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Duality for Non-differentiable Multi-objective Fractional Variational Problem

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Abstract. In this paper, we have introduced efficiency of order m for a class of non-differentiable multi-objective variational problems in which every component of the objective and constraint function contains a term involving the square root of a certain positive semidefinite quadratic form. Necessary optimality conditions are obtained for this solution concept. Parametric dual of non-differentiable multi-objective fractional variational problem is proposed. Duality theorems are proved to relate efficient solutions of order m for primal problem and its dual. These results are obtained using generalized $\rho-$invex functionals of order m. Proposed notion of efficiency of order m leads to stronger results whereas $\rho-$invexity of higher order broadens the domain of the problem.

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1. Introduction

Mathematical programming is tightly interwoven with the classical calculus of variation. Both these subjects have undergone independent development, hence mutual adaptation of ideas and techniques have always been appreciated. The relationship between these two subjects was explored and extended by Hanson [12]. Thereafter variational programming problems [1, 13, 17, 19, 20, 21, 24] have attracted much attention in literature.

Establishing duality results and finding optimality conditions is one of the finest approach to solve such problems. Under different assumptions of convexity and invexity several researchers [3, 4, 15, 18, 22] have used efficiency to establish optimality and duality results for Wolfe as well as Mond-Weir type of duals.

One may come across several type of solution concepts while browsing the literature. One of them is minimizer or maximizer of order m introduced by Auslender [2] and Ward [25]. Jimenez [14] extended the idea of Ward to define notion of strict local efficient solution of order m for vector optimization problem. Bhatia [5] extended this idea further to define global strict minimizer of order m for multi-objective optimization problem. But all these authors have worked for static cases. In this paper, we have introduced efficiency of order m for a class of non-differentiable multi-objective variational problems in which every component of the objective and constraint function contains a term involving the square root of a certain positive semidefinite quadratic form.

Necessary optimality conditions are important because these conditions lay down foundation for many computational techniques in optimization problems as they indicate when a feasible point is not optimal. At the same time these conditions are useful in the development of numerical algorithms for solving certain optimization problems. Further, these conditions are also responsible for the development of duality theory on which there exists an extensive literature and a substantial use of which (duality theory) has been made in theoretical as well as computational applications in many diverse fields. Hence we are motivated to establish necessary optimality conditions for efficient solution of order m for

non-differentiable multi-objective variational problems taking efficiency of order m as optimality criteria. These conditions are further extended to the class of non-differentiable multi-objective fractional variational problem. The parametric dual of above stated problem is given. Weak and strong duality results are established under the assumption generalized $\rho-$invex conditions on the functionals involved.
The paper is organized as follows: In Section 2, some basic definitions and preliminaries are given and necessary optimality conditions for efficiency of order m for multiobjective variational problem are obtained. Section 3, deals with necessary optimality conditions for non-differentiable multiobjective fractional variational problem (NMFVP) using the concept of efficiency of higher order. In Section 4, we propose dual for (NMFVP) for which duality results are obtained under generalized higher order $\rho-$invexity assumptions.

2. Definitions and Preliminaries

Let $\mathbb{R}^n$ denotes a n-dimensional Euclidean space,
$\mathbb{R}^n_+ = \{(x^1, x^2, ..., x^n)^T \in \mathbb{R}^n | x^i \geqslant 0, i = 1, 2, ..., n\}$ and $int\,\mathbb{R}^n_+$ denotes interior of $\mathbb{R}^n_+$ that is $int\,\mathbb{R}^n_+ = \{(x^1, x^2, ..., x^n)^T \in \mathbb{R}^n | x^i > 0, i = 1, 2, ..., n\}$.
For any $x = (x^1, x^2, ..., x^n)^T$, $y = (y^1, y^2, ..., y^n)^T \in \mathbb{R}^n$.

(i) $x = y \Leftrightarrow x^i = y^i$ for all $i = 1, 2, ..., n$.

(ii) $x < y \Leftrightarrow x^i < y^i$ for all $i = 1, 2, ..., n$.

(iii) $x \leqq y \Leftrightarrow x^i \leqslant y^i$ for all $i = 1, 2, ..., n$.

(iv) $x \leqslant y \Leftrightarrow x \leqq y$ and $x \neq y$.

For a given real interval $I = [a, b]$, let $\phi : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable functions with respect to each of its arguments. For notational convenience $\phi(t, x(t), \dot{x}(t))$ will be written as $\phi(t, x, \dot{x})$, where $x : I \to \mathbb{R}^n$ is a piece-wise smooth state function with its derivative $\dot{x}$. We also denote the partial derivative of ϕ with respect to t, x and $\dot{x}$ by $\phi_t, \phi_x, \phi_{\dot{x}}$ respectively. Let $C(I, \mathbb{R}^m)$ be the set of all continuous functions from $I \to \mathbb{R}^m$. Let X be the space of piece-wise smooth state functions

$x : I \to \mathbb{R}^n$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ where the differential operator D is given by $u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds$. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

The Non-differentiable Multi-objective Variational Problem (P1) is defined as follows:

$$\text{(P1) Minimize}\Big(\int_a^b \{f^1(t,x,\dot{x}) + \{x(t)^T B^1(t)x(t)\}^{\frac{1}{2}}\}dt, \dots, \int_a^b \{f^p(t,x,\dot{x}) + \{x(t)^T B^p(t)x(t)\}^{\frac{1}{2}}\}dt\Big)$$

subject to,

$$g^j(t,x,\dot{x}) + \{x(t)^T C^j(t)x(t)\}^{\frac{1}{2}} \leqq 0, t \in I, j \in M = \{1,2,\dots,m\}, \quad (1)$$

$$x(a) = 0, x(b) = 0. \quad (2)$$

where $f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, i \in P = \{1,\dots,p\}$ and $g^j : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, j \in M$, are continuously differentiable functions with respect to each of their arguments. For each $t \in I, B^i(t), i \in P$ and $C^j(t), j \in M$ are $n \times n$ positive semi-definite(symmetric) matrices with $B^i(\cdot)$ and $C^i(\cdot)$ continuous on I. Let X_0 be the set of all feasible solution of (P1) that is

$$X_0 = \{x \in X | g^j(t,x,\dot{x}) + \{x(t)^T C^j(t)x(t)\}^{\frac{1}{2}} \leqq 0, \\ j \in M, t \in I, x(a) = 0, x(b) = 0\}.$$

Definition 2.1. *$\bar{x} \in X_0$ is said to be an efficient solution for (P1) if there is no other $x \in X_0$ such that*

$$\int_a^b \{f^i(t,x,\dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}}\}dt \leqslant \int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt, \text{ for all } i \in P \text{ and,}$$

$$\int_a^b \{f^j(t,x,\dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}}\}dt < \int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt, \text{for at least one } j \in P.$$

Let $m \geqslant 1$ be an integer and ξ be a piece-wise smooth vector valued function on $I \times \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.2. *$\bar{x} \in X_0$ is said to be an efficient solution of order m for (P1) with respect to ξ if there exist $c = (c^1, c^2, ..., c^p) \in int\,\mathbb{R}^p_+$ such that for no other $x \in X_0$*

$$\int_a^b \{f^i(t, x, \dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}}\}dt \leqslant \int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}} + c^i\|\xi(t, x, \bar{x})\|^m\}dt,$$

for all $i \in P$ and,

$$\int_a^b \{f^j(t, x, \dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}}\}dt < \int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}} + c^j\|\xi(t, x, \bar{x})\|^m\}dt,$$

for at least one $j \in P$.

