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On Constraint Qualifications for Multiobjective Problems with Equilibrium Constraints

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Abstract. In this paper, we consider a nonsmooth multiobjective problem with equilibrium constraints (MPEC). We present three constraint qualifications (CQs) and investigate their relations. Furthermore, we derive two types of necessary optimality conditions under these CQs.

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1. Introduction

We consider an optimization problem of the form

$$\begin{aligned} \min \quad & f(x) = (f_1(x), \dots, f_q(x)) \\ \text{s.t.} \quad & h_i(x) = 0, \qquad i = 1, 2, ..., p, \\ & g_j(x) \leqslant 0, \qquad j = 1, 2, ..., r, \\ & G_l(x) \geqslant 0, \quad H_l(x) \geqslant 0, \\ & G_l(x) H_l(x) = 0, \qquad l = 1, ..., m, \\ & x \in X, \end{aligned} \tag{MPEC}$$

with locally Lipschitz functions $f_k(k = 1, \dots, q), h_i(i = 1, \dots, p), g_j(j = 1, \dots, r), G_l, H_l(l = 1, \dots, m) : \mathbb{R}^n \to \mathbb{R}$ and $X \subset \mathbb{R}^n$ is a closed set. This

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problem is called a multiobjective problem with equilibrium (or complementarity) constraints. Mathematical programming with equilibrium constraints is the study of constrained optimization problems where the constraints include variational inequalities or complementarities. This problem is related to the Stackelberg game and is used in the study of engineering design, economic equilibrium and multilevel games; see, e.g.,[1, 7, 11]. Many researchers have contributed to the study of optimality conditions and constraint qualifications for these problems; see, e.g., [4, 5, 6, 9, 13]. Recently Kanzow and Schwartz [6] studied the enhanced M-stationary conditions for a smooth single-objective mathematical problem with equilibrium constraints without an abstract set constraint. They introduced the pseudonormality CQ and investigated the relationship between this new CQ and some existing ones such as the local error bound, exact penalty and Abadie CQ when all the constraint functions were considered continuously differentiable. Later, Ye and Zhang [13] proved the pseudonormality CQ implies the existence of a local error bound where equality and inequality constraints were nonsmooth and equilibrium constraints were assumed to be continuously differentiable functions.

Now, we extend their results for MPEC in terms of Mordukhovich subdifferential under weaker conditions. There are two features for this study: first, we consider a multiobjective problems and our CQs are completely described by the feasible region and objective functions play no role in these definitions, unlike usual in the multiobjective literature. Also, we do not impose continuously differentiable condition on equilibrium constraints.

The organization of the paper is as follows. In Section 2, we provide preliminaries that will be used in the rest of the paper. Section 3 is devoted to the main results of the paper. We present three CQs including the pseudonormality, local error bound and Abadie CQ and discuss about their relations. Also, we derive two types of M-stationary conditions under these CQs.

2. Preliminaries

In this section, we recall some background materials on nonsmooth analysis and preliminary results from [3, 8] which will be used later. Our notations are basically standard. We denote by $\|x\|$, the l_1-norm, i.e., $\|x\| := \sum_{i=1}^{n} |x_i|$. $\mathbb{B}_\varepsilon(x)$ stands for the open ball centered at x with radius $\varepsilon > 0$.

For a function $g : \mathbb{R}^n \to \mathbb{R}$, we denote by $g^+(x) := \max\{0, g(x)\}$ and for the vector-valued function the maximum is taken componentwise. Let S be a nonempty closed subset of $\mathbb{R}^n$, the distance function $\text{dist}_S : \mathbb{R}^n \to \mathbb{R}_+$ is defined by $\text{dist}_S(x) := \inf_{y\in S}\{\|y - x\|\}$, and $\text{int}\ S, \text{bd}\ S$ and $\bar{S}$ stand for the interior, boundary and closure of S, respectively. Next let us present some of the basic concepts of generalized differentiation. We start with tangent and normal cones to sets. The Bouligand tangent cone (or contingent cone) of S at x is defined by

$$T(x; S) := \{v \in \mathbb{R}^n \ : \ \exists v_n \to v, t_n \downarrow 0 \ : \ x + t_n v_n \in S, \ \forall n \in \mathbb{N}\},$$

and the Fréchet normal cone to S at x is given by

$$N^F(x; S) := \big\{\xi \in \mathbb{R}^n \ : \limsup_{y \xrightarrow{S} x} \frac{\langle \xi, y - x\rangle}{\|y - x\|} \leqslant 0\big\}.$$

