Define a binary operation $\odot$ in Γ by $x \odot y = \mu(\mu^{-1}(x) \cdot \mu^{-1}(y))$. Then $(\Gamma, \odot)$ is a group (isomorphic to $(G, \cdot)$) and $F = \mu f \mu^{-1}$ is a function from Γ to Γ. Now we have

$$x \odot_F y = F(x \odot y) = \mu(f(\mu^{-1}(x) \cdot \mu^{-1}(y))) = \mu(\mu^{-1}(x) \cdot_f \mu^{-1}(y))) = x \centerdot y,$$

for all $x, y \in \Gamma$, so $\centerdot = \odot_F$. The converse is clear, thus the proof is complete. $\square$

Now, we are ready to prove another main theorem by using the previous theorem and lemma.

Theorem 4.8. *(a) Every f-grouplike is class united grouplike but the converse is not valid (hence the hypothesis (H_1) holds in every f-grouplike). Therefore, every f-grouplike is a unipotent epigroup and homogroup.*
(b) Let $(\Gamma, \centerdot)$ be an f-grouplike and $(\Gamma, \cdot, e)$ the related ground group. Then, the set of all its identity-likes is equal to $\Delta_f := f^*(\Gamma) = f_*(\Gamma)$ and its standard identity-like is $e_f = f^*(e) \cdot_f f_*(e)$. Moreover, e is an identity-like if and only if f is standard associative (i.e. $f(e) = e$, so $e \centerdot e = e$). Also, we have

$$x \sim_{e_f} y \Leftrightarrow f(f^*(e)x) = f(f^*(e)y) \Leftrightarrow x\Delta_f = y\Delta_f,$$

so $\overline{x} = f^{-1}(\{x\})$.
(c) Putting $\Omega_f := f(\Gamma)$ we have

$$e_f \centerdot \Gamma = e \centerdot \Gamma = \Omega_f \cong (\Gamma, \cdot)/f(e)\Delta_f = (\Gamma, \cdot)/\Delta_f f(e) = \Gamma/\sim_e . \quad (1)$$

Proof. Let $(\Gamma, \centerdot)$ be an f-grouplike, $g : (\Gamma, \cdot) \to (\Gamma, \cdot)$ be the standard strongly associative function in which $f = c \cdot g = f(e) \cdot g$. Then for every x, y in Γ

$$e_f \centerdot x \centerdot y = f(f(f(e)^{-2})f(xy)) = cg(cg(c^{-2})cg(xy))$$
$$= cg(cc^{-2}cxy) = f(xy) = x \centerdot y.$$

Therefore the hypothesis (H_1) holds in $(\Gamma, \centerdot)$ and so it is class united grouplike. Also

$$x \sim_{e_f} y \Leftrightarrow e_f \centerdot x = e_f \centerdot y \Leftrightarrow f(f(f(e)^{-2})x) = f(f(f(e)^{-2})y)$$
$$\Leftrightarrow f(f(e)f(e)^{-2}x) = f(f(e)f(e)^{-2}y) \Leftrightarrow f(f^*(e)x) = f(f^*(e)y)$$
$$\Leftrightarrow x\Delta_g = y\Delta_g \Leftrightarrow x\Delta_f f(e) = y\Delta_f f(e).$$

The function $f : (\Gamma, \cdot) \to (\Omega, \centerdot)$, defined by $f(x) = e \centerdot x$ is an epimorphism and its kernel is $f(e)\Delta_f = \Delta_f f(e)$, and if f is strongly associative, then the kernel is Δ_f.
Now, consider the class united grouplike $\Gamma = \{0, 1, \varepsilon\}$ with the following multiplication table

$\centerdot$	0	1	ε
0	0	1	0
1	1	0	1
ε	0	1	0

We have $\varepsilon \centerdot \Gamma \cong \mathbb{Z}_2$. If there exists a binary operation "$\cdot$" such that $(\Gamma, \cdot)$ is group and $f : (\Gamma, \cdot) \to (\Gamma, \cdot)$ is associative, then $|\Gamma| = |f^*(\Gamma)||f(\Gamma)|$, so $|f(\Gamma)|$ is 1 or 3. On the other hand if e is its identity element, then

$$f(\Gamma) = e \cdot_f \Gamma = e \centerdot \Gamma = \{0, 1\},$$

for every possible cases of $e \in \Gamma$. But this is a contradiction, so Γ is not f-grouplike.
The proof of the other parts is easy, by using Theorem 4.1. $\square$

Example 4.9. Put $(R)_b = \{(r)_b | r \in R\}$ and $[R]_b = \{[r]_b | r \in R\}$ for every subset R of real numbers. If R is an additive sub-group of $\mathbb{R}$ and $b \in R \setminus \{0\}$, then $(R)_b = R \cap \mathbb{R}_b$, $(R, +_b)$ is an f-grouplike, where

$f = (\ )_b|_R$, $((R)_b, +_b)$ is a group (the group $e\Gamma = 0 +_f \Gamma$) and

$$\frac{R}{\langle b\rangle} = \frac{R}{[R]_b} \cong (R)_b = R \cap \mathbb{R}_b \cong (\frac{1}{b}(R)_b, +_1).$$

Specially if $1 \in R$, then $\frac{R}{\mathbb{Z}} \cong (R)_1 = R \cap [0,1)$. Therefore, $\frac{\mathbb{R}}{\mathbb{Z}} \cong ([0,1), +_1)$ that is also a result in [5].

Remark 4.10. *For every f-grouplike $(\Gamma, \centerdot)$ we have the following related items:*

(A) The unipotent homogroup and epigroup $(\Gamma, \centerdot)$,
(B) The ground group $(\Gamma, \cdot)$,
(C) The normal subgroup $f(e)\Delta_f$ of $(\Gamma, \cdot)$,
(D) The group $(\Omega_f, \cdot_f) = (e \centerdot \Gamma, \cdot_f)$,
(E) The factor subset Ω_f of the group $(\Gamma, \cdot)$.

So, there exist several aspects of the topic for future researches.

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