Lemma 2.3. *(Chankong and Haimes [7]) Let $\bar{x}$ be an efficient solution of order m for (P1) with respect to ξ. For each $k \in P$, define*

$$P(k)\ \text{Minimize} \int_a^b \{f^k(t, x, \dot{x}) + \{x(t)^T B^k(t)x(t)\}^{\frac{1}{2}}\}dt$$

subject to,

$$g^j(t, x, \dot{x}) + \{x(t)^T C^j(t)x(t)\}^{\frac{1}{2}} \leqq 0, t \in I, j \in M,$$

$$\int_a^b \{f^j(t, x, \dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}}\}dt \leqslant \int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt, j \in P, j \neq k,$$

$$x(a) = 0, x(b) = 0.$$

Then $\bar{x}$ is an optimal solution of order m for $P(k)$ with respect to ξ, $k \in P$.

Proof. *Fix $k \in P$. If possible suppose $\bar{x}$ is not an optimal solution of m for $P(k)$ with respect to $\xi, k \in P$, that is for each $\alpha(k) \in \mathbb{R}_+ - \{0\}$ there exist feasible solution for the problem $P(k)$ namely $\hat{x}$ such that*

$$\int_a^b \{f^k(t, \hat{x}, \dot{\hat{x}}) + \{\hat{x}(t)^T B^k(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt$$
$$< \int_a^b \{f^k(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^k(t)\bar{x}(t)\}^{\frac{1}{2}} + \alpha(k)\|\xi(t, \hat{x}, \bar{x})\|^m\}dt,$$

Feasibility of $\hat{x}$, yields

$$\int_a^b \{f^j(t, \hat{x}, \dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt \leqslant \int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt,$$

$$\leqslant \int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}} + \alpha(j)\|\xi(t, \hat{x}, \bar{x})\|^m\}dt,$$

for each $\alpha(j) \in \mathbb{R}_+ - \{0\}, j \in P, j \neq k$.

Using above inequalities we arrive a contradiction to the fact that $\bar{x}$ is an efficient solution of order m for (P1) with respect to ξ. Hence the result follows. $\square$

Lemma 2.4. *Fix $k \in P$, $\bar{x}$ be an optimal solution of order m for $P(k)$ with respect to ξ. Then there exist $\tau = (\tau^1, \tau^2, \ldots, \tau^p) \in \mathbb{R}^p_+$, piece-wise smooth functions $\lambda^j : I \to \mathbb{R}, j \in M, z^i : I \to \mathbb{R}^n, i \in P, w^i : I \to \mathbb{R}^n$, $i \in M$, such that*

$$\sum_{i=1}^{p} \tau^i (f^i_{\bar{x}}(t) + B^i(t) z^i(t)) + \sum_{j=1}^{m} \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t) w^j(t)) \tag{3}$$
$$= \frac{d}{dt}\Big[\sum_{i=1}^{p} \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^{m} \lambda^j(t) g^j_{\dot{\bar{x}}}(t)\Big],\ t \in I,$$

$$\int_a^b \sum_{j=1}^m \lambda^j(t)\{g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T C^j(t) w^j(t)\}dt = 0, \tag{4}$$

$$\tau \geqq 0, \lambda^j(t) \geqq 0, j \in M, (\tau, \lambda^1(t), \dots, \lambda^m(t)) \neq 0\,, t \in I, \tag{5}$$

$$z^i(t)^T B^i(t) z^i(t) \leqslant 1, (\bar{x}(t)^T B^i(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T B^i(t) z^i(t), i \in P \tag{6}$$

$$w^j(t)^T C^j(t) w^j(t) \leqslant 1, (\bar{x}(t)^T C^j(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T C^j(t) w^i(t), j \in M. \tag{7}$$

Proof. Let $\bar{x}$ be an optimal solution of order m for $P(k)$ with respect to ξ. Let
$F^i(x) = \int_a^b \{f^i(t,x,\dot{x})\}dt, i \in P$,
$J^i(x) = \int_a^b \{x(t)^T B^i(t) x(t)\}^{\frac{1}{2}} dt, i \in P$,
$G(x)(t) = (g^1(t,x,\dot{x}) + \{x(t)^T C^1(t) x(t)\}^{\frac{1}{2}}, \dots,$
$g^m(t,x,\dot{x}) + \{x(t)^T C^m(t) x(t)\}^{\frac{1}{2}})$,
$L^2(I,\mathbb{R}) = \{f : I \to \mathbb{R} | f \text{ is measurable and } \int_a^b |f(t)|^2 d\mu(t) < \infty\}$, where μ is Lebesgue measure.
$K = C_+(I,\mathbb{R}^m) = \{s \in C(I,\mathbb{R}^m) | s(t) \geqq 0)\} \subset L^2(I,\mathbb{R}^m)$
The problem $P(k)$ may be rewritten as Cone Constrained Problem (CCP):

$$\text{(CCP) Minimize } F^k(x) + J^k(x)$$

subject to,

$$-G(x) \in K,\ x \in X.$$

$$-F^j(x) - J^j(x) + F^j(\bar{x}) + J^j(\bar{x}) \in \mathbb{R}_+, j \in P, j \neq k.$$

Since $\bar{x}$ is an optimal solution of P(k), so is of (CCP). By Fritz John Theorem [9], there exist $\tau^k \in \mathbb{R}_+, \rho \in K^*$(polar cone of K)$, \tau^j \in (\mathbb{R}_+)^* = \mathbb{R}_+, j \in P, j \neq k$ not all zero such that

$$0 \in \tau^k\{\partial(F^k + J^k)(\bar{x}))\} + \rho\partial G(\bar{x}) + \sum_{j \in P, j\neq k} \tau^j\{\partial(F^j + J^j)(\bar{x})\}, \tag{8}$$

where $\partial(F^i + J^i)(\bar{x}), i \in P, \partial G(\bar{x})$ denote sub-differential of $F^i + J^i, G$ at $\bar{x}$, respectively.

$$\rho G(\bar{x}) = 0. \tag{9}$$

Since $\rho \in K^* \Rightarrow \rho \in L^2(I, \mathbb{R}^m)$, by Riesz representation theorem [16] there exist $\lambda = (\lambda^1, \ldots, \lambda^m) \in L^2(I, \mathbb{R}^m)$ such that

$$\rho(s) = \int_a^b \lambda(t)^T s(t)dt, \text{for all}\, s = (s^1, \ldots, s^m) \in L^2(I, \mathbb{R}^m). \tag{10}$$