The cone

$$N(x; S) := \{\xi \in \mathbb{R}^n \ : \ \exists\ x_n \to x, \text{ and } \xi_n \to \xi \text{ with } \xi_n \in N^F(x_n; S)\}$$

is called the limiting (Mordukhovich) normal cone to S at x. If S is a convex set, then $N^F(x; S) = N(x; S)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near x. The Fréchet subdifferential of f at x is defined by

$$\partial^F f(x) := \Big\{\xi \in \mathbb{R}^n : \liminf_{y\to x} \frac{f(y) - f(x) - \langle \xi, y - x\rangle}{\|y - x\|} \geqslant 0\Big\}.$$

The limiting (Mordukhovich) subdifferential of f at x is given by

$$\partial f(x) := \{\xi \in \mathbb{R}^n \ : \ \exists\ x_n \to x \text{ and } \xi_n \to \xi \text{ with } \xi_n \in \partial^F f(x_n)\}.$$

The limiting normal cone can always be described via the cone spanned on the generalized gradient of the distance function

$$N(x;S) = \bigcup_{\lambda \geqslant 0} \lambda \partial \mathrm{dist}_S(x).$$

The following proposition contains the sum and chain rules and a formula for computing limiting subdifferential of the maximum function.

Proposition 2.1.

(i) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* x *and* $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *be lower semi continuous and finite at* x *and let* c_1, c_2 *be nonnegative scalars. Then*

$$\partial(c_1 f + c_2 g)(x) \subset c_1 \partial f(x) + c_2 \partial g(x).$$

(ii) *Let* $\phi : \mathbb{R}^m \to \mathbb{R}^n$ *be Lipschitz near* x*, and let* $g : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* $\phi(x)$*. Then the function* $f(x') := g(\phi(x'))$ *is Lipschitz near* x*, and we have*

$$\partial f(x) \subset \{\partial \langle \gamma, \phi(.)\rangle (x) : \ \gamma \in \partial g(\phi(x))\}.$$

(iii) *Let for each* $i = 1, \ldots, k$ *the function* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* x*, and set* $f(x') := \max_{1 \leqslant i \leqslant k} f_i(x')$*. If* $\xi \in \partial f(x)$*, then there exist* $\gamma_i \geqslant 0$ $(i = 1, \ldots, n)$ *with* $\sum_{i=1}^k \gamma_i = 1$ *and* $\gamma_i = 0$ *for* $i \notin M(x)$ *such that* $\xi \in \partial(\sum_{i=1}^k \gamma_i f_i)(x)$*, where* $M(x) := \{i : f(x) = f_i(x)\}$.

Next we recall a necessary condition for a point to be a local minimizer.

Proposition 2.2. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* x *and* S *be closed subset of* $\mathbb{R}^n$*. If* x *is a local minimizer of* f *on* S*, then* $0 \in \partial f(x) + N(x;S)$.

We denote the set of feasible set by C and to facilitate the notation, we define the following index sets for an arbitrary $x^* \in C$:

$$\begin{aligned}
\mathfrak{B} &:= \{1, 2, ..., p\},\\
A(x^*) &:= \{j : g_j(x^*) = 0\},\\
I_{00}(x^*) &:= \{l : G_l(x^*) = 0, H_l(x^*) = 0\},\\
I_{0+}(x^*) &:= \{l : G_l(x^*) = 0, H_l(x^*) > 0\},\\
I_{+0}(x^*) &:= \{l : G_l(x^* > 0, H_l(x^*) = 0\}.
\end{aligned}$$

A feasible point $x^* \in C$ is called a Pareto optimal solution for MPEC if there exists no feasible solution x such that $f_k(x) \leqslant f_k(x^*)$ for each $k = 1, \dots, q$ and $f_{k_0}(x) < f_{k_0}(x^*)$ for at least one index k_0. A feasible point x^* is called a weak Pareto optimal solution for MPEC if there exists no feasible solution x such that for each $k = 1, \dots, q$, $f_k(x) < f_k(x^*)$. A feasible point x^* is a local Pareto optimal (local weak Pareto optimal) solution for MPEC if there exists $\delta > 0$ such that x^* is a Pareto optimal (weak Pareto optimal) solution in $\mathbb{B}_\delta(x^*) \cap C$.

