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A Characterization for Fuzzy Normed Linear Spaces and its Applications

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Abstract. In this paper we give a characterization of fuzzy normed linear space and then two examples are introduced. By using this characterization some fuzzy fixed point theorems are established.

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1. Introduction

The idea of fuzzy Hilbert spaces was introduced and studied by many authors see [5, 7, 12, 13]. This notion, in fact, is a generalization of the notion of probabilistic inner product spaces(see [1]). we consider Mukherjee and Bag [10] redefined fuzzy real inner product space introduced by Goudarzi and Vaezpour [8] in order to establish a characterization theorem. Based on this theorem, it has been possible to rewrite some theorems namely Browder-Petryshyn, D. de Figueiredo etc, in fuzzy setting (see [9]) and some other theorems.
The organization of the paper is as follows: Section 2, provides some preliminary results which are used in this paper. In Section 3, characterization of fuzzy normed linear space is given. Some fixed point theorems in fuzzy inner product spaces are established in Section 4.

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2. Preliminaries

In this section some definitions and preliminary results are given which are used in this work.
Throughout this paper, the symbols $\wedge$ and $\vee$ mean the min and the max, respectively.

Definition 2.1. *[2] Let V be a linear space over real numbers. A fuzzy subset N of $V \times \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is called a fuzzy norm on V if for all $x, y \in V$ and $s \in \mathbb{R}$, following conditions are satisfied:*
$(N1)$ *for all $r \in \mathbb{R}$ with $r \leqslant 0, N(x,r) = 0$;*
$(N2)$ *(for all $r \in \mathbb{R}, r > 0, N(x,r) = 1$) iff $x = 0$;*
$(N3)$ *for all $r \in \mathbb{R}, r > 0, N(sx,r) = N(x, \frac{r}{|s|})$ if $s \neq 0$;*
$(N4)$ *for all $r, t \in \mathbb{R}, x, y \in V$;*
$N(x+y, r+t) \geqslant min\{N(x,r), N(y,t)\}$;
$(N5)N(x,.)$ *is a non-decreasing function of R and $\lim_{r\to\infty} N(x,r) = 1$.*
The pair (V, N) will be referred to as a fuzzy normed linear space.

Definition 2.2. *[11] Let V be a linear space over real numbers.*
Define $m : V \times V \times \mathbb{R} \to [0,1]$ such that for all $x, y, z \in V, r \in \mathbb{R}$ the following conditions are satisfied:
$(FIP-1)$ *for all $r \in \mathbb{R}$ with $r \leqslant 0, m(x,x,r) = 0$;*
$(FIP-2)$ *for all $r \in \mathbb{R}, r > 0, m(x,x,r) = 1$ iff $x = 0$;*
$(FIP-3)$ $m(x,y,r) = m(y,x,r)$;
$(FIP-4)$ *For any scalar k,*

$$m(kx,y,r) = \begin{cases} 1 - m(x,y,\frac{r}{k}) & \text{if } k < 0 \\ H(r) & \text{if } k=0 \\ m(x,y,\frac{r}{k}) & \text{else,} \end{cases}$$

where

$$H(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{else.} \end{cases}$$

$(FIP-5)$ $m(x+y,z,r+t) \geqslant m(x,z,r) \wedge (y,z,t)$;
$(FIP-6)$ $\lim_{r\to\infty} m(x,y,r) = 1$.
Then m is said to be a fuzzy inner product and (V, m) is a fuzzy inner product space.

Theorem 2.3. *[2] Let (V, N) be a fuzzy normed linear space. Assume further that,*
$(N6)$ *(for all $r > 0, N(x,r) > 0$) implies $x = 0$.*
Define $\|x\|_\alpha = \wedge\{r > 0 : N(x,r) \geqslant \alpha\}, \alpha \in (0,1)$. Then $\{\| \ \|_\alpha : \alpha \in (0,1)\}$

is an ascending family of norms on V and they are called $\alpha-$norms on V corresponding to the fuzzy norm N on V.

