Journal of Mathematical Extension
Vol. 11, No. 3, (2017), 43-67
ISSN: 1735-8299
URL: http://www.ijmex.com

Two New Generalizations for F-Contraction on Closed Ball and Fixed Point Theorems with Application

E. Ameer*
International Islamic University H-10

M. Arshad
International Islamic University H-10

Abstract. We introduce the notion of modified Hardy-Rogers type F-contraction on closed ball, F-multiplicative contraction on closed ball and we obtain some new fixed point results for such contractions. Some comparative examples are constructed to illustrate these results. The existence of the solution of family of Volterra type integral equations is shown via fixed point methods.

AMS Subject Classification: 47H10; 54H25
Keywords and Phrases: B-metric-like space, multiplicative metric space, F-contraction

1. Introduction and Preliminaries

Banach contraction Principle states that any contraction on a complete metric space has a unique fixed point. This principle guarantees the existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many

Received: October 2016; Accepted: May 2017
*Corresponding author

directions. One of the most interesting generalization of it was given by Wardowski [29]. Later on, Wardowski and Van Dung [30] gave the idea of F-weak contraction and proved a theorem concerning F-weak contraction. After wards, Abbas et al. [1] further generalized the concept of F-contraction and proved certain fixed point results. Hussain and Salimi [14] introduced an α-GF contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces. Batra et al. [7, 8] extended the concept of F-contraction on graphs and altered distances. They proved some fixed point and coincidence point results by illustrating them with some examples. Recently, Cosentino and Vetro [11] followed the approach of F-contraction and obtained some fixed point theorems of Hardy-Rogers-type for self-mappings in complete metric spaces and complete ordered metric spaces.

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X then by imposing a subtle restriction on the choice of x_0, one may force Picard iterative iterative sequence to stay eventually in Y. In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T.

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$.

Definition 1.1. *[9] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}_0^+$ is called b-metric if there exists a real number $s \geqslant 1$ such that for every $x, y, z \in X$, we have*

(i) If $d(x, y) = 0$, if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(ii) $d(x, y) \leqslant s\,[d(x, z) + d(z, y)]$. In this case, the pair (X, d) is called a b-metric space.

Definition 1.2. *[20] Let X be a nonempty set and let $d : X \times X \to \mathbb{R}_0^+$ be a function, called a b-dislocated metric if there exists a real number*

$s \geqslant 1$ *such that, the following conditions hold for every* $x, y, z \in X$:

(i) If $d(x,y) = 0$, *then* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(ii) $d(x,y) \leqslant s\left[d(x,z) + d(z,y)\right]$.

The pair (X,d) *is called a b-dislocated metric space(b-metric-like space).*

First we recall the concept of F-contraction, which was introduced by Wardowski [29], later we will mention his result.

Definition 1.3. *A mapping* $T : X \to X$, *is said to be F-contraction if it satisfies following condition:*

$$(d(T(x),T(y)) > 0 \Rightarrow \tau + F(d(T(x),T(y)) \leqslant F(d(x,y))). \qquad (1)$$

for all $x, y \in X$ *and some* $\tau > 0$. *Where* $F : \mathbb{R}^+ \to \mathbb{R}$ *is a mapping satisfying following properties:*

(F1): *F is strictly increasing.*

(F2): *For each sequence* $\{a_n\}$ *of positive numbers* $\lim_{n\to\infty} a_n = 0$ *if and only if* $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3): *There exists* $\theta \in (0,1)$ *such that* $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

Theorem 1.4. *[29] Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a F-contraction. Then T has a unique fixed point* $\upsilon \in X$ *and for every* $x_0 \in X$ *a sequence* $\{T^n(x_0)\}$ $\forall n \in N$ *is convergent to* υ.

Theorem 1.5. *[29] Let* $F : \mathbb{R}^+ \to \mathbb{R}$ *be given by the formula* $F(\alpha) = \ln\alpha$. *It is clear that F satisfied* $(F_1) - (F_3)$ *for any* $\kappa \in (0,1)$. *Each mapping* $T : X \to X$ *satisfying (1) is a F-contraction such that*

$$d(T(x),T(y)) \leqslant e^{-\tau} d(x,y), \text{ for all } x, y \in X, T(x) \neq T(y).$$

It is clear that for $x, y \in X$ *such that* $T(x) = T(y)$, *the inequality* $d(T(x),T(y)) \leqslant e^{-\tau} d(x,y)$, *also holds, i.e. T is a Banach contraction.*

Remark 1.6. *From* (F_1) *and (1) it is easy to conclude that every F-contraction is necessarily continuous.*

Secelean [24] proved the following lemma and replaced condition $(F2)$ *by an equivalent but a more simple condition* $(F2')$.

Lemma 1.7. *[24] Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an increasing map and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:*

(*a*) *if* $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ *then* $\lim\limits_{n\longrightarrow\infty} \alpha_n = 0$;

(*b*) *if* $\inf F = -\infty$ *and* $\lim\limits_{n\longrightarrow\infty} \alpha_n = 0$, *then* $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$.

He replaced the following condition.

$(F2')$ $\inf F = -\infty$

or, also, by

$(F2'')$ *there exists a sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *of positive real numbers such that* $\lim\limits_{n\longrightarrow\infty} F(\alpha_n) = -\infty$.

Recently Piri [19] replaced the following condition $(F3')$ *instead of the condition* $(F3)$ *in Definition 1.3.*

$(F3')$ *F is continuous on* $(0, \infty)$.

We denote by Δ_F *the set of all functions satisfying the conditions* $(F1)$, $(F2)$ *and* $(F3')$.

Definition 1.8. *Let X be a nonempty set. Multiplicative metric [26] is a mapping* $d : X \times X \longrightarrow \mathbb{R}$ *satisfying the following conditions:*

(m1) $d(x, y) > 1$ *for all* $x, y \in X$ *and* $d(x, y) = 1$ *if and only if* $x = y$,

(m2) $d(x, y) = d(y, x) > 1$ *for all* $x, y \in X$,

(m3) $d(x, y) \leqslant d(x, z).d(z, y)$ *for all* $x, y, z \in X$ *(multiplicative triangle inequality).*

Also (X, d) *is called a multiplicative metric space.*

On the other hand, Ozavsar and Cervikel [23] generalized the celebrated Banach contraction mapping principle in the setup of multiplicative metric spaces.