3. Main Results

In this section three constraint qualifications including the pseudonormality, local error bound and Abadie are presented for MPEC. The relations between these CQs are also investigated. Then two types of M-stationary conditions are derived under the weakest and strongest ones.

First, we define the pseudonormality CQ for MPEC which is an extension of what was given in [6, 13] for single-objective problems with continuously differentiable complementarity constraints.

Definition 3.1. *We say that the pseudonormality CQ is satisfied at x^* if there is no nonzero vector $(\lambda, \mu, \gamma, \nu) \in \mathbb{R}^{p+r+m+m}$ such that*

(i) $0 \in \sum_{i=1}^{p} \partial(\lambda_i h_i)(x^*) + \sum_{j=1}^{r} \mu_j \partial g_j(x^*) + \sum_{l=1}^{m} [\partial(\gamma_l G_l)(x^*) + \partial(\nu_l H_l)(x^*)] + N(x^*; X)$;

(ii) $\mu_j \geqslant 0 \ \forall j = 1, \dots, r,\ \gamma_l = 0 \ \forall l \in I_{+0}(x^*),\ \nu_l = 0 \ \forall l \in I_{0+}(x^*)$, *and either* $\gamma_l < 0, \nu_l < 0$ or $\gamma_l \nu_l = 0 \ \forall l \in I_{00}(x^*)$;

(iii) *There exists a sequence $\{x^n\} \subset X$ converging to x^* such that for all n,*

$$\sum_{i=1}^{p} \lambda_i h_i(x^n) + \sum_{j=1}^{r} \mu_j g_j(x^n) + \sum_{l=1}^{m} [\gamma_l G_l(x^n) + \nu_l H_l(x^n)] > 0.$$

Next, we recall the definition of the local error bound CQ for MPEC. Following [9], we say that the local error bound CQ is satisfied at x^* if there exist some positive scalars δ, σ such that for each $x \in \mathbb{B}_\delta(x^*) \cap X$, one has

$$\operatorname{dist}_C(x) \leqslant \sigma\Big(\|h(x)\| + \|g^+(x)\| + \sum_{l=1}^{m} \operatorname{dist}_\Omega(G_l(x), H_l(x))\Big),$$

where $\Omega := \{(a,b) \in \mathbb{R}^2 : a \geqslant 0,\ b \geqslant 0, ab = 0\}$.
Inspired by [9], we define the Abadie CQ for MPEC. As we will show later, our definition coincide with the one in [9] where $q = 1$.

Definition 3.2. *Let all the constraint functions be directionally differentiable at x^*. We say that the Abadie CQ holds at x^* if $T(x^*; S) = L(x^*)$, where*

$$\begin{aligned} L(x^*) \ := \ \Big\{d \in T(x^*; X) \ : \ & h_i^{'}(x^*; d) = 0,\ \forall i \in \mathfrak{B}, \\ & g_j^{'}(x^*; d) \leqslant 0,\ \ \forall j \in A(x^*), \\ & G_l^{'}(x^*; d) = 0,\ \ \forall l \in I_{0+}(x^*), \\ & H_l^{'}(x^*; d) = 0,\ \ \forall l \in I_{+0}(x^*), \\ & G_l^{'}(x^*; d) \geqslant 0,\ \ \forall l \in I_{00}(x^*), \\ & H_l^{'}(x^*; d) \geqslant 0,\ \ \forall l \in I_{00}(x^*), \\ & (G_l^{'}(x^*; d))(H_l^{'}(x^*; d)) = 0,\ \ \forall l \in I_{00}(x^*)\Big\}, \end{aligned}$$

and for some $\sigma > 0$ and each $d \in T(x^*; X)$ one has

$$\begin{aligned} \operatorname{dist}_{L(x^*)}(d) \leqslant \sigma\Big(&\|h_i^{'}(x^*; d)\|_{i\in\mathfrak{B}} + \|g_j^{'+}(x^*; d)\|_{j\in A(x^*)} \\ &+ \|G_l^{'}(x^*; d)\|_{l\in I_{0+}(x^*)} + \|H_l^{'}(x^*; d)\|_{l\in I_{+0}(x^*)} \\ &+ \sum_{l\in I_{00}(x^*)} \operatorname{dist}_\Omega(G_l^{'}(x^*; d), H_l^{'}(x^*; d))\Big). \end{aligned} \tag{1}$$

In [6], the authors showed that the pseudonormality CQ yields the local error bound CQ and the Abadie CQ with continuously differentiable data in the absence of an abstract set constraint for a single-objective mathematical program with equilibrium constraints. Later, Ye and Zhang in [13] generalized their results about the implication of the local error bound CQ by the pseudonormality CQ with continuously differentiable equilibrium constraints. Now, we want to establish the connection between the above CQs for MPEC with nonsmooth data. For this purpose, we do not require continuously differentiable constraint functions. We need some auxiliary results in order to achieve our goals.