Take $s = G(\bar{x})$ in (10) along with (9), we get

$$\int_a^b \sum_{j=1}^m \lambda^j(t)\{g^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T C^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt = 0. \tag{11}$$

(8) implies

$$0 = \tau^k (F^k)'(\bar{x}) + \tau^k \mu^k + \rho\nu + \sum_{j \in P, j \neq k} \tau^j \{(F^j)'(\bar{x}) + \mu^j\}, \tag{12}$$

where $\mu^k \in \partial J^k(\bar{x}), k \in P, \nu \in \partial G(\bar{x})$.

$$(F^i)'(\bar{x})(v) = \int_a^b \{f^i_{\bar{x}}(t)^T v(t) + f^i_{\dot{\bar{x}}}(t)^T \dot{v}(t)\}dt, i \in P, \text{for all}\, v \in X. \tag{13}$$

As $\mu^i \in \partial J^i(\bar{x}), i \in P$, from [8, 10]

$$\mu^i(v) = \int_a^b \{(B^i(t)z^i(t))^T v(t)\}dt, \text{for all}\, v \in X, \tag{14}$$

where $z^i : I \to \mathbb{R}^n,$ with $z^i(t)^T B^i(t) z^i(t) \leqslant 1, (\bar{x}(t)^T B^i(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T B^i(t) z^i(t), i \in P$. As $\nu \in \partial G(\bar{x})$, from [8, 10]

$$\begin{aligned} \nu(v)(t) = ((&g^1_{\bar{x}}(t) + C^1(t)w^1(t))^T v(t) + g^1_{\dot{\bar{x}}}(t)^T \dot{v}(t), \ldots, \\ &(g^m_{\bar{x}}(t) + C^1(t)w^1(t))^T v(t) + g^m_{\dot{\bar{x}}}(t)^T \dot{v}(t)) \end{aligned} \tag{15}$$

for all $v \in X$,

where $w^i : I \to \mathbb{R}^n$, with

$$w^i(t)^T C^i(t) w^i(t) \leqslant 1, (\bar{x}(t)^T C^i(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T C^i(t) w^i(t), i \in M. \tag{16}$$

Using $(10), (13), (14)$ and (15) in (12)

$$\int_a^b \Big\{ \sum_{i=1}^p \tau^i (f^i_{\bar{x}}(t) + B^i(t) z^i(t)) + \sum_{j=1}^m \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t) w^j(t)) \Big\}^T v(t) + \Big\{ \sum_{i=1}^p \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t) \Big\}^T \dot{v}(t) \Big\} dt = 0. \tag{17}$$

for all $v \in X$.

Integrating by parts the following function and using boundary condition of v,

$$\int_a^b \Big\{ \sum_{i=1}^p \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t) \Big\}^T \dot{v}(t) = -\int_a^b \left\{ \frac{d}{dt} \left[\sum_{i=1}^p \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t) \right] \right\}^T v(t) dt.$$

Using above equation in (17), we get

$$\int_a^b \Big\{ \sum_{i=1}^p \tau^i (f^i_{\bar{x}}(t) + B^i(t) z^i(t)) + \sum_{j=1}^m \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t) w^j(t)) - \frac{d}{dt} \Big[\sum_{i=1}^p \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t) \Big] \Big\}^T v(t) dt = 0.$$

for all $v \in X$.

By fundamental theorem of calculus of variation [11],

$$\sum_{i=1}^p \tau^i (f^i_{\bar{x}}(t) + B^i(t) z^i(t)) + \sum_{j=1}^m \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t) w^j(t)) = \frac{d}{dt} \Big[\sum_{i=1}^p \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t) \Big],\ t \in I.$$

(4) follows from (11) and (16). Result now follows by proceeding as in [6]. $\square$

Theorem 2.5. *(Necessary optimality conditions) Let $\bar{x}$ be an efficient solution of order m for (P1) with respect to ξ. Then there exist $\tau = (\tau^1, \tau^2, \ldots, \tau^p) \in \mathbb{R}^p_+$, piece-wise smooth functions $\lambda^j : I \to \mathbb{R}, j \in M, z^i : I \to \mathbb{R}^n, i \in P, w^i : I \to \mathbb{R}^n, i \in M$, such that*

$$\sum_{i=1}^{p} \tau^i (f^i_{\bar{x}}(t) + B^i(t) z^i(t)) + \sum_{j=1}^{m} \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t) w^j(t)) = \frac{d}{dt}\Big[\sum_{i=1}^{p} \tau^i f^i_{\dot{\bar{x}}}(t) + \sum_{j=1}^{m} \lambda^j(t) g^j_{\dot{\bar{x}}}(t)\Big],\ t \in I, \tag{18}$$

$$\int_a^b \sum_{j=1}^{m} \lambda^j(t)\{g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T C^j(t) w^j(t)\} dt = 0, \tag{19}$$

$$\tau \geqq 0, \lambda^j(t) \geqq 0, j \in M, (\tau, \lambda^1(t), \ldots, \lambda^m(t)) \neq 0\,, t \in I, \tag{20}$$

$$z^i(t)^T B^i(t) z^i(t) \leqslant 1, (\bar{x}(t)^T B^i(t) \bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T B^i(t) z^i(t), i \in P, \tag{21}$$

$$w^j(t)^T C^j(t) w^j(t) \leqslant 1, (\bar{x}(t)^T C^j(t) \bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T C^j(t) w^j(t), j \in M. \tag{22}$$

Proof. By Lemma 2.3, $\bar{x}$ is an optimal solution of order m for P(k)(arbitrary chosen but fixed k) with respect to ξ. Results follows Lemma 2.4. $\square$

3. Non-Differentiable Multiobjective Fractional Variational Problem

A class of fractional programming problem, in which objective function is the ratio of two functions, has received considerable importance during past few decades. Because of its ratio structure, it finds its application in various fields like economics, informational theory, engineering, heat exchange networking and numerical analysis. We refer readers to [23] for both the theoretical progress in fractional programming and for mathematical and technical problems where this theory can applied. Now, consider the non-differentiable multiobjective fractional variational problem

$$(P2)\ \text{Minimize}\left(\frac{\int_a^b\{f^1(t,x,\dot{x})+\{x(t)^T B^1(t)x(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^1(t,x,\dot{x})-\{x(t)^T E^1(t)x(t)\}^{\frac{1}{2}}\}dt},\right.$$
$$\left.\ldots,\frac{\int_a^b\{f^p(t,x,\dot{x})+\{x(t)^T B^p(t)x(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^p(t,x,\dot{x})-\{x(t)^T E^p(t)x(t)\}^{\frac{1}{2}}\}dt}\right)$$

subject to

$$g^j(t,x,\dot{x})+\{x(t)^T C^j(t)x(t)\}^{\frac{1}{2}}\leqq 0, t\in I, j\in M, \tag{23}$$

$$x(a)=0, x(b)=0, \tag{24}$$

where $k^i : I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}, i\in P$ are continuously differentiable functions with respect to each of their arguments. For each $t\in I, E^i(t), i\in P$ are $n\times n$ positive semi definite(symmetric) matrices with, $E^i(\cdot)$ continuous on I.
Assume that $\int_a^b\{f^i(t,x,\dot{x})+\{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}}\}dt\geqslant 0$ and $\int_a^b\{k^i(t,x,\dot{x})-\{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}}\}dt>0$, for all $i\in P$, for all $x\in X$.