Theorem 2.4. *[11] Let (V, m) be a fuzzy inner product space. Assume further that,*
$(FIP-7)$ $m(x, y, st) \geqslant m(x, x, s^2) \wedge m(y, y, t^2), \forall s, t \in \mathbb{R}, x, y \in V.$
Define a function $N : V \times \mathbb{R} \rightarrow [0, 1]$ by

$$N(cx, t) = \begin{cases} m(|c|x, |c|x, t^2) & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then N is a fuzzy norm on V.
This norm is called as induced norm of m.

Theorem 2.5. *[11] Let (V, m) be a fuzzy inner product space. Assume further that,*
$(FIP-8)$ $-\infty < \{\wedge t, \vee t : m(x, y, t) \geqslant \alpha\} < +\infty$ $\forall \alpha \in (0, 1)$ *and*
$\{\forall t > 0 : m(x, x, t) > 0\}$ *implies* $x = 0$.
Define for $\alpha \in (0, 1)$,
$< x, y >_\alpha = \{\wedge t : m(x, x, t) \geqslant \alpha\}$.
Then $\{< x, y >_\alpha : \alpha \in (0, 1)\}$ is a family of inner product in V. We call these inner products as $\alpha-$ inner products corresponding to the fuzzy inner product m.

Theorem 2.6. *[2] If (V, m) be a fuzzy inner product space satisfying $(FIP-7), (FIP-8)$ and N be its induced fuzzy norm. The $\alpha-$norms derived from induced fuzzy norm and from $\alpha-$inner products,$\alpha \in (0, 1)$ are same m [2].*

If $(V,\ N)$ be a fuzzy normed linear space satisfying $(N6)$. If $\{x_n\}$ be an $\alpha-$convergent sequence in $(V,\ N)$. Then $\|x_n - x\|_\alpha \rightarrow 0$ as $n \rightarrow \infty$ ($\|\ \|_\alpha$ denotes the $\alpha-$norm of N) see [4].

Definition 2.7. *[3] Let $(V,\ N)$ be a fuzzy normed linear space. A subset H of V is said to be l-fuzzy closed if for each $\alpha \in (0,\ 1)$ and for any sequence $\{x_n\}$ in H and $x \in V, (\underline{lim}_{n\rightarrow\infty} N(x_n - x, r) > \alpha\ \forall r > 0) \Rightarrow x \in H$.*

If $(V,\ N)$ be a fuzzy normed linear space satisfying $(N6)$ and $H \subset V$. Then H is l-fuzzy closed iff H is closed w.r.t. $\|\ \|_\alpha$ ($\alpha-$norm of N) for each $\alpha(0,\ 1)$ see [4].

Definition 2.8. *[3] Let $(V,\ N)$ be a fuzzy normed linear space. A subset A of V is said to be bounded iff there exists $r > 0$ and $0 < s < 1$ such that $N(x, r) > 1 - s\ \ \forall x \in A$. V is said to be l-fuzzy bounded if for each $\alpha \in (0,\ 1)$ there exists $r(\alpha)$such that $N(x, r(\alpha)) > \alpha\ \forall x \in V$.*

If $(V,\ N)$ be a fuzzy normed linear space satisfying $(N6)$ and $H \subset V$. Then V is l-fuzzy bounded iff V is bounded w.r.t. $\|\ \|_\alpha$ ($\alpha-$norm of N) for each $\alpha(0,\ 1)$ see [3].

Definition 2.9. *[4] Let $(V,\ N)$ be a fuzzy normed linear space and $\alpha \in (0,\ 1)$. A sequence $\{x_n\}$ in V is said to be $\alpha-$convergent in V if there exists $x \in V$ such that $\lim_{n\to\infty} N(x_n - x, r) > \alpha\ \forall r > 0$ and x is called the limit of $\{x_n\}$. It is said to be $\alpha-$ complete if any $\alpha-$Cauchy sequence in V, $\alpha-$converges to a point in V. It is said to be l-fuzzy complete if it is $\alpha-$complete for all $\alpha \in (0,\ 1)$.*

Let $(V,\ N)$ be a fuzzy normed linear space. A mapping $f : (V,\ N) \to (V,\ N)$ is said to be fuzzy non-expansive if [3].

$$N(f(x) - f(y); r) \geqslant N(x - y,\ r)\ \forall\ x,\ y \in V\ \forall r \in R.$$

Let $(V,\ N)$ be a fuzzy normed linear space satisfying $(N6)$ and $f : V \to V$ be a fuzzy non-expansive mapping. Then f is a non-expansive mapping w.r.t. each $\alpha-$norm of N, where $\alpha \in (0,\ 1)$ see [4].