Definition 1.9. *[29] Let* (X, d) *be a multiplicative metric space,* $x \in X$ *and* $\varepsilon > 1$. *We now define a set*

$$B_\varepsilon(x) = \{y \in X \backslash d(x, y) < \varepsilon\},$$

which is called a multiplicative open ball of radius ε with center x. Similarly, one can describe a multiplicative closed ball as

$$\overline{B_\varepsilon(x)} = \{y \in X \backslash d(x,y) \leqslant \varepsilon\}.$$

Definition 1.10. *[27] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. If, for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number N such that $n \geqslant N \Longrightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to x, denoted by $x_n \longrightarrow x$ as $n \longrightarrow \infty$.*

Lemma 1.11. *[27] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \longrightarrow x$ as $n \longrightarrow \infty$ if and only if $d(x_n, x) \longrightarrow 1$ as $n \longrightarrow \infty$.*

Definition 1.12. *[27] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for $m, n \geqslant N$.*

Theorem 1.13. *[27] Let (X, d_X) and (X, d_Y) be two multiplicative metric spaces, $T : X \to X$ be a mapping and $\{x_n\}$ be any sequence in X. Then T is multiplicative continuous at the point $x \in X$ if and only if $Tx_n \to Tx$ for every sequence $\{x_n\}$ with $x_n \longrightarrow x$ as $n \longrightarrow \infty$.*

Lemma 1.14. *[27] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m, x_n) \longrightarrow 1$ as $m, n \longrightarrow \infty$.*

Definition 1.15. *[27] Let (X, d) be a multiplicative metric space. The multiplicative metric paces X is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X for all $n \in \mathbb{N}$ converges in X.*

Definition 1.16. *[27] Let (X, d) be a multiplicative metric space. A self mapping $T : X \to X$ is said to be multiplicative contraction if there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \leqslant (d(x, y))^\lambda \tag{2}$$

$\forall x, y \in X$.

Theorem 1.17. *[27] Let (X, d) be a complete multiplicative metric space and $T : X \to X$ be multiplicative contraction, then T has a unique fixed point.*

Theorem 1.18. *[16, Theorem 5.1.4] Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leqslant kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)} \tag{3}$$

and $d(x_0, T(x_0)) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

2. Modified Hardy-Rogers Type F-Contraction on Closed Ball

In this section, we introduce the concept of modified Hardy-Rogers type F-contractions on closed ball in b-metric like spaces, and establish new fixed point theorems for such contraction.

Definition 2.1. Let (X, d) be a b-metric like space. The mapping $T : X \to X$ is called modified Hardy-Rogers type F-contraction on closed ball if there exists $F \in \Delta_F$ and $\tau > 0$ such that

$$\tau + F\left(d\left(Tx, Ty\right)\right) \leqslant$$

$$F\left(\begin{array}{c} \alpha d\left(x, y\right) + \beta d\left(x, Ty\right) + \gamma d\left(y, Tx\right) + \delta d\left(y, Ty\right) + \\ \eta \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \lambda \frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)} + \mu \frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)} \end{array}\right); \tag{4}$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$ with $d\left(Tx, Ty\right) > 0$, where $\alpha, \beta, \gamma, \delta, \eta, \lambda, \mu \geqslant 0$ such that $s\alpha + \left(s^2 + s\right)\beta + 2s^2\gamma + \delta + \eta + \lambda + s\mu < 1$.

we now introduce a new type of fixed point theorem for modified Hardy-Rogers type F-contraction on closed ball in a complete b-metric like space.

Theorem 2.2. *Let (X, d) be a complete b-metric like space, T be a continuous modified Hardy-Rogers type F-contraction on closed ball $\overline{B(x_0, r)} \subseteq X$ and x_0 be an arbitrary point in X. Moreover,*

$$d(x_0, Tx_0) \leqslant (1-\theta)r, \tag{5}$$

where $\theta = \frac{\alpha+s\beta+2s\gamma+\mu}{1-s\beta-\delta-\eta-\lambda}$ and $r > 0$. Then there exist a point x^*in

$\overline{B(x_0, r)}$ such that

$Tx^* = x^*$. Furthermore, if $\alpha + \beta + \gamma + \lambda \leqslant 1$, then the fixed point of T is unique.

Proof. Choose a point x_1 in X such that $x_1 = Tx_0$. continuing in this way, so we get $x_{n+1} = Tx_n$, for all $n \geqslant 0$ and this implies that (x_n) is a nonincreasing sequence. Now we will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$, by using mathematical induction. Since from (5), we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leqslant (1-\theta)r < r,$$

thus, $x_1 \in \overline{B(x_0, r)}$. Suppose $x_2....x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Thus from (4), we obtain

$$F(d(x_j, x_{j+1})) = F(d(Tx_{j-1}, Tx_j)) \leqslant$$

$$\begin{aligned}
F[&\alpha d(x_{j-1}, x_j) + \beta d(x_{j-1}, Tx_j) + \gamma d(x_j, Tx_{j-1}) + \delta d(x_j, Tx_j) \\
&+\eta \frac{d(x_j, Tx_j)\left[1 + d(x_{j-1}, Tx_{j-1})\right]}{1 + d(x_{k-1}, x_k)} + \lambda \frac{d(x_j, Tx_j) + d(x_j, Tx_{j-1})}{1 + d(x_j, Tx_j) d(x_j, Tx_{j-1})} \\
&+\mu \frac{d(x_{j-1}, Tx_{j-1})\left[1 + d(x_j, Tx_{j-1})\right]}{1 + d(x_{j-1}, x_j) + d(x_j, Tx_j)}] - \tau
\end{aligned}$$

$$\begin{aligned}
F[&\alpha d(x_{j-1}, x_j) + \beta d(x_{j-1}, x_{j+1}) + \gamma d(x_j, x_j) + \delta d(x_j, x_{j+1}) \\
&+\eta \frac{d(x_j, x_{j+1})\left[1 + d(x_{j-1}, x_j)\right]}{1 + d(x_{j-1}, x_j)} + \lambda \frac{d(x_j, x_{j+1}) + d(x_j, x_j)}{1 + d(x_j, x_{j+1}) d(x_j, x_j)} \\
&+\mu \frac{d(x_{j-1}, x_j)\left[1 + d(x_j, x_j)\right]}{1 + d(x_{j-1}, x_j) + d(x_j, x_{j+1})}] - \tau
\end{aligned}$$