Definition 3.1. *A point $\bar{x}\in X_0$ is said to be an efficient solution for (P2) if there is no other $x\in X_0$ such that*

$$\frac{\int_a^b\{f^i(t,x,\dot{x})+\{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^i(t,x,\dot{x})-\{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}}\}dt}\leqslant\frac{\int_a^b\{f^i(t,\bar{x},\dot{\bar{x}})+\{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^i(t,\bar{x},\dot{\bar{x}})-\{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt},$$

for all $i\in P$ and

$$\frac{\int_a^b\{f^j(t,x,\dot{x})+\{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^j(t,x,\dot{x})-\{x(t)^T E^j(t)x(t)\}^{\frac{1}{2}}\}dt}<\frac{\int_a^b\{f^j(t,\bar{x},\dot{\bar{x}})+\{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b\{k^j(t,\bar{x},\dot{\bar{x}})-\{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt},$$

for at least one $j\in P$.

Let $m \geqslant 1$ be an integer and $\xi : I \times X \times X \to \mathbb{R}^n$ be a piece-wise continuous function.

Definition 3.2. *A point $\bar{x} \in X_0$ is said to be an efficient solution of order m with respect to ξ for (P2) if there exist $c = (c^1, c^2, ..., c^p) \in int\,\mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\,\mathbb{R}^p_+$ such that for no other $x \in X_0$*

$$\frac{\int_a^b \{f^i(t,x,\dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}} - c^i\|\xi(t,x,\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,x,\dot{x}) - \{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}} + d^i\|\xi(t,x,\bar{x})\|^m\}dt}$$
$$\leqslant \frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{ for all } i \in P \text{ and}$$

$$\frac{\int_a^b \{f^j(t,x,\dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}} - c^j\|\xi(t,x,\bar{x})\|^m\}dt}{\int_a^b \{k^j(t,x,\dot{x}) - \{x(t)^T E^j(t)x(t)\}^{\frac{1}{2}} + d^j\|\xi(t,x,\bar{x})\|^m\}dt}$$
$$< \frac{\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^j(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{for at least one } j \in P.$$

Consider the following parametric non-differentiable multi-objective variational problem ($NMFVP_v$)

$$\textit{Minimize}\Bigg(\int_a^b \{f^1(t,x,\dot{x}) + \{x(t)^T B^1(t)x(t)\}^{\frac{1}{2}} - v^1[k^1(t,x,\dot{x}) - \{x(t)^T E^1(t)x(t)\}^{\frac{1}{2}}]\}dt,$$

$$\ldots, \int_a^b \{f^p(t,x,\dot{x}) + \{x(t)^T B^p(t)x(t)\}^{\frac{1}{2}} - v^p[k^p(t,x,\dot{x}) - \{x(t)^T E^p(t)x(t)\}^{\frac{1}{2}}]\}dt\Bigg)$$

subject to

$$g^j(t,x,\dot{x}) + \{x(t)^T C^j(t)x(t)\}^{\frac{1}{2}} \leqq 0, t \in I, j \in M, \tag{25}$$

$$x(a) = 0, x(b) = 0, \tag{26}$$

$$v = (v^1, v^2, ..., v^p) \in \mathbb{R}^p_+.$$

Lemma 3.3. *If $\bar{x}$ is an efficient solution of order m for (P2) with respect to ξ then there exist $\bar{v} = (\bar{v}^1, \bar{v}^2, ..., \bar{v}^p) \in \mathbb{R}^p_+$ such that $\bar{x}$ is an efficient solution of order m for (NMFVP$_{\bar{v}}$) with respect to ξ.*

Proof. Let $\bar{x}$ be an efficient solution of order m for (P2) with respect to ξ.
Take $\bar{v}^i = \frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, i = 1, 2, \ldots, p.$

If possible suppose, $\bar{x}$ is not an efficient solution of order m for (NMFV$P_{\bar{v}}$) with respect to ξ. Then for any $\rho = (\rho^1, \rho^2, ..., \rho^p) \in int\, \mathbb{R}^p_+$, there exist $\hat{x} \in X_0$ such that

$$\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}} - \bar{v}^i[k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}]\}dt$$
$$\leqslant \rho^i \int_a^b \|\xi(t,\hat{x},\bar{x})\|^m dt, \text{ for all } i \in P \text{ and}$$

$$\int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}} - \bar{v}^j[k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}}]\}dt$$
$$< \rho^j \int_a^b \|\xi(t,\hat{x},\bar{x})\|^m dt, \text{for at least one } j \in P.$$

Which is

$$\frac{\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt} \leqslant \bar{v}^i + \frac{\rho^i \int_a^b \{\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}, \tag{27}$$

for all $i \in P$ and

$$\frac{\int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt} < \bar{v}^j + \frac{\rho^j \int_a^b \{\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}, \tag{28}$$

for at least one $j \in P$.

Case(i) If $\|\xi(t,\hat{x},\bar{x})\|^m = 0$. Then (27) and (28) became

$$\frac{\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt} \leqslant \bar{v}^i, \text{ for all } i \in P,$$

and

$$\frac{\int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt} < \bar{v}^j \text{ for at least one } j \in P.$$

Then for any $c = (c^1, c^2, ..., c^p) \in int\, \mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\, \mathbb{R}^p_+$, there exist $\hat{x} \in X_0$ such that

$$\begin{aligned}&\frac{\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}} - c^i\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}} + d^i\|\xi(t,\hat{x},\bar{x})\|^m\}dt} \\ &\leqslant \frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{for all } i \in P \text{ and}\end{aligned}$$

$$\begin{aligned}&\frac{\int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}} - c^j\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}} + d^j\|\xi(t,\hat{x},\bar{x})\|^m\}dt} \\ &< \frac{\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^j(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{for at least one } j \in P.\end{aligned}$$

Which is contradiction to the fact that $\bar{x}$ is an efficient solution of order m for (P2) with respect to ξ.