Let $V,\ W$ be two linear spaces over the real numbers. Let N_1 and N_2 be two fuzzy norms on $V,\ W$ respectively. A mapping $f : (V,\ N_1) \to (W, N_2)$ is said to be sectional fuzzy continuous at $x_0 \in U$, if there exists $\alpha \in (0,\ 1)$ such that for each $\varepsilon > 0$, there exists $\delta > 0$,

$$N_1(x - x_0, \delta) \geqslant \alpha \Longrightarrow N_2(f(x) - f(x_0), \varepsilon) \geqslant \alpha\ \forall x \in V.$$

If f is sectional fuzzy continuous at each point of V, then f is said to be sectional fuzzy continuous on V [3].

If $(V,\ N_1)$ and $(V,\ N_2)$ are fuzzy normed linear spaces satisfying $(N6)$. Then a mapping $f : (V,\ N_1) \to (V,\ N_2)$ is sectional fuzzy continuous iff $f : (V,\ \|\ \|^1_\alpha) \to (V,\ \|\ \|^2_\alpha)$ is continuous for some $\alpha \in (0,\ 1)$ see [4].

3. A Characterization Theorem

Now we are ready to establish a characterization theorem.

Theorem 3.1. *Let (V, N) be a fuzzy normed linear space. Then the following assertions are equivalent:*

1. *(V, N) satisfying $(N6)$.*
2. *There exists a fuzzy inner product (V, m) satisfying $(FIP-7), (FIP-8)$ gives N as induced norm.*

Proof. ($\Rightarrow$) Let (V, N) be a fuzzy normed linear space satisfying $(N6)$. Define a function $m : V \times V \times \mathbb{R} \to [0, 1]$ by

$$m(kx, y, r) = \begin{cases} \frac{N(kx,r)+N(ky,r)}{2} & \text{if r} \geqslant 0 \text{ and k} > 0, \\ 1 - \frac{N(kx,r)+N(ky,r)}{2} & \text{if } r \geqslant 0, k < 0, \\ 0 & \text{if } r \leqslant 0, \end{cases}$$

To check the conditions of $FIP-$space definition, we only examine the conditions $(FIP - 5)$. The other conditions clearly holds.
In $(FIP - 5)$, let $x, y, z \in V, t, s \in \mathbb{R}^+$,

$$m(x+y, z, r+s) = \frac{N(x,r) + N(y,r)}{2} \geqslant \frac{1}{2}\min\{N(x,r), N(y,s)\} + \frac{1}{2}N(z, r+s)$$

$$\geqslant \frac{1}{2}\min\{N(x,r), N(y,s)\} + \frac{1}{2}\min\{N(z,r), N(z,s)\} \geqslant$$

$$\min\{\frac{N(x,r) + N(z,r)}{2}, \frac{N(y,s) + N(z,s)}{2}\} = \min\{m(x,z,r), m(y,z,s)\}.$$

To show $FIP-7$ assume that $s, r > 0$ and $s \leqslant r$ then $s^2 \leqslant rs$ and since $N(x, .)$ is a non-decreasing function of $\mathbb{R}$ we have $N(x, sr) \geqslant N(s^2)$. This follows that

$$m(x, y, sr) = \frac{N(x, sr) + N(y, sr)}{2} \geqslant \frac{N(x, s^2) + N(y, s^2)}{2} \geqslant m(x, y, s^2) \wedge m(x, y, r^2).$$

The other case, $r \leqslant s$, can be proved similarly.

Also the condition $FIP - 8$ clearly hold.

Therefore $m(x, y, r)$ satisfying in all conditions of definition $FIP - space$ and also conditions in Theorems 2.2,2.3. Then we have induced norm of m and $M(cx, r) = m(|c|x, |c|x, r^2) = N(|c|x, r^2)$.