$$F(d(x_j,x_{j+1}))\leqslant F((\alpha+s\beta+2s\gamma+\mu)\,d(x_{j-1},x_j)$$

$$+\,(s\beta+\delta+\eta+\lambda)\,d(x_j,x_{j+1}))-\tau,$$

this implies

$$F(d(x_j,x_{j+1}))<F((\alpha+s\beta+2s\gamma+\mu)\,d(x_{j-1},x_j)$$

$$+\,(s\beta+\delta+\eta+\lambda)\,d(x_j,x_{j+1})),$$

for all $n\in\mathbb{N}$. As F is strictly increasing, so we have

$$d(x_j,x_{j+1})<(\alpha+s\beta+2s\gamma+\mu)\,d(x_{j-1},x_j)$$

$$+\,(s\beta+\delta+\eta+\lambda)\,d(x_j,x_{j+1}).$$

Which implies

$$(1-s\beta-\delta-\eta-\lambda)\,d(x_j,x_{j+1})<(\alpha+s\beta+2s\gamma+\mu)\,d(x_{j-1},x_j).$$

$$d(x_j,x_{j+1})<\left(\frac{\alpha+s\beta+2s\gamma+\mu}{1-s\beta-\delta-\eta-\lambda}\right)d(x_{j-1},x_j).$$

Here $\theta=\frac{\alpha+s\beta+2s\gamma+\mu}{1-s\beta-\delta-\eta-\lambda}<1$. Hence,

$$d(x_j,x_{j+1})<\theta d(x_{j-1},x_j)<\theta^2 d(x_{j-2},x_{j-1})<\cdots<\theta^j d(x_0,x_1). \quad (6)$$

Now,

$$\begin{aligned} d(x_0,x_{j+1}) &\leqslant s\,[d(x_0,x_1)+...+d(x_j,x_{j+1})] \\ &\leqslant sd(x_0,x_1)[1+\theta+...+\theta^j] \\ &\leqslant s(1-\theta)r\frac{(1-\theta^{j+1})}{1-\theta}<r. \end{aligned}$$

Thus $x_{j+1}\in\overline{B(x_0,r)}$. Hence $x_n\in\overline{B(x_0,r)}$ for all $n\in\mathbb{N}$. Continuing this process, we get

$$F(d(x_n,x_{n+1}))=F(d(Tx_{n-1},Tx_n))\leqslant$$

$$F[\alpha d(x_{n-1},x_n)+\beta d(x_{n-1},Tx_n)+\gamma d(x_n,Tx_{n-1})+\delta d(x_n,Tx_n)$$
$$+\eta\frac{d(x_n,Tx_n)\left[1+d(x_{n-1},Tx_{n-1})\right]}{1+d(x_{n-1},x_n)}+\lambda\frac{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}{1+d(x_n,Tx_n)d(x_n,Tx_{n-1})}$$
$$+\mu\frac{d(x_{n-1},Tx_{n-1})\left[1+d(x_n,Tx_{n-1})\right]}{1+d(x_{n-1},x_n)+d(x_n,Tx_n)}]-\tau$$

$$F[\alpha d(x_{n-1},x_n)+\beta d(x_{n-1},x_{n+1})+\gamma d(x_n,x_n)+\delta d(x_n,x_{n+1})$$
$$+\eta\frac{d(x_n,x_{n+1})\left[1+d(x_{n-1},x_n)\right]}{1+d(x_{n-1},x_n)}+\lambda\frac{d(x_n,x_{n+1})+d(x_n,x_n)}{1+d(x_n,x_{n+1})d(x_n,x_n)}$$
$$+\mu\frac{d(x_{n-1},x_n)\left[1+d(x_n,x_n)\right]}{1+d(x_{n-1},x_n)+d(x_n,x_{n+1})}]-\tau$$

$$F(d(x_n,x_{n+1}))\leqslant F((\alpha+s\beta+2s\gamma+\mu)\,d(x_{n-1},x_n)$$
$$+(s\beta+\delta+\eta+\lambda)\,d(x_n,x_{n+1}))-\tau,$$

this implies

$$F(d(x_n,x_{n+1}))<F((\alpha+s\beta+2s\gamma+\mu)\,d(x_{n-1},x_n)$$
$$+(s\beta+\delta+\eta+\lambda)\,d(x_n,x_{n+1})),$$

for all $n\in\mathbb{N}$. As F is strictly increasing, so we have

$$d(x_n,x_{n+1})<(\alpha+s\beta+2s\gamma+\mu)\,d(x_{n-1},x_n)$$
$$+(s\beta+\delta+\eta+\lambda)\,d(x_n,x_{n+1}).$$

Which implies

$$(1-s\beta-\delta-\eta-\lambda)\,d(x_n,x_{n+1})<(\alpha+s\beta+2s\gamma+\mu)\,d(x_{n-1},x_n).$$

$$d(x_n,x_{n+1})<\left(\frac{\alpha+s\beta+2s\gamma+\mu}{1-s\beta-\delta-\eta-\lambda}\right)d(x_{n-1},x_n).$$

Here $\theta=\frac{\alpha+s\beta+2s\gamma+\mu}{1-s\beta-\delta-\eta-\lambda}<1$. Hence,

$$d(x_n,x_{n+1})<\theta d(x_{n-1},x_n)<d(x_{n-1},x_n). \tag{7}$$

Consequently,

$$\tau + F\left(d(x_n, x_{n+1})\right) \leqslant F\left(d(x_{n-1}, x_n)\right).$$

which implies,

$$\begin{aligned} F\left(d(x_n, x_{n+1})\right) &\leqslant F\left(d(x_{n-1}, x_n)\right) - \tau \\ &= F\left(d(Tx_{n-2}, Tx_{n-1})\right) - \tau \\ &\leqslant F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau \\ &= F\left(d(Tx_{n-3}, Tx_{n-2})\right) - 2\tau \\ &\leqslant F\left(d(x_{n-3}, x_{n-2})\right) - 3\tau \\ &\;\;\vdots \\ &\leqslant F\left(d(x_0, x_1)\right) - n\tau. \end{aligned}$$

This implies that

$$F\left(d(x_n, x_{n+1})\right) \leqslant F\left(d(x_0, x_1)\right) - n\tau. \tag{8}$$

And so $\lim\limits_{n \longrightarrow \infty} F\left(d\left(x_n, x_{n+1}\right)\right) = -\infty$. By $(F2)$, we find that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

We shall prove that (x_n) is Cauchy in (X, d). So, it suffices to show that $\lim\limits_{n \to \infty} d(x_n, x_m) = 0$. We argue by contradiction. Suppose there exist $\varepsilon > 0$ and sequences $(n(p))$ and $(m(p))$ of natural numbers such that

$$n(p) > m(p) > p, \quad d(x_{n(p)}, x_{m(p)}) \geqslant \varepsilon,$$

$$d(x_{n(p)-1}, x_{m(p)}) < \varepsilon \quad \text{for all } p \in \mathbb{N}. \qquad 10$$