Case(ii) If $\|\xi(t,\hat{x},\bar{x})\|^m \neq 0$. For any $c = (c^1, c^2, ..., c^p) \in int\, \mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\, \mathbb{R}^p_+, i \in P$ define

$$\rho^i = \frac{c^i \int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt + d^i \int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}}\}dt + d^i \int_a^b \|\xi(t,\hat{x},\bar{x})\|^m dt}.$$

Substituting this in (27) and (28) yields

$$\begin{aligned}&\frac{\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^i(t)\hat{x}(t)\}^{\frac{1}{2}} - c^i\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^i(t)\hat{x}(t)\}^{\frac{1}{2}} + d^i\|\xi(t,\hat{x},\bar{x})\|^m\}dt} \\ &\leqslant \frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{for all } i \in P \text{ and}\end{aligned}$$

$$\frac{\int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + \{\hat{x}(t)^T B^j(t)\hat{x}(t)\}^{\frac{1}{2}} - c^j\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^j(t,\hat{x},\dot{\hat{x}}) - \{\hat{x}(t)^T E^j(t)\hat{x}(t)\}^{\frac{1}{2}} + d^j\|\xi(t,\hat{x},\bar{x})\|^m\}dt}$$
$$< \frac{\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^j(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, \text{for at least one } j \in P.$$

Which is contradiction to the fact that $\bar{x}$ is an efficient solution of order m for (P2) with respect to ξ. Hence the result follows. □

Theorem 3.4. *(Necessary optimality condition) Let $\bar{x}$ be an efficient solution of order m for (P2) with respect to ξ. Then there exist $\tau = (\tau^1, \tau^2, \dots, \tau^p) \in \mathbb{R}^p_+, v = (v^1, v^2, \dots, v^p)$ piece-wise smooth functions $\lambda^j : I \to \mathbb{R}, j \in M, z^i : I \to \mathbb{R}^n, y^i : I \to \mathbb{R}^n, i \in P, w^i : I \to \mathbb{R}^n, i \in M,$ such that*

$$\sum_{i=1}^p \tau^i(f^i_{\bar{x}}(t) - v^i k^i_{\bar{x}}(t) + B^i(t)z^i(t) + v^i E^i(t)y^i(t)) + \sum_{j=1}^m \lambda^j(t)(g^j_{\bar{x}}(t) + C^j(t)w^j(t))$$
$$= \frac{d}{dt}\Big[\sum_{i=1}^p \tau^i(f^i_{\dot{\bar{x}}}(t) - v^i k^i_{\dot{\bar{x}}}(t)) + \sum_{j=1}^m \lambda^j(t) g^j_{\dot{\bar{x}}}(t)\Big], \; t \in I, \quad (29)$$

$$\int_a^b \sum_{j=1}^m \lambda^j(t)\{g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T C^j(t)w^j(t)\}dt = 0, \quad (30)$$

$$\tau \geqq 0, \lambda^j(t) \geqq 0, j \in M, (\tau, \lambda^1(t), \dots, \lambda^m(t)) \neq 0\,, t \in I, \quad (31)$$

$$v^i = \frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}, i \in P, \quad (32)$$

$$z^i(t)^T B^i(t)z^i(t) \leqslant 1, (\bar{x}(t)^T B^i(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T B^i(t)z^i(t), i \in P \quad (33)$$

$$y^i(t)^T E^i(t)y^i(t) \leqslant 1, (\bar{x}(t)^T E^i(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T E^i(t)y^i(t), i \in P \quad (34)$$

$$w^j(t)^T C^j(t) w^j(t) \leqslant 1, (\bar{x}(t)^T C^j(t)\bar{x}(t))^{\frac{1}{2}} = \bar{x}(t)^T C^j(t) w^j(t), j \in M. \tag{35}$$

Proof. Follows from Lemma 3.3 and Theorem 2.5. □

Definition 3.5. *$\bar{x} \in X_0$ is said to be a normal efficient solution of order m for (P2) with respect to ξ if it is efficient solution of order m for (P2) with respect to ξ and $\sum_{i=1}^{p} \tau^i = 1$.*

4. Duality Results

Parametric Dual
Following the parametric approach of Bector [4], the dual (D) to multiobjective fractional variational problem is defined as follows:
(D) Maximize $v = (v^1, v^2, \ldots, v^p)$
subject to

$$\sum_{i=1}^{p} \tau^i (f_u^i(t) - v^i k_u^i(t) + E^i(t) z^i(t) + v^i B^i(t) z^i(t)) + \sum_{j=1}^{m} \lambda^j(t)(g_u^j(t) + C^j(t) w^j(t))$$
$$= \frac{d}{dt}\Bigg[\sum_{i=1}^{p} \tau^i (f_{\dot{u}}^i(t) - \bar{v}^i k_{\dot{u}}^i(t)) + \sum_{j=1}^{m} \lambda^j(t) g_{\dot{u}}^j(t)\Bigg], t \in I, \tag{36}$$

$$\int_a^b \sum_{j=1}^{m} \lambda^j(t)\{g^j(t, u, \dot{u}) + u(t)^T C^j(t) w^j(t)\} dt \geqslant 0, \tag{37}$$

$$z^i(t)^T B^i(t) z^i(t) \leqslant 1, (u(t)^T B^i(t) u(t))^{\frac{1}{2}} = u(t)^T B^i(t) z^i(t), i \in P, \tag{38}$$

$$y^i(t)^T E^i(t) y^i(t) \leqslant 1, (u(t)^T E^i(t) u(t))^{\frac{1}{2}} = u(t)^T E^i(t) y^i(t), i \in P, \tag{39}$$

$$w^j(t)^T C^j(t) w^j(t) \leqslant 1, (u(t)^T C^j(t) u(t))^{\frac{1}{2}} = u(t)^T C^j(t) w^j(t), j \in M. \tag{40}$$

$$\int_a^b \{f^i(t,u,\dot{u}) + u(t)^T B^i(t) z^i(t) - v^i\{k^i(t,u,\dot{u}) - u(t)^T E^i(t) y^i(t)\}\}dt \geqslant 0, i \in P, \tag{41}$$

$$u \in X, \tau^i \geqslant 0, \sum_{i=1}^{p} \tau^i = 1, \lambda^j(t) \geqq 0, j \in M, t \in I, v^i \geqslant 0,\ i \in P. \tag{42}$$

$$u(a) = 0,\ u(b) = 0. \tag{43}$$

Let U be the set of all feasible solutions of (D).
In order to prove weak duality theorem, the existing class of functionals is not sufficient. Hence we need to generalized this class further. The notion of generalized $\rho-$invexity of higher order solves the purpose.
Let $\Phi : X \to \mathbb{R}$ defined by $\Phi(x) = \int_a^b \phi(t,x,\dot{x})dt$ be Frechet differentiable, where $\phi(t,x,\dot{x})$ is a scalar function with continuous derivatives upto and including second order with respect to each of its arguments. Let there exist a real number ρ and a differentiable vector function $\eta : I \times X \times X \to \mathbb{R}^n$ with

$$\eta(t,x,\bar{x}) = 0 \text{ at } t \text{ if } x(t) = \bar{x}(t). \tag{44}$$

For the sake of convenience, $\phi_x(t)$ represent $\phi_x(t,x(t),\dot{x}(t))$ and $\phi_{\dot{x}}(t)$ represent $\phi_{\dot{x}}(t,x(t),\dot{x}(t))$.

Definition 4.1. *A functional $\Phi(x)$ is said to be ρ-pseudoinvex type 2 of order m at $\bar{x}$ with respect to η and ξ if*
$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt \geqslant 0 \Rightarrow$
$\Phi(x) \geqslant \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt$, *for all $x \in X$.*

Or equivalently

$\Phi(x) < \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt \Rightarrow$
$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt < 0$, *for all $x \in X$.*

Definition 4.2. *A functional $\Phi(x)$ is said to be ρ-quasiinvex type 2 of order m at $\bar{x}$ with respect to η and ξ if*

$\Phi(x) \leqslant \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt \Rightarrow$
$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt \leqslant 0$, *for all* $x \in X$.