($\Leftarrow$) Conversely, suppose that there exists a fuzzy inner product on V satisfying $(FIP-7), (FIP-8)$. We show that induced fuzzy norm satisfying $(N6)$. Since (V, m) satisfy $(FIP - 7)$, the induced fuzzy norm N of m given by:

$$N(kx, r) = \begin{cases} m(|k|x, |k|x, t^2) & \text{if r} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now $(FIP - 8)$ gives $\{\forall r > 0; m(|c|x, |c|x, r^2)\} \Rightarrow x = 0$.
Therefore $\{\forall r > 0; N(x, r) = m(|c|x, |c|x, r^2)\} \Rightarrow x = 0$. That is N satisfy $(N6)$. $\square$

If $(V, \ < ., . >)$ be an ordinary inner product, we can define the standard fuzzy inner product induced by the inner product $< ., . >$(see [8]).

In the following example we will replace inner product with $\alpha-$norm.

Example 3.2. Let (V, N) be a fuzzy normed linear space satisfying $(N6)$. Then we get $\alpha-$norm on V by Theorem 2.1 for $\alpha \in (0,1)$. we define a mapping F as follows

$$F(kx, y, r) = \begin{cases} \frac{\sqrt{r}}{\sqrt{r}+\sqrt{\|k\|\|x\|_\alpha \|y\|_\alpha}} & \text{if k} \geqslant 0 \text{ and r} > 0, \\ H(r) & \text{if k=0} \\ 1 - \frac{\sqrt{r}}{\sqrt{r}+\sqrt{\|k\|\|x\|_\alpha \|y\|_\alpha}} & \text{else,} \\ 0 & \text{if } r \leqslant 0, \end{cases}$$

where

$$H(r) = \begin{cases} 1 & \text{if r} > 0 \\ 0 & \text{else.} \end{cases}$$

Define $a * b = \min\{a,\ b\}$, we show that F is fuzzy inner product on V. We only examine the condition $(FIP-5)$. Other conditions are clear and for seeing details see [8].

In $(FIP-5)$, for each $x, y \in V$ and $t, r \in \mathbb{R}$,either

$$\sqrt{1+\frac{r}{t}} \leqslant \sqrt{\frac{\|x+y\|_\alpha}{\|x\|_\alpha}},$$

or

$$\sqrt{\frac{\|x+y\|_\alpha}{\|x\|_\alpha}} \leqslant \sqrt{1+\frac{r}{t}}.$$

Since $F(kx, z, r) = F(x, z, \frac{r}{k})$, we can suppose that $\|x\|_\alpha = 1, \|x+y\|_\alpha = 1$.

Then the inequality

$$\sqrt{1+\frac{r}{t}} \leqslant \sqrt{\frac{\|x+y\|_\alpha}{\|x\|_\alpha}}$$

is not true.

From the other case we get

$$\sqrt{t\|z\|_\alpha \|x+y\|_\alpha} \leqslant \sqrt{(r+t)\|z\|_\alpha \|x\|_\alpha} \rightarrow$$

$$\sqrt{t^2+tr} + \sqrt{t\|z\|_\alpha \|x+y\|_\alpha} \leqslant \sqrt{t^2+tr} + \sqrt{(r+t)\|z\|_\alpha \|x\|_\alpha} \rightarrow$$

$$\frac{\sqrt{r+t}}{\sqrt{r+t}+\sqrt{\|z\|_\alpha \|x+y\|_\alpha}} \geqslant \frac{\sqrt{t}}{\sqrt{t}+\sqrt{\|z\|_\alpha \|x\|_\alpha}}.$$

That is $m(x+y, z, r+t) \geqslant m(x, z, t) \geqslant \min\{m(x, z, t),\ m(y, z, r)\}$ and the condition $FIP-5$ holds.

We give other example which shows all ordinary normed linear space could have a fuzzy set satisfying in $(N6)$.

We guess the linear combination of exponential map plays a major role in fuzzy sets.