By triangular inequality, we have

$$\begin{aligned} d(x_{n(p)}, x_{m(p)}) &\leqslant s[d(x_{n(p)}, x_{n(p)-1}) + d(x_{n(p)-1}, x_{m(p)})] \\ &< s\varepsilon + sd(x_{n(p)}, x_{n(p)-1}) \\ &= s\varepsilon + sd(x_{n(p)-1}, Tx_{n(p)-1}). \end{aligned} \tag{11}$$

From (9), there exists $p_1 \in \mathbb{N}$ such that for all $p \geqslant p_1$

$$d(x_{n(p)-1}, Tx_{n(p)-1}) < \varepsilon. \tag{12}$$

Combining (11) to (12) yields that

$$d(x_{n(p)}, x_{m(p)}) < 2s\varepsilon \quad \text{for all } p \geqslant p_1. \tag{13}$$

On the other hand, by definition of a modified Hardy-Rogers type F-contraction on closed ball,

$$\begin{aligned}
F(\varepsilon) \leqslant F(d(Tx_{n(p)}, Tx_{m(p)})) &\leqslant F[\alpha d(x_{n(p)}, x_{m(p)}) + \beta d(x_{n(p)-1}, Tx_{m(p)}) \\
&+ \gamma d(x_{m(p)}, Tx_{n(p)-1}) + \delta d(x_{m(p)}, Tx_{m(p)}) \\
&+ \eta \frac{d(x_{m(p)}, Tx_{m(p)}) \left[1 + d(x_{n(p)-1}, Tx_{n(p)-1})\right]}{1 + d(x_{n(p)-1}, x_{m(p)})} \\
&+ \lambda \frac{d(x_{m(p)}, Tx_{m(p)}) + d(x_{m(p)}, Tx_{n(p)-1})}{1 + d(x_{m(p)}, Tx_{m(p)}) d(x_{m(p)}, Tx_{n(p)-1})} \\
&+ \mu \frac{d(x_{n(p)-1}, Tx_{n(p)-1}) \left[1 + d(x_{m(p)}, Tx_{n(p)-1})\right]}{1 + d(x_{n(p)-1}, x_{m(p)}) + d(x_{m(p)}, Tx_{m(p)})}] - \tau \\
&\leqslant F[\alpha d(x_{n(p)}, x_{m(p)}) + \beta d(x_{n(p)-1}, x_{m(p)+1}) \\
&+ \gamma d(x_{m(p)}, x_{n(p)}) + \delta d(x_{m(p)}, x_{m(p)+1}) \\
&+ \eta \frac{d(x_{m(p)}, x_{m(p)+1}) \left[1 + d(x_{n(p)}, x_{n(p)})\right]}{1 + d(x_{n(p)-1}, x_{m(p)})} \\
&+ \lambda \frac{d(x_{m(p)}, x_{m(p)+1}) + d(x_{m(p)}, x_{n(p)})}{1 + d(x_{m(p)}, x_{m(p)+1}) d(x_{m(p)}, x_{n(p)})} \\
&+ \mu \frac{d(x_{n(p)-1}, x_{n(p)}) \left[1 + d(x_{m(p)}, x_{n(p)})\right]}{1 + d(x_{n(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{m(p)+1})}] - \tau.
\end{aligned}$$

Which implies,

$$F(\epsilon) \leqslant F\left[s\alpha\varepsilon + \left(s^2 + s\right)\beta\varepsilon + 2s^2\gamma\varepsilon + \delta\varepsilon + \eta\varepsilon + \lambda\varepsilon + s\mu\varepsilon\right] - \tau.$$

As, $s\alpha + \left(s^2 + s\right)\beta + 2s^2\gamma + \delta + \eta + \lambda + s\mu < 1$, so we get

$$s\alpha\varepsilon + \left(s^2 + s\right)\beta\varepsilon + 2s^2\gamma\varepsilon + \delta\varepsilon + \eta\varepsilon + \lambda\varepsilon + s\mu\varepsilon < \varepsilon,$$

we deduce that

$$F(\varepsilon) < F(\varepsilon),$$

which is a contradiction. Thus, (x_n) is a Cauchy sequence in the complete b-metric like space $\left(\overline{B(x_0, r)}, d\right)$. Since $\left(\overline{B(x_0, r)}, d\right)$ is a complete metric space, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. Since T is continuous, then,

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Hence $x^* = T(x^*)$ that is x^* is fixed point of T. Finally, to prove x^* is a unique fixed point of T, let $x, y \in \overline{B(x_0, r)}$ and $x \neq y$ be any two fixed point of T. Then we have

$$\begin{aligned}
&\tau + F\left(d(x, y)\right)) = \tau + F\left(d(Tx, T(y))\right) \leqslant F[\alpha d(x, y) \\
&+\beta d(x, Ty) + \gamma d(y, Tx) + \delta d(y, Ty) \\
&+\eta \frac{d(y, Ty)\left[1 + d(x, Tx)\right]}{1 + d(x, y)} + \lambda \frac{d(y, Ty) + d(y, Tx)}{1 + d(y, Ty)d(y, Tx)} \\
&+\mu \frac{d(x, Tx)\left[1 + d(y, Tx)\right]}{1 + d(x, y) + d(y, Ty)}]. \\
&= F\left[\left(\alpha + \beta + \gamma + \lambda\right) d(x, y)\right]
\end{aligned}$$

Which implies

$$d(x, y) < \left(\alpha + \beta + \gamma + \lambda\right) d(x, y) < d(x, y),$$

which is a contradiction. Thus $d(x, y) = 0$. Hence $x = y$. This completes the proof. □

Example 2.3. Let $X = [0, 1]$ and $d\left(x, y\right) = \left(\max\left\{x, y\right\}\right)^2$. Then (X, d) is a complete b-metric like space with a constant $s = 2$. Define the mapping $T : X \to X$ by,

$$T\left(x\right) = \begin{cases} \frac{x}{4} & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ 5 & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Set $\theta = 0.26$, $x_0 = \dfrac{1}{15}$, $r = \dfrac{1}{4}$, then $\overline{B(x_0, r)} = \left[0, \frac{1}{2}\right]$. If $F(t) = \ln(t)$, $t > 0$, then

$$d(x_0, T(x_0)) = \left|\frac{1}{15} - \frac{1}{60}\right| = \frac{3}{60} < (1-\theta)r.$$