Or equivalently

$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt > 0 \Rightarrow$
$\Phi(x) > \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt$, *for all* $x \in X$.

Definition 4.3. *A functional* $\Phi(x)$ *is said to be strictly* ρ*-pseudoinvex type 2 of order* m *at* $\bar{x}$ *with respect to* η *and* ξ *if*
$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt \geqslant 0 \Rightarrow$
$\Phi(x) > \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt$, *for all* $x \in X - \{\bar{x}\}$.

Or equivalently

$\Phi(x) \leqslant \Phi(\bar{x}) + \rho \int_a^b \|\xi(t,x,\bar{x})\|^m dt \Rightarrow$
$\int_a^b \{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + [\frac{d\eta(t,x,\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)\}dt < 0$, *for all* $x \in X - \{\bar{x}\}$.

Lemma 4.4. *[17] Let* $A(t)$ *be* $n \times n$ *positive semi definite (symmetric) matrix, with* $A(\cdot)$ *continuous on* I *and* $s(t)^T A(t)s(t) \leqslant 1$. *Then,*

$$\int_a^b \{x(t)^T A(t)x(t)\}^{\frac{1}{2}} dt \geqslant \int_a^b \{x(t)^T A(t)s(t)\}dt.$$

Various duality results connecting efficient solutions of primal and its dual problem are established in the sequel.

Theorem 4.5. *(Weak duality) Let* $x \in X_0$ *and*

$(u, \tau^1, \ldots, \tau^p, \bar{v}^1, \ldots, \bar{v}^p, \lambda^1, \ldots, \lambda^m, z^1, \ldots, z^p, y^1, \ldots, y^p, w^1, \ldots, w^m) \in U$,

let us write

$$\theta^i(x) = \int_a^b \{f^i(t,x,\dot{x}) + x(t)^T B^i(t)z^i(t) - \bar{v}^i\{k^i(t,x,\dot{x}) - x(t)^T E^i(t)y^i(t)\}\}dt, i \in P,$$

$$G(x) = \int_a^b \sum_{j=1}^m \lambda^j(t)\{g^j(t,x,\dot{x}) + x(t)^T C^j(t)w^j(t)\}dt.$$

Suppose $\theta^i(x)$, *for* $i \in P$ *are strictly* ρ^i*-pseudoinvex type 2 functionals of order* m *at* $\bar{x}$ *with respect to* η *and* ξ *and* $G(x)$ *is* ρ'*-quasiinvex type*

2 functional of order m at $\bar{x}$ with respect to η and ξ, where ρ', $\rho^i \in int\,\mathbb{R}_+$, for $i \in P$. Then the following cannot hold:

$$\frac{\int_a^b \{f^i(t,x,\dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}} - c^i\|\xi(t,x,u)\|^m\}dt}{\int_a^b \{k^i(t,x,\dot{x}) - \{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}} + d^i\|\xi(t,x,u)\|^m\}dt} \leqslant \bar{v}^i,$$

for all $i \in P$ and

$$\frac{\int_a^b \{f^j(t,x,\dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}} - c^j\|\xi(t,x,u)\|^m\}dt}{\int_a^b \{k^j(t,x,\dot{x}) - \{x(t)^T E^j(t)x(t)\}^{\frac{1}{2}} + d^j\|\xi(t,x,u)\|^m\}dt} < \bar{v}^j,$$

for at least one $j \in P$,

for some $c = (c^1, c^2, ..., c^p) \in int\,\mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\,\mathbb{R}^p_+$.

Proof. Contrary to the result, assume that for any $c = (c^1, c^2, ..., c^p) \in int\,\mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\,\mathbb{R}^p_+$

$$\frac{\int_a^b \{f^i(t,x,\dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}} - c^i\|\xi(t,x,u)\|^m\}dt}{\int_a^b \{k^i(t,x,\dot{x}) - \{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}} + d^i\|\xi(t,x,u)\|^m\}dt} \leqslant \bar{v}^i,$$

for all $i \in P$ and

$$\frac{\int_a^b \{f^j(t,x,\dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}} - c^j\|\xi(t,x,u)\|^m\}dt}{\int_a^b \{k^j(t,x,\dot{x}) - \{x(t)^T E^j(t)x(t)\}^{\frac{1}{2}} + d^j\|\xi(t,x,u)\|^m\}dt} < \bar{v}^j,$$

for at least one $j \in P$,

$$\int_a^b \{f^i(t,x,\dot{x}) + \{x(t)^T B^i(t)x(t)\}^{\frac{1}{2}} - \bar{v}^i[k^i(t,x,\dot{x}) - \{x(t)^T E^i(t)x(t)\}^{\frac{1}{2}}]\}dt$$

$$\leqslant (c^i + d^i\bar{v}^i)\int_a^b \|\xi(t,x,u)\|^m dt, \text{for all}\, i \in P\, \text{and}$$

$$\int_a^b \{f^j(t,x,\dot{x}) + \{x(t)^T B^j(t)x(t)\}^{\frac{1}{2}} - \bar{v}^j[k^j(t,x,\dot{x}) - \{x(t)^T E^j(t)x(t)\}^{\frac{1}{2}}]\}dt$$

$$< (c^j + d^j\bar{v}^j)\int_a^b \|\xi(t,x,u)\|^m dt, \text{for at least one } j \in P.$$

Choose $\rho^i = c^i + d^i\bar{v}^i, i \in P$. Using Lemma 4. along with (41) implies

$$\theta^i(x) \leqslant \theta^i(u) + \int_a^b \{\rho^i \|\xi(t,x,u)\|^m\}dt, \text{for all}\, i \in P \text{ and}$$

$$\theta^j(x) < \theta^j(u) + \int_a^b \{\rho^i \|\xi(t,x,u)\|^m\}dt, \text{for at least one}\, j \in P.$$

Since $\theta^i(x)$, for $i \in P$ are strictly ρ^i-pseudoinvex type 2 functionals of order m at $\bar{x}$ with respect to η and ξ, we obtain

$$\int_a^b \{\eta(t,x,u)[f_u^i(t)+B^i(t)z^i(t) - \bar{v}^i(k_u^i(t) - E^i(t)y^i(t))] \\ + \frac{d\eta(t,x,u)}{dt}[f_{\dot{u}}^i(t) - \bar{v}^i k_{\dot{u}}^i(t)]\}dt < 0, \text{for all}\, i \in P. \tag{45}$$

Multiplying (45) by $\tau^i, i \in P$ and by summing over $i \in P$, we get

$$\int_a^b \Big\{\eta(t,x,u)\Big[\sum_{i=1}^p \tau^i\{f_u^i(t) + B^i(t)z^i(t) - \bar{v}^i(k_u^i(t) - E^i(t)y^i(t)\}\Big] \\ + \frac{d\eta(t,x,u)}{dt}\Big[\sum_{i=1}^p \tau^i\{f_{\dot{u}}^i(t) - \bar{v}^i k_{\dot{u}}^i(t)\}\Big]\Big\}dt < 0. \tag{46}$$