Lemma 3.3. *Assume that X is a convex polyhedral set. Then our definition of the Abadie CQ is equivalent to those given in [9].*

Proof. Let $I(x)$ be the identity function,

$$F(x) := \begin{pmatrix} h_i(x)_{i=1,\ldots,p} \\ g_j(x)_{j=1,\ldots,r} \\ \begin{pmatrix} G_l(x) \\ H_l(x) \end{pmatrix}_{l=1,\ldots,m} \end{pmatrix}, \text{ and } \Lambda := \begin{pmatrix} \{0\}^p \\ (-\infty, 0]^r \\ \Omega^m \end{pmatrix}.$$

It is sufficient to show that

$$T((F,I)(x^*); \Lambda \times X) = T(F(x^*); \Lambda) \times T(x^*; X), \tag{2}$$

in order to prove that $L(x^*)$ in Definition 3.2 is equal to the linearized cone defined in [9], in the presence of the constraint set. The inclusion " $\subseteq$ " follows immediately from [10, Proposition 6.41]. To prove the inclusion " $\supseteq$ ", consider an arbitrary vector $d \in T(F(x^*); \Lambda) \times T(x^*; X)$, thus $d = (d_1, d_2)$ with $d_1 \in T(F(x^*); \Lambda)$ and $d_2 \in T(x^*; X)$. The latter gives us, the existence of sequences $d_1^n \to d_1, d_2^n \to d_2, t_1^n \downarrow 0$ and $t_2^n \downarrow 0$ such that $F(x^*) + t_1^n d_1^n \in \Lambda$, and $x^* + t_2^n d_2^n \in X$, for each n. Defining $d^n := (d_1^n, d_2^n) \to (d_1, d_2)$, it remains to find a sequence $t^n \downarrow 0$ such that for each n, $(F(x^*), x^*) + t^n (d_1^n, d_2^n) \in \Lambda \times X$. Taking $t^n := \min\{t_1^n, t_2^n\}$, for each n, the similar argument to [6, Lemma 5.3] implies that $F(x^*) + t^n d_1^n \in \Lambda$, for all n sufficiently large. On the other hand, since X is a convex polyhedral set, it has the following representation

$$X = \{x : a_i x \leqslant b_i, i = 1, \ldots, k\}.$$

Now, if $x^* \in \mathrm{int} X$, trivially $x^* + t^n d_2^n \in X$, for all n sufficiently large. If $x^* \in \mathrm{bd} X$, hence $a_i x^* = b_i$, and also $a_i(x^* + t_2^n d_2^n) \leqslant b_i$, for all $i = 1, \ldots, k$. Consequently, for all i, $a_i t_2^n d_2^n \leqslant 0$, i.e., $a_i d_2^n \leqslant 0$, for each n, which, in turn, implies that $a_i(x^* + t^n d_2^n) \leqslant b_i$, for all $i = 1, \ldots, k$. It means that $x^* + t^n d_2^n \in X$, for each n. Therefore, we arrive at (2).

Furthermore, obviously if the second condition in the Abadie CQ in [9] holds, then (1) also holds. Conversely, suppose that d be an arbitrary vector in $\mathbb{R}^n$. Then, there exists $\hat{d} \in T(x^*; X)$, such that $\operatorname{dist}_{T(x^*;X)}(d) = \|d - \hat{d}\|$. Since the distance function is Lipschitz of rank 1, we have