For $x, y \in \overline{B(x_0, r)}$, we get

$$\begin{aligned}
d(Tx, Ty) &= (\max\{Tx, Ty\})^2 = \left(\max\left\{\frac{x}{4}, \frac{y}{4}\right\}\right)^2 \\
&= \frac{1}{16}(\max\{x, y\})^2 \\
&< \frac{1}{6}(\max\{x, y\})^2 + 0\left(\max\left\{x, \frac{y}{4}\right\}\right)^2 \\
&+ \frac{1}{60}\left(\max\left\{y, \frac{x}{4}\right\}\right)^2 + \frac{1}{20}\left(\max\left\{y, \frac{y}{4}\right\}\right)^2 \\
&+\frac{1}{50}\frac{\left(\max\left\{y, \frac{y}{4}\right\}\right)^2\left[1 + \left(\max\left\{x, \frac{x}{4}\right\}\right)^2\right]}{1 + (\max\{x, y\})^2} \\
&+ \frac{1}{30}\frac{\left(\max\left\{y, \frac{y}{4}\right\}\right)^2 + \left(\max\left\{y, \frac{x}{4}\right\}\right)^2}{1 + \left(\max\left\{y, \frac{y}{4}\right\}\right)^2\left(\max\left\{y, \frac{x}{4}\right\}\right)^2} \\
&+0\frac{\left(\max\left\{x, \frac{x}{4}\right\}\right)^2\left[1 + \left(\max\left\{y, \frac{x}{4}\right\}\right)^2\right]}{1 + (\max\{x, y\})^2 + \left(\max\left\{y, \frac{y}{4}\right\}\right)^2} \\
&= \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx) + \delta d(y, Ty) + \\
&\eta\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \lambda\frac{d(y, Ty) + d(y, Tx)}{1 + d(y, Ty)\,d(y, Tx)} \\
&+\mu\frac{d(x, Tx)[1 + d(y, Tx)]}{1 + d(x, y) + d(y, Ty)},
\end{aligned}$$

Thus,

$$\begin{aligned}
d(T(x), T(y)) &< \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx) + \delta d(y, Ty) + \\
&\eta\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \lambda\frac{d(y, Ty) + d(y, Tx)}{1 + d(y, Ty)\,d(y, Tx)} \\
&+\mu\frac{d(x, Tx)[1 + d(y, Tx)]}{1 + d(x, y) + d(y, Ty)}.
\end{aligned}$$

Which implies

$$\tau{+}\ln\left(d(T(x),T(y))\right)$$

$$\leqslant \ln\left(\begin{array}{c}\alpha d\left(x,y\right)+\beta d\left(x,Ty\right)+\gamma d\left(y,Tx\right)+\delta d\left(y,Ty\right)+\\ \eta\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\lambda\frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)}+\mu\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}\end{array}\right).$$

That is

$$\tau{+}F\left(d(T(x),T(y))\right)$$

$$\leqslant F\left(\begin{array}{c}\alpha d\left(x,y\right)+\beta d\left(x,Ty\right)+\gamma d\left(y,Tx\right)+\delta d\left(y,Ty\right)+\\ \eta\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\lambda\frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)}+\mu\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}\end{array}\right).$$

for $\tau=\left(0,\frac{10}{96}\right]$, $\alpha=\frac{1}{6}$, $\beta=\mu=0$, $\gamma=\frac{1}{60}$, $\delta=\frac{1}{20}$, $\eta=\frac{1}{50}$ and $\lambda=\frac{1}{30}$. Thus T satisfies all the conditions of Theorem 2.2 on closed ball and T has a unique fixed point in $\overline{B\left(\frac{1}{15},\frac{1}{4}\right)}$. Now if $x=\frac{3}{4}, y=1\in\left(\frac{1}{2},1\right]$, then

$$\tau{+}F\left(d(T(x),T(y))\right)$$

$$> F\left(\begin{array}{c}\alpha d\left(x,y\right)+\beta d\left(x,Ty\right)+\gamma d\left(y,Tx\right)+\delta d\left(y,Ty\right)+\\ \eta\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\lambda\frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)}+\mu\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}\end{array}\right).$$

and consequently, condition (4) does not hold on X. .

Corollary 2.4. *Let T be a continuous selfmap in a complete b-metric like space (X,d) and x_0 be an arbitrary point in X and $r>0$. Assume that $F\in\Delta_F$ and $\tau>0$ for all $x,y\in\overline{B(x_0,r)}\subseteq X$ with $d\left(TxTy\right)>0$such that*

$$\tau+F\left(d\left(Tx,Ty\right)\right)\leqslant F\left(\alpha d\left(x,y\right)\right);\quad\text{where } 0\leqslant\alpha<1.$$

Moreover,

$$d(x_0, Tx_0) \leqslant (1-\alpha)r, \tag{14}$$

Then there exist a point x^* *in* $\overline{B(x_0, r)}$ *such that* $Tx^* = x^*$.

Example 2.5. Let $X = [0,2]$ and $d(x,y) = |x-y|^2 + \frac{x}{10} + \frac{y}{10}$. Then (X,d) is a complete b-metric like space. Define the mapping $T : X \to X$ by,

$$T(x) = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1]; \\ x - \frac{1}{2} & \text{if } x \in (1,2]. \end{cases}$$

Set $\alpha = \dfrac{3}{10}$, $x_0 = \dfrac{1}{2}$, $r = \dfrac{1}{2}$, then $\overline{B(x_0, r)} = [0,1]$. If $F(t) = \ln(t)$, $t > 0$, then

$$d(x_0, T(x_0)) = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4} < (1-\alpha)r.$$