$(u, \tau^1, \ldots, \tau^p, \bar{v}^1, \ldots, \bar{v}^p, \lambda^1, \ldots, \lambda^m, z^1, \ldots, z^p, y^1, \ldots, y^p, w^1, \ldots, w^m) \in U$, $x \in X_0$ and $\rho' \|\xi(t,x,\bar{x})\|^m \geqslant 0$, yields

$$G(x) \leqslant 0 \leqslant G(u) + \textstyle\int_a^b \{\rho' \|\xi(t,x,u)\|^m\}dt$$

Since $G(x)$ is ρ'-quasiinvex type 2 functional of order m at $\bar{x}$ with respect to η and ξ, we obtain

$$\int_a^b \{\eta(t,x,u)[\sum_{j=1}^m \lambda^j(t)\{g_u^j(t) + C^j(t)w^j(t)]\} + \frac{d\eta(t,x,u)}{dt}[\sum_{j=1}^m \lambda^j(t)g_{\dot{u}}^j(t)]\}dt \leqslant 0, \tag{47}$$

Adding inequalities (46) and (47), we get

$$\int_a^b \Bigg\{\eta(t,x,u)\Bigg[\sum_{i=1}^{p}\tau^i\{f_u^i(t)+B^i(t)z^i(t)-\bar{v}^i(k_u^i(t)-E^i(t)y^i(t))\} \\ +\sum_{j=1}^{m}\lambda^j(t)\{g_u^j(t)+C^j(t)w^j(t)\}\Bigg] \\ +\frac{d\eta(t,x,u)}{dt}\Bigg[\sum_{i=1}^{p}\tau^i\{f_{\dot{u}}^i(t)-\bar{v}^ik_{\dot{u}}^i(t)\}+\sum_{j=1}^{m}\lambda^j(t)g_{\dot{u}}^j(t)\Bigg]\Bigg\}dt<0. \quad (48)$$

Using (36), we get

$$\int_a^b \Bigg\{\eta(t,x,u)\frac{d}{dt}\Bigg[\sum_{i=1}^{p}\tau^i\{f_{\dot{u}}^i(t)-\bar{v}^ik_{\dot{u}}^i(t)\}+\sum_{j=1}^{m}\lambda^j(t)g_{\dot{u}}^j(t)\Bigg] \\ +\frac{d\eta(t,x,u)}{dt}\Bigg[\sum_{i=1}^{p}\tau^i\{f_{\dot{u}}^i(t)-\bar{v}^ik_{\dot{u}}^i(t)\}+\sum_{j=1}^{m}\lambda^j(t)g_{\dot{u}}^j(t)\Bigg]\Bigg\}dt<0.$$

$$\int_a^b \frac{d}{dt}\Big\{\eta(t,x,u)\Big[\sum_{i=1}^{p}\tau^i\{f_{\dot{u}}^i(t)-\bar{v}^ik_{\dot{u}}^i(t)\}+\sum_{j=1}^{m}\lambda^j(t)g_{\dot{u}}^j(t)\Big]\Big\}dt<0,$$

$$\Big\{\eta(t,x,u)\Big[\sum_{i=1}^{p}\tau^i\{f_{\dot{u}}^i(t)-\bar{v}^ik_{\dot{u}}^i(t)\}+\sum_{j=1}^{m}\lambda^j(t)g_{\dot{u}}^j(t)\Big]\Big\}\Bigg|_a^b<0.$$

Conditions (24), (43) and (44) lead to contradiction. Hence result follows. □

Theorem 4.6. *(Strong duality) Let $\bar{x}$ be a normal efficient solution of order m for (P2) with respect to ξ. Then there exist $\tau=(\tau^1,\tau^2,\ldots,\tau^p)\in\mathbb{R}_+^p, v=(v^1,v^2,\ldots,v^p)\in\mathbb{R}_+^p$, piece-wise smooth functions $\lambda^j: I\to\mathbb{R}, j\in M, z^i: I\to\mathbb{R}^n, y^i: I\to\mathbb{R}^n, i\in P, w^i: I\to\mathbb{R}^n, i\in M$, such that $(\bar{x},\tau,v,\lambda^1,\ldots,\lambda^m,z^1,\ldots,z^p,y^1,\ldots,y^p,w^1,\ldots,w^m)\in U$. Further if weak duality theorem holds. Then*

$$(\bar{x},\tau,v,\lambda^1,\ldots,\lambda^m,z^1,\ldots,z^p,y^1,\ldots,y^p,w^1,\ldots,w^m),$$

is an efficient solution of order m for (D) with respect to ξ.

Proof. Since $\bar{x}$ is an efficient solution of order m with respect to ξ for (P2), hence by Theorem 3.4, there exist $\tau = (\tau^1, \tau^2, \ldots, \tau^p) \in \mathbb{R}^p_+, v = (v^1, v^2, \ldots, v^p) \in \mathbb{R}^p_+$, piece-wise smooth functions $\lambda^j : I \to \mathbb{R}, j \in M, z^i : I \to \mathbb{R}^n, y^i : I \to \mathbb{R}^n, i \in P, w^i : I \to \mathbb{R}^n, i \in M$, such that $(\bar{x}, \tau, v, \lambda^1, \ldots, \lambda^m, z^1, \ldots, z^p, y^1, \ldots, y^p, w^1, \ldots, w^m) \in U$.
Let if possible, $(\bar{x}, \tau, v, \lambda^1, \ldots, \lambda^m, z^1, \ldots, z^p, y^1, \ldots, y^p, w^1, \ldots, w^m)$ is not an efficient solution of order m for (D) with respect to ξ , then for any $\rho = (\rho^1, \rho^2, ..., \rho^p) \in int\, \mathbb{R}^p_+$, there exist $(\hat{x}, \hat{\tau}, \hat{v}, \hat{\lambda}^1, \ldots, \hat{\lambda}^m, \hat{z}^1, \ldots, \hat{z}^p, \hat{y}^1, \ldots, \hat{y}^p, \hat{w}^1, \ldots, \hat{w}^m) \in U$ such that

$$\hat{v}^i + \rho^i \int_a^b \|\xi(t, \hat{x}, \bar{x})\|^m dt \geqslant v^i = \frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t) \bar{x}(t)\}^{\frac{1}{2}}\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t) \bar{x}(t)\}^{\frac{1}{2}}\} dt} \tag{49}$$

for all $i \in P$ and,

$$\hat{v}^j + \rho^j \int_a^b \|\xi(t, \hat{x}, \bar{x})\|^m dt > v^j = \frac{\int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t) \bar{x}(t)\}^{\frac{1}{2}}\} dt}{\int_a^b \{k^j(t, \bar{x}, \dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t) \bar{x}(t)\}^{\frac{1}{2}}\} dt} \tag{50}$$

for at least one $j \in P$.