$$\begin{aligned}
\operatorname{dist}_{L(x^*)}(d) &\leqslant \operatorname{dist}_{L(x^*)}(\hat{d}) + \|d - \hat{d}\| \\
&\leqslant k\Big(\|h_i'(x^*; \hat{d})\|_{i\in\mathfrak{B}} + \|{g_j'}^{+}(x^*; \hat{d})\|_{j\in A(x^*)} \\
&+ \|G_l'(x^*; \hat{d})\|_{l\in I_{0+}(x^*)} + \|H_l'(x^*; \hat{d})\|_{l\in I_{+0}(x^*)} \\
&+ \sum_{l\in I_{00}(x^*)} \operatorname{dist}_\Omega(G_l'(x^*; \hat{d}), H_l'(x^*; \hat{d}))\Big) + \|d - \hat{d}\| \\
&\leqslant k\Big(\|h_i'(x^*; d)\|_{i\in\mathfrak{B}} + \|{g_j'}^{+}(x^*; d)\|_{j\in A(x^*)} \\
&+ \|G_l'(x^*; d)\|_{l\in I_{0+}(x^*)} + \|H_l'(x^*; d)\|_{l\in I_{+0}(x^*)} \\
&+ \sum_{l\in I_{00}} \operatorname{dist}_\Omega(G_l'(x^*; d), H_l'(x^*; d)) + l\|d - \hat{d}\|\Big) + \|d - \hat{d}\| \\
&\leqslant k'\Big(\|h_i'(x^*; d)\|_{i\in\mathfrak{B}} + \|{g_j'}^{+}(x^*; d)\|_{j\in A(x^*)} \\
&+ \|G_l'(x^*; d)\|_{l\in I_{0+}(x^*)} + \|H_l'(x^*; d)\|_{l\in I_{+0}(x^*)} \\
&+ \sum_{l\in I_{00}(x^*)} \operatorname{dist}_\Omega(G_l'(x^*; d), H_l'(x^*; d)) + \operatorname{dist}_{T(x^*;X)}(d)\Big),
\end{aligned}$$

where l is the sum of the Lipschitz constant of $h_i', i \in \mathfrak{B}, {g_j'}^{+}, j \in A(x^*)$, $G_l', l \in I_{0+}(x^*) \cup I_{00}(x^*), H_l', l \in I_{+0}(x^*) \cup I_{00}(x^*)$ and $k' = \max\{k, kl+1\}$, and the proof is complete. □

Lemma 3.4. *Let the pseudonormality CQ hold at x^*. Then there are positive scalars δ, c such that for all $x \in \mathbb{B}_\delta(x^*) \cap X$ and $x \notin C$ the following holds:*

$$\frac{1}{c} \leqslant \|\xi\| \qquad \forall \xi \in \partial\Big(\|h(x)\| + \|g^+(x)\| + \sum_{l=1}^{m} \operatorname{dist}_\Omega(G_l(x), H_l(x))\Big). \quad (3)$$

Proof. Combining the proofs techniques of [12, Lemma 2] and [6, Lemma 4.3], the result can be obtained. □

Lemma 3.5. *Assume that there exist* $\delta, c > 0$ *such that for all* $x \in \mathbb{B}_\delta(x^*) \cap X$ *and* $x \notin C$, (3) *holds. Then*

$$\operatorname{dist}_C(x) \leqslant nc\Big(\|h(x)\|+\|g^+(x)\|+\sum_{l=1}^{m}\operatorname{dist}_\Omega(G_l(x), H_l(x))\Big), \quad \forall x \in \mathbb{B}_{\delta/2}(x^*)\cap X.$$

Proof. The proof follows immediately from the proof of [6, Lemma 4.4]. □

Now, we are ready to establish the relations between the aforementioned CQs. Using Lemmas 3.4 and 3.5, we are able to find some conditions that guarantee the existence of a local error bound.

Theorem 3.6. *Assume that the pseudonormality CQ is satisfied at* x^**, then the local error bound CQ is also satisfied at this point.*

As it was shown in [9], the local error bound CQ implies the Abadie CQ. On the other hand, due to Theorem 3.6 the pseudonormality CQ implies the local error bound CQ. Thus from Lemma 3.9, we obtain the following result.

Theorem 3.7. *Assume that all the constraint functions in MPEC are directionally differentiable at* x^* *and* X *is a convex polyhedral set. If the pseudonormality CQ holds at* x^**, then the Abadie CQ also holds at this point.*

Summing up the above theorems, we have the following diagram:

$$\textit{pseudonormalityCQ} \Longrightarrow \text{local error bound CQ} \Longrightarrow \text{Abadie CQ}.$$

In the next example that we take from [9], we will show that the Abadie CQ is strictly weaker than the pseudonormality CQ.