For $x, y \in \overline{B(x_0, r)}$, the inequality

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + \frac{Tx}{10} + \frac{Ty}{10} \\ &= \left|\frac{x}{3} - \frac{y}{3}\right|^2 + \frac{x}{30} + \frac{y}{30} = \frac{1}{9}|x-y|^2 + \frac{x}{30} + \frac{y}{30} \\ &< \frac{1}{3}\left[|x-y|^2 + \frac{x}{10} + \frac{y}{10}\right], \end{aligned}$$

holds. Thus,

$$d(T(x), T(y)) < \alpha d(x,y).$$

Which implies

$$\tau + \ln(d(T(x), T(y))) \leqslant \ln(\alpha d(x,y)).$$

That is

$$\tau + F(d(T(x), T(y))) \leqslant F(\alpha d(x,y))$$

for $\tau = \frac{1}{100}$, $\frac{1}{3} \leqslant \alpha < \frac{1}{s} < 1$ and $\beta = \gamma = \delta = \eta = \lambda = \mu = 0$. Thus T satisfies all the conditions of Corollary 2.4 on closed ball and "0" is the unique fixed point. Now if $x = 1.5, y = 2 \in (1, 2]$, then

$$\begin{aligned} d(T(x), T(y)) &= \left| x - \frac{1}{2} - y + \frac{1}{2} \right|^2 + \frac{x - \frac{1}{2}}{10} + \frac{y - \frac{1}{2}}{10} \\ &= |x - y|^2 + \frac{2x - 1}{20} + \frac{2y - 1}{20} \\ &= |x - y|^2 + \frac{x}{10} + \frac{y}{10} - \frac{1}{10} \\ &> |x - y|^2 + \frac{x}{10} + \frac{y}{10} = d(x, y). \end{aligned}$$

Which implies

$$d(T(x), T(y)) > d(x, y)$$

$$\tau + \ln\left(d(T(x), T(y))\right) \nleq \ln\left(d(x, y)\right)$$

and consequently, F-contraction condition (1) does not hold on X.

3. F-Multiplicative Contraction on Closed Ball

In this section, we define a new contraction called F-multiplicative contraction on closed ball and obtained a new fixed point theorem for such contraction in complete multiplicative metric spaces.

Definition 3.1. *Let (X, d) be a multiplicative metric space. The mapping $T : X \to X$ is called F-multiplicative contraction on closed ball if there exists $\tau > 1$ such that*

$$\tau . F\left(d\left(Tx, Ty\right)\right) \leqslant F\left(d\left(x, y\right)\right); \tag{15}$$

and

$$d(x_0, Tx_0) \leqslant r^{(1-\lambda)}, \text{ where } \lambda \in [0, 1) \text{ and } r > 0. \tag{16}$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$ with $d\left(Tx, Ty\right) > 1$, where $F : (1, \infty) \to (1, \infty)$ is a mapping satisfying the following conditions:

(F1) F is non-decreasing, i.e. for all $x, y \in [1, \infty)$ such that $x \leqslant y$, we have $F(x) \leqslant F(y)$;*

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $[1, \infty)$, $\lim_{n\to\infty} \alpha_n = 1$ if and only if $\lim_{n\to\infty} F(\alpha_n) = 0$;*

(F3) F is continuous mapping.*

we now establish a new type of fixed point theorem for F-multiplicative contraction on closed ball in a complete multiplicative metric space.

Theorem 3.2. *Let (X, d) be a complete multiplicative metric space. Let $T : X \to X$ be an F-multiplicative contraction mapping on a closed ball $\overline{B(x_0, r)}$ and x_0 be an arbitrary point in X. Moreover, $\prod_{j=0}^{\mathbb{N}} d(x_0, Tx_0) \leqslant r$ for all $j \in \mathbb{N}$, Then there exist a unique point x^* in $\overline{B(x_0, r)}$ such that $T(x^*) = x^*$.*

Proof. Choose a point x_1 in X such that $x_1 = Tx_0$. continuing in this way, so we get $x_{n+1} = Tx_n$, for all $n \geqslant 0$ and this implies that (x_n) is a nonincreasing sequence. Now we will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$, by using mathematical induction. Since from (16), we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leqslant r^{(1-\lambda)} \leqslant r,$$

thus, $x_1 \in \overline{B(x_0, r)}$. Suppose $x_2....x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Thus from (15), we obtain

$$F(d(x_j, x_{j+1})) = F(d(Tx_{j-1}, Tx_j)) \leqslant \frac{1}{\tau} F[d(x_{j-1}, x_j)] < F[d(x_{j-1}, x_j)]$$

$$d(x_j, x_{j+1}) < d(x_{j-1}, Tx_{j-1}) \tag{17}$$

Now, using triangular inequality and (17), we get

$$\begin{aligned} d(x_0, x_{j+1}) &\leqslant d(x_0, x_1).....d(x_j, x_{j+1}) \\ &\leqslant \prod_{j=0}^{\mathbb{N}} d(x_0, Tx_0) \\ &\leqslant r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$ continuing this process, we get

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Tx_{n-1}, Tx_n)) \leqslant \frac{1}{\tau} F[d(x_{n-1}, x_n)] \\ &\leqslant \frac{1}{\tau} F[d(Tx_{n-2}, Tx_{n-1})] \\ &\leqslant \frac{1}{\tau^2} F[d(x_{n-2}, x_{n-1})] \\ &\vdots \\ &\leqslant \frac{1}{\tau^n} F[d(x_0, x_1)]. \end{aligned}$$

And so $\lim_{n\longrightarrow\infty} F\left(d\left(x_n, x_{n+1}\right)\right) = 0$. By $(F2)$, we find that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 1. \tag{18}$$

We shall prove that (x_n) is Cauchy in (X, d). So, it suffices to show that $\lim_{n\to\infty} d(x_n, x_m) = 1$. We argue by contradiction. Suppose there exist $\varepsilon > 0$ and $n(p) > m(p) > p$, where $n(p)$ is least number greater than $m(q)$ such that

$$d(x_{n(p)}, x_{m(p)}) \geqslant \varepsilon, \quad d(x_{n(p)-1}, x_{m(p)}) < \varepsilon \quad \text{for all } p \in \mathbb{N}. \tag{19}$$

By triangular inequality, we have

$$\begin{aligned} \varepsilon \leqslant d(x_{n(p)}, x_{m(p)}) &\leqslant d(x_{n(p)}, x_{n(p)-1}).d(x_{n(p)-1}, x_{m(p)}) \\ &< d(x_{n(p)}, x_{n(p)-1}).\varepsilon. \end{aligned} \tag{20}$$

Which as $p \to \infty$ implies that $\lim_{p\longrightarrow\infty} d(x_{n(p)}, x_{m(p)}) = \epsilon$.

$$d(x_{n(p)}, x_{m(p)}) \leqslant d(x_{n(p)}, x_{n(p)-1}).d(x_{n(p)-1}, x_{m(p)-1}).d\left(x_{m(p)-1}, x_{m(p)}\right)$$

and

$$d(x_{n(p)-1}, x_{m(p)-1}) \leqslant d(x_{n(p)-1,}x_{n(p)}).d(x_{n(p)}, x_{m(p)}).d\left(x_{m(p)}, x_{m(p)-1}\right)$$