Example 3.3. Let $(X, \|.\|)$ be an ordinary normed linear space. We define a fuzzy set of $X \times \bar{\mathbb{R}}$, as follows, which satisfying in $(N6)$.

$$N(x,r) = \begin{cases} \frac{\exp(\frac{-\|x\|}{|t|})}{\exp(\frac{\|x\|}{|t|})-\exp(\frac{-\|x\|}{|t|})+1} & \text{if } \ t \neq +\infty, \\ 0 & \text{if } t = +\infty \ \text{ or } \ t \leqslant 0. \end{cases}$$

We choose $t-$norm "$*$" as 'product' and $\bar{\mathbb{R}}$ is the extended real line $[-\infty, +\infty]$.

The conditions of $(N1), (N2), (N3), (N5)$ clearly holds.

To show $(N6)$ suppose, by contrapositive way, $x \neq 0$ then for $t = +\infty$ we have $N(x,t) = 0$.

For condition $(N4)$, first we note that the inequalities are holds:

1.
$$\frac{-\|x+y\|}{t+s} \geqslant \frac{-\|x\| - \|y\|}{t+s} = \frac{-\|x\|}{t+s} - \frac{\|y\|}{t+s} \geqslant \frac{-\|x\|}{t} - \frac{\|y\|}{s}.$$

2.
$$\varepsilon_1\varepsilon_2 + \varepsilon_1 \exp(\frac{\|x\|}{t}) + \varepsilon_2 \exp(\frac{\|y\|}{s}) + \exp(\frac{-\|x\|}{t} - \frac{\|y\|}{s}) \geqslant 1,$$

where $\varepsilon_1 = 1 - \exp(-\frac{\|y\|}{s}),\ \varepsilon_2 = 1 - \exp(-\frac{\|x\|}{t})$.

Inequality (1) is obviously true.

All factors in the inequality (2) are positive numbers. Also if $\frac{\|x\|}{t} \geqslant 1$ or $\frac{\|y\|}{s} \geqslant 1$ then $\varepsilon_1 \exp(\frac{\|x\|}{t}) \geqslant 1$ or $\varepsilon_2 \exp(\frac{\|y\|}{s}) \geqslant 1$ and the inequality is hold. Suppose that $\frac{\|x\|}{t} \to 0^+$, $\frac{\|y\|}{s} \to 0^+$ then $\exp(-\frac{\|x\|}{t} - \frac{\|y\|}{s}) \to 1$ and again the inequality is hold.

We will use this inequality as follows:

$$-\exp(-\frac{\|x\|}{t} - \frac{\|y\|}{s}) + 1 \leqslant$$

$$(1-\exp(\frac{-\|x\|}{t})(1-\exp(\frac{-\|y\|}{s}))+(1-\exp(\frac{-\|x\|}{t}))\exp(\frac{\|x\|}{t})+(1-\exp(-\frac{\|y\|}{t}))\exp(\frac{\|x\|}{t}).$$

Now to show $(N4)$ assume that $s, t > 0$ and $x, y \in X$.

$$N(x+y, t+s) = \frac{\exp(-\frac{\|x+y\|}{t+s})}{\exp(\frac{\|x+y\|}{t+s}) - \exp(-\frac{\|x+y\|}{t+s}) + 1} \geqslant \frac{\exp(-\frac{\|x\|}{t} - \frac{\|y\|}{s})}{\exp(\frac{\|x\|}{t+s} + \frac{\|y\|}{t+s}) - \exp(-\frac{\|x\|}{t} - \frac{\|y\|}{s}) + 1} \geqslant$$

$$\frac{\exp(-\frac{\|x\|}{t} - \frac{\|y\|}{s})}{\exp(\frac{\|x\|}{t} + \frac{\|y\|}{s}) + (1 - \exp(\frac{-\|x\|}{t}))(1 - \exp(\frac{-\|y\|}{s})) + (1 - \exp(\frac{-\|x\|}{t}))\exp(\frac{\|x\|}{t}) + (1 - \exp(-\frac{\|y\|}{t}))\exp(\frac{\|x\|}{t}}$$

$$\geqslant \frac{\exp(-\frac{\|x\|}{t}) \times \exp(\frac{-\|y\|}{s})}{(\exp(\frac{\|x\|}{t}) - \exp(\frac{-\|x\|}{t}) + 1)(\exp(\frac{\|y\|}{s}) - \exp(\frac{-\|y\|}{s}) + 1)},$$

where in the denominator of the third fraction we note that

$$s, t > 0, \ \frac{\|x+y\|}{t+s} \leqslant \frac{\|x\|}{t+s} + \frac{\|y\|}{t+s} \leqslant \frac{\|x\|}{t} + \frac{\|y\|}{s}.$$