Example 3.8. Consider the following system

$$\begin{aligned} G_1(x_1,x_2,x_3) &= 1 - \|(x_1,x_2,x_3)\|, \quad & H_1(x_1,x_2,x_3) &= \|(x_1,x_2)\|, \\ G_2(x_1,x_2,x_3) &= \max\{x_1^2, x_2\}, & H_2(x_1,x_2,x_3) &= \|(x_1,x_2)\|. \end{aligned}$$

Similar to what was done in [9], one can check that the Abadie CQ is satisfied at $(0,0,0)$. But the point $(0,0,0)$ does not satisfy the pseudonormality CQ. Because an easy calculation shows that

$\partial H_1(0,0,0) = \{(\xi,\eta,0)|\ \xi,\eta \in [-1,1]\}$, by considering $\gamma_1 = \gamma_2 = \nu_2 = 0,\ \nu_1 = 1$, we have the following relations

$$(0,0,0) \in \partial(\nu_1 H_1(0,0,0)) \text{ and } \nu_1 H_1(x_1,x_2,x_3) > 0,\ \forall (x_1,x_2) \neq 0.$$

In the sequel, we turn our attention to the necessary optimality conditions. The following result describes a relation between the limiting subdifferential of a function f at a point x^* and the limiting subdifferential of the function $v \mapsto f^-(x^*;v)$ at the direction $v = 0$.

Proposition 3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near x^*. Then*

$$\partial f^{'}(x^*;0) \subseteq \partial f(x^*).$$

Proof. For a given vector $\xi \in \partial f^{'}(x^*,0)$, there exist sequences $v_n \to 0$ and $\xi_n \to \xi$ such that for each n, $\xi_n \in \partial^F f^{'}(x^*,v_n)$ which implies that

$$\liminf_{v\to v_n} \frac{f^{'}(x^*,v) - f^{'}(x^*,v_n) - \langle \xi_n, v - v_n\rangle}{\|v - v_n\|} \geqslant 0.$$

For any $\varepsilon > 0$, the latter gives us for each n, the existence of some positive scalar $\delta_n > 0$ such that

$$f^{'}(x^*,v) - f^{'}(x^*,v_n) > \langle \xi_n, v - v_n\rangle - \varepsilon\|v - v_n\|,\ \forall v \in \mathbb{B}_{\delta_n}(v_n). \quad (4)$$

We consider a sequence $\{t_n^k\}_{k=1}^{\infty}$ such that for each n, $\lim_{k\to\infty} t_n^k = 0$ and

$$f^-(x^*;v_n) = \lim_{k\to\infty} \frac{f(x^* + t_n^k v_n) - f(x^*)}{t_n^k}, f^-(x^*;v) = \lim_{k\to\infty} \frac{f(x^* + t_n^k v) - f(x^*)}{t_n^k}.$$

From (4) we obtain

$$\lim_{k\to\infty} \frac{f(x^* + t_n^k v) - f(x^* + t_n^k v_n)}{t_n^k} - \langle \xi_n, v - v_n\rangle + \varepsilon\|v - v_n\| > 0,\ \forall v \in \mathbb{B}_{\delta_n}(v_n). \quad (5)$$

Therefore, (5) implies that for each n, there exists $k_n < k_{n+1}$ such that

$$\frac{f(x^* + t_n^{k_n} v) - f(x^* + t_n^{k_n} v_n)}{t_n^{k_n}} - \langle \xi_n, v - v_n\rangle + \varepsilon\|v - v_n\| > 0,\ \forall v \in \bar{\mathbb{B}}_{\frac{\delta_n}{2}}(v_n).$$

If we set $t_n := t_n^{k_n}$, hence for each n, and each $v \in \mathbb{B}_{\frac{\delta_n}{2}}(v_n)$, we get

$$f(x^* + t_n v) - f(x^* + t_n v_n) - \langle \xi_n, t_n(v - v_n)\rangle + t_n\varepsilon\|v - v_n\| > 0. \quad (6)$$

Defining $x_n := x^* + t_n v_n$, we have $x_n \to x^*$. Finally we show that for each n, $\xi_n \in \partial^F f(x_n)$. For each $y \in \mathbb{B}_{t_n\frac{\delta_n}{2}}(x_n)$, taking $v := \frac{y-x^*}{t_n} \in \mathbb{B}_{\frac{\delta_n}{2}}(v_n)$, we deduce from (6) that

$$f(y) - f(x_n) - \langle \xi_n, y - x_n\rangle > -\varepsilon\|y - x_n\|.$$

Since $\varepsilon > 0$ was chosen arbitrary, then for each n, $\xi_n \in \partial^F f(x_n)$, where $x_n \xrightarrow{f} x^*$. Thus $\xi \in \partial f(x^*)$ and the proof is complete.