Thus, as $n \to \infty$ from above two inequalities, we have $\lim_{p\longrightarrow\infty} d(x_{n(p)-1}, x_{m(p)-1}) = \epsilon$. Put $x = x_{n(p)-1,}$ and $y = x_{m(p)-1}$ in (3.1) we get

$$\begin{aligned} F(d(x_{n(p)}, x_{m(p)})) &\leqslant \frac{1}{\tau} F(d(x_{n(p)-1}, x_{m(p)-1})) \\ &< F(d(x_{n(p)-1}, x_{m(p)-1})), \end{aligned}$$

which further by continuity of F and $\tau > 1$ gives $F(\varepsilon) < F(\varepsilon)$ Hence $\{x_n\}$ is a Cauchy sequence in the complete multiplicative metric space $\left(\overline{B(x_0, r)}, d\right)$. Since $\left(\overline{B(x_0, r)}, d\right)$ is a complete multiplicative metric space, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. Since T is multiplicative continuous, then,

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Hence $x^* = T(x^*)$ that is x^* is fixed point of T. Finally, the uniqueness of the fixed point follows from condition (3.1).

Example 3.3. Let $X = \mathbb{R}$ and $d : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ be a multiplicative metric defined by $d(x, y) = e^{|x-y|}$. Note that $(\mathbb{R}, d)$ is a complete multiplicative metric space. Define mapping, $T : X \to X$, and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 1], \\ x - 1 & \text{otherwise.} \end{cases}$$

$F(t) = \ln(t)$ with $\tau = 2$. Set $\lambda = \frac{5}{21}$ $x_0 = \frac{2}{7}$, $r = 4$ then $\overline{B(x_0, r)} = [0, 1]$. Now

$$r^{(1-\lambda)} = 4^{\left(\frac{11}{16}\right)},$$

and so $d(x_0, Tx_0) \leqslant r^{(1-\lambda)}$ holds. Note also that, For if $x, y \in \overline{B(x_0, r)}$, we have

$$e^{|Tx-Ty|} = e^{\left|\frac{x}{4}-\frac{y}{4}\right|} = e^{\frac{1}{4}|x-y|} < e^{|x-y|},$$

which implies,

$$d(T(x), T(y)) = e^{\frac{1}{4}|x-y|} < e^{|x-y|} = d(x, y).$$

That is,

$$d(T(x), T(y)) < d(x, y).$$

Consequently,

$$\tau . \ln\left(d(T(x), T(y))\right) \leqslant \ln\left(d(x, y)\right).$$

If $x \notin \overline{B(x_0, r)}$ or $y \notin \overline{B(x_0, r)}$ then

$$d(T(x), T(y)) = e^{|x-1-y+1|} = e^{|x-y|} = d(x, y),$$

this implies,

$$d(T(x), T(y)) > d(x, y)^{\lambda}, \text{ where } \lambda \in [0, 1),$$

and consequently, contractive condition (2) does not hold on X. But, hypotheses of Theorem 3.2 hold on closed ball and $x = 0$ is a fixed point of T.

4. Application

In this section, we discuss an application of fixed point Theorem, proved in the previous section, in solving the family of Volterra type integral equations given below.

$$u(t) = \int_0^t G(t,s)\, K(s, w(s)) ds. \tag{21}$$

for $t \in [0, a]$, where $a > 0$. Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on closed ball $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define sup norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau t}\}$, where $\tau > 0$. Let $C([0, a], \mathbb{R})$ be endowed with the metric,

$$d_\tau(u, v) = \sup_{t\in[0,a]} [|u(t)| + |v(t)|]^2, \tag{22}$$

for all $u, v \in C([0, a], \mathbb{R})$. Obviously, $(C([0, a], \mathbb{R}, \| \cdot \|_\tau)$ is a complete b-metric like space. Define the operator $T : C([0, a], \mathbb{R}) \to C([0, a], \mathbb{R})$ by

$$Tu(t) = \int_0^t G(t,s)\, K(s, u(s)) ds.$$

Suppose that

$$\sup_{t\in[0,a]} \int_0^t G(t,s)\, ds \leqslant 1.$$

To show the existence of solution of integral equation (21), we shall show, with the application of Theorem 2.2, that T has a fixed point in $C([0, a], \mathbb{R})$.

Now we prove the following Theorem to ensure the existence of solution of integral equation (21).

For more details on such applications we refer the reader to [6, 18].

Theorem 4.1. *Assume the following conditions are satisfied:*
(1) $K : [0,a] \times \mathbb{R} \to \mathbb{R}$ *and* $G : [0,a] \times [0,a] \to \mathbb{R}^+$ *are continuous functions;*
(2) There exists $\tau \geqslant 1$*, such that*

$$|K(t,s,u)| + |K(t,s,v)| \leqslant \left[\tau e^{-\tau} h(u,v)\right]^{\frac{1}{2}}$$

for all $t, s \in [0,a]$ *and* $u, v \in C([0,a], \mathbb{R})$*, where*

$$\begin{aligned} h(u,v) &= \alpha\left[|u(t)| + |v(t)|\right]^2 + \beta\left[|u(t)| + |Tv(t)|\right]^2 + \\ &\quad \gamma\left[|v(t)| + |Tu(t)|\right]^2 + \delta\left[|v(t)| + |Tv(t)|\right]^2 + \\ &\quad \eta\frac{\left[|v(t)| + |Tv(t)|\right]^2\left[1 + \left[|u(t)| + |Tu(t)|\right]^2\right]}{\left[|u(t)| + |v(t)|\right]^2} + \\ &\quad \lambda\frac{\left[|v(t)| + |Tv(t)|\right]^2 + \left[|v(t)| + |Tu(t)|\right]^2}{1 + \left[|v(t)| + |Tv(t)|\right]^2\left[|v(t)| + |Tu(t)|\right]^2} \\ &\quad + \mu\frac{\left[|u(t)| + |Tu(t)|\right]^2\left[1 + \left[|v(t)| + |Tu(t)|\right]^2\right]}{1 + \left[|u(t)| + |v(t)|\right]^2 + \left[|v(t)| + |Tv(t)|\right]^2}, \end{aligned}$$

where, $\alpha, \beta, \gamma, \delta, \eta, \lambda, \mu \geqslant 0$ *with* $s\alpha + \left(s^2 + s\right)\beta + 2s^2\gamma + \delta + \eta + \lambda + s\mu < 1$*. Then the integral equation (4.1) has a solution.*