Case(i) If $\|\xi(t, \hat{x}, \bar{x})\|^m = 0$. Then for any $c = (c^1, c^2, ..., c^p) \in int\, \mathbb{R}^p_+$ and $d = (d^1, d^2, ..., d^p) \in int\, \mathbb{R}^p_+$, we have

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t) \bar{x}(t)\}^{\frac{1}{2}} - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t) \bar{x}(t)\}^{\frac{1}{2}} + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} \leqslant \hat{v}^i$$

for all $i \in P$ and,

$$\frac{\int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t) \bar{x}(t)\}^{\frac{1}{2}} - c^j \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^j(t, \bar{x}, \dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t) \bar{x}(t)\}^{\frac{1}{2}} + d^j \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} < \hat{v}^j,$$

for at least one $j \in P$.

Which is contradiction to weak duality theorem.
Case(ii) If $\|\xi(t, \hat{x}, \bar{x})\|^m \neq 0$. For any $c = (c^1, c^2, ..., c^p) \in int\, \mathbb{R}^p_+$ and

$d = (d^1, d^2, ..., d^p) \in int\, \mathbb{R}^p_+, i \in P$, define

$\rho^i = \frac{c^i \int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt + d^i \int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}{(\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}} + d^i \|\xi(t,\hat{x},\bar{x})\|^m\}dt) \int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}}\}dt}$,

Substituting this in (49) and (50) yields,

$$\frac{\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^i(t)\bar{x}(t)\}^{\frac{1}{2}} - c^i \|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^i(t)\bar{x}(t)\}^{\frac{1}{2}} + d^i \|\xi(t,\hat{x},\bar{x})\|^m\}dt} \leqslant \hat{v}^i$$

for all $i \in P$ and

$$\frac{\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + \{\bar{x}(t)^T B^j(t)\bar{x}(t)\}^{\frac{1}{2}} - c^j \|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^j(t,\bar{x},\dot{\bar{x}}) - \{\bar{x}(t)^T E^j(t)\bar{x}(t)\}^{\frac{1}{2}} + d^j \|\xi(t,\hat{x},\bar{x})\|^m\}dt} < \hat{v}^j,$$

for at least one $j \in P$.

Which contradicts weak duality theorem. Thus $(\bar{x}, \tau, v, \lambda^1, \ldots, \lambda^m, z^1, \ldots, z^p, y^1, \ldots, y^p, w^1, \ldots, w^m)$ is an efficient solution of order m for (D) with respect to ξ. $\square$

Remark 4.7. *Sufficient optimality conditions can be prove proceeding on similar lines of weak duality.*

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References

[1] T. Antczak, On efficiency and mixed duality for new class of nonconvex multiobjective variational control problems, *Journal of Global Optimization*, 59 (2014), 754-785.

[2] A. Auslender, Stability in mathematical programming with non-different iable data, *SIAM Journal on Control and Optimization*, 22 (1984), 239-254.

[3] C. R. Bector, Duality in nonlinear fractional programming, *Zeitschrift Fur Operation Research*, 7 (1973) , 183-193.

[4] C. R. Bector, S. Chandra, and I. Husain, Optimality condition and subdifferentiable multiobjective fractional programming, *Journal of Optimization Theory and Applications*, 79 (1993), 105-125.

[5] G. Bhatia, Optimality and mixed saddle point criteria in multiobjective optimization, *Journal of Mathematical Analysis and Applications*, 342 (2008), 135-145.

[6] S. Chandra, B. D. Craven, and I. Husain, A class of nondifferentiable continuous programming problems, *Journal of Mathematical Analysis and Applications*, 107 (1985), 122-131.

[7] V. Chankong and Y. Y. Haimes, Multiobjective Decision making: Theory and Methodology, North Holland, New York, (1983).

[8] F. H. Clarke, Generalized gradients of Lipschitz functionals, *Advances in Mathematics*, 40 (1981),52-67.

[9] B. D. Craven and B. Mond, Lagrangean conditions for quasidifferentiable optimization, in Proceedings, 9th Intern. Conf. on Mathematical Programming, Budapest, 1976, in Survey of Mathematical Programming (A. Prekopa, Ed.), Vol. 1, pp. 177-191, Akad. Kiado, Budapest, and North-Holland, Amsterdam, (1979).

[10] B. D. Craven and B. Mond, Sufficient Fritz John optimality conditions for nondifferentiable convex programming, *J. Austral. Math. Soc.* , Ser. B 19 (1976), 462-468.

[11] I. M. Gelfand and S. V. Fomin, *Calculus of Variations, Translated from Russian,* 34-35, Prentice-hall, Inc. Englewood Cliffs, New Jersey, (1963).

[12] M. A. Hanson, Bounds for functionally convex optimal control problems, *Journal of Mathematical Analysis and Applications*, 8 (1964), 84-89.

[13] A. Jayswal, I. M. Stancu-Minasian, and S. Choudhury, Second order duality for variational problems involving generalized convexity, *Opsearch*, 52 (3) (2015), 582-596.

[14] B. Jimenez, Strict efficiency in vector optimization, *Journal of Mathematical Analysis and Applications*, 265 (2002), 264-284.

[15] R. N. Kaul and V. Lyall, A note on nonlinear fractional vector maximization, *Opsearch*, 26 (1989), 108-121.

[16] E. Kreyszig, *Introductory Functional Analysis with Applications*,188, John Wiley & Sons, New York, (1978).

[17] J. C. Liu, Duality for nondifferentiable static multiobjective variational problems involving generalized (F, ρ)-convex functions, *Comput. Math. Appl.*, 31 (12) (1996), 77-89.

[18] Z. A. Liang, H. X. Huang, and P. M. Pardalos, Efficiency condition and duality for a class of multiobjective fractional programming problem, *Journal of Global Optimization*, 27 (2003), 447-471.

[19] S. K. Mishra and R. N. Mukherjee, Duality for multiobjective fractional variational problems, *Journal of Mathematical Analysis and Applications*, 186 (1994), 711-725.

[20] S. Mititelu and I. M. Stancu-Minasian, Efficiency and duality for multiobjective fractional variational problems with (ρ, b)-quasiinvexity, *Yugoslav Journal of Operations Research*, 19 (2009), 85-99.

[21] B. Mond and M. A. Hanson, Duality for variational problem, *Journal of Mathematical Analysis and Applications*, 18 (1967), 355-364.

[22] A. M. Stancu, *Mathematical Programming with Type-I Functions,* Matrix Rom., Bucharest, (2013).

[23] I. M. Stancu-Minasian, *Fractional Programming: Theory, Methods and Applications,* Kluwer Academic Publishers, Dordrecht, The Netherlands, (1997).

[24] I. M. Stancu-Minasian and S. Mititelu, Multiobjective fractional variational problems with (ρ, b)-quasiinvexity, *Proc. Rom. Acad. Series A Math. Phys. Tech. Sci. Inf. Sci.,* 9(1) (2008), 5-11.

[25] D. E. Ward, Characterization of strict local minima and necessary conditions for weak sharp minima, *Journal of Optimization Theory and Applications*, 80 (1994), 551-571.

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