Now we are ready to prove the necessary optimality conditions at a local weak Pareto optimal solution of MPEC. Let us start with considering the following scalar optimization problem which is used in the sequel:

$$\begin{aligned} \min \quad & \theta(x) \\ \text{s.t.} \quad & x \in C, \end{aligned} \qquad \text{(P1)}$$

where $\theta(x) := \max\{f_k(x) - f_k(x^*) : k = 1, \ldots, q\}$ and C is the feasible set for MPEC. If x^* is a local weak Pareto optimal solution of MPEC, then one can easily verify that x^* is a local minimum of problem P1.

Theorem 3.10. *Suppose that x^* is a local weak Pareto optimal solution of MPEC and all the constraint and objective functions are directionally differentiable at this point. If the Abadie CQ is satisfied at x^*, then the M-stationary conditions hold: there are multipliers $\alpha, \lambda, \mu, \gamma, \nu \in \mathbb{R}^{q+p+r+2m}$ such that $\alpha \neq 0$ and*

(i) $0 \in \sum_{k=1}^q \alpha_k \partial f_k(x^*) + \sum_{i=1}^p \partial(\alpha_i h_i)(x^*) + \sum_{j=1}^r \mu_j \partial g_j(x^*)$
$+ \sum_{l=1}^m [\partial(\gamma_l G_l)(x^*) + \partial(\nu_l H_l)(x^*)] + N(x^*; X)$;

(ii) $\alpha_k \geqslant 0,\ \mu_j \geqslant 0\ \forall j = 1, \ldots, r, \mu_j = 0,\ \forall j \notin A(x^*)\ \gamma_l = 0\ \forall l \in I_{+0}(x^*),\ \nu_l = 0\ \forall l \in I_{0+}(x^*)$, *and either* $\gamma_l < 0, \nu_l < 0$ *or* $\gamma_l\nu_l = 0\ \forall l \in I_{00}(x^*)$.

Proof. Similar to the proof of [2, Proposition 2.3.2], one can prove that the function θ is directionally differentiable at x^*. First we claim that the following problem attains its minimum at $v = 0$,

$$\begin{aligned} \min \quad & \theta^{'}(x^*; v) \\ \text{s.t.} \quad & v \in L(x^*). \end{aligned} \tag{7}$$

Suppose by contradiction that, there exists a vector $w \in L(x^*)$ such that

$$\theta^{'}(x^*; w) < 0. \tag{8}$$

According to the Abadie CQ and the definition of the tangent cone, we can find sequences $w_n \to w$ and $t_n \downarrow 0$ such that for each n, one has $x^* + t_n w_n \in S$. Now using the Lipschitzness of θ and (8) we obtain $\theta(x^* + t_n w_n) < \theta(x^*)$, for all n sufficiently large, which has a contradiction with the fact x^* is a local minimum of problem P1 hence this contradiction shows that the assertion is true.

Next we prove that the function $v \mapsto \Theta(v)$ attains its minimum on $T(x^*; X)$ at $v = 0$, where

$$\begin{aligned} \Theta(v) := \theta^{'}(x^*; v) + l_\theta \sigma \Big(& \|h_i^{'}(x^*; v)\|_{i \in \mathfrak{B}} + \|g_j^{'+}(x^*; v)\|_{j \in A(x^*)} \\ & + \|G_l^{'}(x^*; v)\|_{l \in I_{0+}(x^*)} + \|H_l^{'}(x^*; v)\|_{l \in I_{+0}(x^*)} \\ & + \sum_{l \in I_{00}(x^*)} \operatorname{dist}_\Omega(G_l^{'}(x^*; v), H_l^{'}(x^*; v)) \Big). \end{aligned}$$

and l_θ is the Lipschitz constant of the function θ near x^*. For a given vector $v \in T(x^*; X)$, considering $\bar{v} := \operatorname{proj}_{L(x^*)} v$ and Lipschitz property of $\theta^{'}(x^*; .)$ together with inequality in (1) we get