Proof. By assumption (2)

$$\begin{aligned} \left[|Tu(t)| + |Tv(t)|\right]^2 &= \left(\left|\int_0^t G(t,s)\, K_1(s,u(s)ds\right| + \left|\int_0^t G(t,s)\, K_2(s,v(s))ds\right|\right)^2 \\ &\leqslant \left(\int_0^t G(t,s)\left|K_1(s,u(s)\right| ds + \int_0^t G(t,s)\left|K_2(s,v(s))\right| ds\right)^2 \\ &= \left(\int_0^t G(t,s)\left(\left|K_1(s,u(s)\right| + \left|K_2(s,v(s))\right|\right) ds\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\int_0^t G(t,s) \left(e^{-\tau} h(u,v) \right)^{\frac{1}{2}} ds \right)^2 \\
&\leqslant e^{-\tau} h(u,v) \left(\int_0^t G(t,s)\, ds \right)^2 \\
&\leqslant e^{-\tau} h(u,v).
\end{aligned}
$$

This implies

$$[|Tu(t)| + |Tv(t)|]^2 \leqslant e^{-\tau} h(u,v),$$

which further implies

$$\tau + \ln \left([|Tu(t)| + |Tv(t)|]^2 \right) \leqslant \ln \left(h(u,v) \right).$$

So all the conditions of Theorem 2.2 are satisfied. Hence, in the light of Theorem 2.2, (4.1) has a solution.

Conflict of Interests
The authors declare that they have no competing interests.

Acknowledgements
The authors would like to thank the referees for their valuable comments and suggestions.

References

[1] M. Abbas, B. Ali, and S. Romaguera, Fixed and periodic points of generalized contractions in metric spaces, *Fixed Point Theory Appl.*, 243 (2013).

[2] M. Abbas, B. Ali, and Y. I. Suleiman, Common Fixed Points of Locally Contractive Mappings in Multiplicative Metric Spaces with Application, *International Journal of Mathematics and Mathematical Sciences,* (2015), Article ID 218683, 7 pages.

[3] M. A. Alghamdi, N. Hussain, and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, *Journal of Inequalities and Applications,* 402 (2013), 312-345.

[4] E. Ameer and M. Arshad, Two New Fixed Point Results for Generalized Wardowski Type Contractions, *Journal of Analysis and Number Theory,* 5 (1) (2017), 1-9.

[5] M. Arshad, A. Shoaib, and I. Beg, Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space, *Fixed Point Theory and Appl.*, (2013), 115-130.

[6] M. Arshad, A. Shoaib, and P. Vetro, Common Fixed Points Of a pair Of Hardy Rogers Type Mappings on a closed ball in ordered dislocated Metric Spaces, *Journal of Function Spaces and Appl.*, (2013), article ID 638181, 9 pages.

[7] M. Arshad, E. Ameer, and A. Hussain, Hardy Rogers Type Fixed Point Theorems for α-GF-Contractions, *Archivum Mathematicum,* 51 (2015), 129-141.

[8] A. Augustynowicz, Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function, *Serdica Mathematical Journal,* 23 (1997), 203-210.

[9] R. Batra and S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, *Int. J. Comput. Math.*, 91 (2014), 1-8.

[10] R. Batra, S. Vashistha, and R. Kumar, A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance, *J. Math. Comput. Sci.*, 4 (5) (2014), 826-833.

[11] I. A. Bakhtin, The contraction principle in quasimetric spaces, *In: Functional Analysis,* 30 (1989), 26-37.

[12] A. E. Bashirov, E. M. Karplnara, and A. Ozyaplcl, Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications,* 337 (2008), 36-48.

[13] V. Berinde, *Iterative Approximation of Fixed Points,* Springer-Verlag, Berlin Heidelberg, 2007.

[14] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-Type,* Filomat 28 (4) (2014), 715-722.

[15] N. Hussain and P. Salimi, Suzuki-wardowski type fixed point theorems for α-GF-contractions, *Taiwanese J. Math.,* 20 (20) (2014). doi: 10.11650/tjm.18.2014.4462

[16] N. Hussain, E. Karapınar, P. Salimi, and F. Akbar, α-admissible mappings and related fixed point theorems, *J. Inequal. Appl.,* 114 (2013), 1-11.

[17] N. Hussain and P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions, *Taiwan. J. Math.,* (2014). doi:10.11650/tjm.18.2014.4462

[18] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.,* 60 (1968), 71-76.

[19] E. Kryeyszig, *Introductory Functional Analysis with Applications,* John Wiley & Sons, New York, (Wiley Classics Library Edition), 1989.

[20] P. S. Kumari and D. Panthi, Connecting various type of cyclic contractions and contractive self-mappings with Hardy-Rogers self-mappings, *Fixed Point Theory Appl.,* 15 (2016), 201-220.

[21] G. Minak, A. Halvaci, and I. Altun, Ćirić type generalized $F-$contractions on complete metric spaces and fixed point results, *Filomat,* 28 (6) (2014), 1143-1151.

[22] D. ÓRegan and A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, *Journal of Mathematical Analysis and Applications,* 341 (2008), 1241-1252.

[23] M. Ozavsar and A. C. Cervikel, *Fixed points of multiplicative contraction mappings on multiplicative metric spaces,* http://arxiv.org/abs/1205.5131v1.

[24] H. Piri and P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Poin Theory Appl.,* 210 (2014), 201-225.

[25] M. Sgroi and C. Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, *Filomat,* 27 (7) (2013), 1259-1268.

[26] P. Salimi, A. Latif, and N. Hussain, Modified $\alpha-\psi$-Contractive mappings with applications, *Fixed Point Theory Appl.,* 151 (2013), 342-355.

[27] B. Samet, C. Vetro, and P. Vetro, Fixed point theorems for $\alpha-\psi$-contractive type mappings, *Nonlinear Anal.,* 75 (2012), 2154-2165.

[28] N. A. Secelean, Iterated function systems consisting of F contractions, *Fixed Point Theory Appl.,* Article ID 277 (2013). doi:10.1186/1687-1812-2013-277

[29] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.,* (2012), Article ID 94.

[30] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric space, *Demonstr. Math.,* (1) (2014), 146-155.

Eskandar Ameer
Department of Mathematics and Statistics
Ph.D Student in Pure Mathematics
International Islamic university H-10
Islamabad, Pakistan
E-mail: eskandarameer@gmail.com

Muhammad Arshad
Department of Mathematics and Statistics
Assistant Professor of Mathematics
International Islamic university H-10
Islamabad, Pakistan
E-mail: marshadzia@iiu.edu.pk