$$\begin{aligned} \Theta(0) = \theta^{'}(x^*; 0) \leqslant \theta^{'}(x^*; \bar{v}) & \leqslant \theta^{'}(x^*; v) + l_\theta \|\bar{v} - v\| \\ & = \theta^{'}(x^*; v) + l_\theta \operatorname{dist}_{L(x^*)}(v) \leqslant \Theta(v), \end{aligned}$$

which shows that $v = 0$ is a minimum of the function Θ. Now using Proposition 2.2, we obtain

$$0 \in \partial \Theta(0) + N(0; T(x^*; X)).$$

Thus, we deduce from Propositions 2.1 and 3.9 that

$$0 \in \partial\theta'(x^*;0) + l_\theta\sigma\Big(\sum_{j\in A(x^*)} \partial {g_j'}^{+}(x^*;0) + \sum_{i=1}^{p} \partial|h_i'(x^*;0)| + \sum_{l\in I_{0+}(x^*)} \partial|G_l'(x^*;0)|$$

$$\sum_{l\in I_{+0}(x^*)} \partial|H_l'(x^*;0)| + \sum_{l\in I_{00}(x^*)} \partial\mathrm{dist}_\Omega(G_l'(x^*;0), H_l'(x^*;0))\Big)$$

$$\subseteq \partial\theta'(x^*,0) + \sum_{j\in J}\mu_j\partial g_j'(x^*,0) + \sum_{i=1}^{p}\partial(\lambda_i h_i')(x^*,0) + \sum_{l=1}^{m}[\partial(\gamma_l G_l')(x^*,0)$$

$$+ \partial(\nu_l H_l')(x^*;0)] + N(0;T(x^*;X))$$

$$\subseteq \partial\theta(x^*) + \sum_{j=1}^{r}\mu_j\partial g_j(x^*) + \sum_{i=1}^{p}\partial(\lambda_i h_i)(x^*) + \sum_{l=1}^{m}[\partial(\gamma_l G_l)(x^*) + \partial(\nu_l H_l)(x^*)]$$

$$+ N(x^*;X),$$

where $\mu_j \geqslant 0 (j \in A(x^*))$, $\mu_j := 0$ $(j \notin A(x^*))$ and $\gamma_l = 0$ $\forall l \in I_{+0}(x^*)$, $\nu_l = 0$ $\forall l \in I_{0+}(x^*)$, and either $\gamma_l < 0, \nu_l < 0$ *or* $\gamma_l\nu_l = 0$ $\forall l \in I_{00}(x^*)$. Again by using Proposition 2.1 (iii), we can find nonnegative scalars α_k such that $\sum_{k=1}^{q}\alpha_k = 1$ and condition (i) is fulfilled, this completes the proof of the theorem. □

The last theorem states the enhanced M-stationary conditions at a local weak Pareto optimal solution.

Theorem 3.11. *Let x^* be a local weak Pareto optimal solution of MPEC and the pseudonormality CQ be satisfied at this point. Then the enhanced M-stationary conditions hold: there are multipliers $\alpha, \lambda, \mu, \gamma, \nu \in \mathbb{R}^{q+p+r+2m}$ such that $\alpha \neq 0$ and conditions (i)-(ii) in Theorem 3.10 hold and also if $\lambda, \mu, \gamma, \nu$ are not all equal to zero, then there exists a sequence $\{x^n\} \subset X$ converging to x^* such that for all n,*

$$\begin{array}{ll} if\ \lambda_i \neq 0,\ then\ \lambda_i h_i(x^n) > 0, & if\ \mu_j > 0,\ then\ \mu_j g_j(x^n) > 0, \\ if\ \gamma_l \neq 0,\ then\ \gamma_l G_l(x^n) > 0, & if\ \nu_l \neq 0,\ then\ \nu_l H_l(x^n) > 0. \end{array} \tag{9}$$

Proof. Applying problem P1, the proof follows from the proofs of [12, Theorem 1], [6, Theorem 3.1] and Proposition 2.1. □

The assertion of Theorem 3.11 is named Enhanced M-stationary conditions because conditions in (9) develop the M-stationary conditions, which provide a more precise characterization of the constraints that correspond to nonzero multipliers by replacing the complementary slackness condition with a stronger condition, which is called complementarity violation condition (9).

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