Also in the denominator of this fraction we have, by inequality (2)

$$\exp(\frac{\|x\|}{t} + \frac{\|y\|}{s}) + (1 - \exp(\frac{-\|x\|}{t}))(1 - \exp(\frac{-\|y\|}{s})) + (1 - \exp(\frac{-\|x\|}{t}))\exp(\frac{\|x\|}{t}) + (1 - \exp(-\frac{\|y\|}{t}))\exp(\frac{\|x\|}{t})$$

$$= \exp(\frac{\|x\|}{t} + \frac{\|y\|}{s}) + 1 - \exp(\frac{-\|y\|}{s}) - \exp(\frac{\|x\|}{t}) + \exp(\frac{-\|x\|}{t} - \frac{\|y\|}{s}) + \exp(\frac{\|x\|}{t}) - 1 + \exp(\frac{\|y\|}{s}) - 1 \leqslant$$

$$\exp(\frac{\|x\|}{t} + \frac{\|y\|}{s}) - \exp(\frac{-\|y\|}{s}) - \exp(\frac{-\|x\|}{t}) + \exp(\frac{-\|x\|}{t} - \frac{\|y\|}{s}) + (\exp(\frac{\|x\|}{t}) - \exp(\frac{\|x\|}{t} - \frac{\|y\|}{s})) +$$

$$(\exp(\frac{\|y\|}{s}) - \exp(\frac{\|y\|}{s} - \frac{\|x\|}{t})) + 1 = (\exp(\frac{\|x\|}{t} - \exp(-\frac{\|x\|}{t})) + 1)(\exp(\frac{\|y\|}{t} - \exp(-\frac{\|y\|}{t})) + 1),$$

where added factors, $(\exp(\frac{\|x\|}{t} - \exp(-\frac{\|x\|}{t})) + 1)$ and $(\exp(\frac{\|y\|}{t} - \exp(-\frac{\|y\|}{t})) + 1)$,
are positive.

4. Some Fixed Point Theorems

Browder-Petryshyin and D. de Figueiredo theorems have been studied in fuzzy setting in [10]. In this Section we rewrite them. Some other fixed point theorems are also established in this Section.

Theorem 4.1. *[10] (Browder-Petryshyin). Let (V, m) be an l-fuzzy complete inner product space satisfying $(FIP-7), (FIP-8)$ and C be a l-fuzzy closed, convex and bounded subset of V. If $f : C \to C$ is a non-expansive mapping on C, then f has a fixed point in C.*

Theorem 4.2. *[10] Let* $(V,\ m)$ *be an l-fuzzy complete inner product space satisfying* $(FIP-7),(FIP-8)$ *and* C *be a l-fuzzy closed, convex and bounded subset of* V*. Let* $m(x,\ x,\ .)$ *is upper semicontinuous for each* $x \in V$*. If* $f : C \to C$ *is non-expansive mapping then* $F(f_t)$*, the set of fixed point of* f_t *is nonempty, where* $f_t(x) = tx + (1-t)f(x); t \in (0,\ 1)$*. Moreover* f_t *have the same fixed points as of* f*.*

Theorem 4.3. *[10] (D. de Figueiredo). Let* $(V,\ m)$ *be an l-fuzzy complete inner product space satisfying* $(FIP-7),(FIP-8)$ *and* C *be an l-fuzzy closed, convex and fuzzy bounded subset of* V *containing* 0*. If* $f : C \to C$ *is a fuzzy non-expansive mapping, then for any* x_0 *in* C *the sequence* $\{x_n\}$*, with*

$$x_n = f_n^{n^2} x_{n-1}, n = 1, 2, 3, ...$$

and $f_n x = \frac{n}{n+1} fx$ *there exists* $\alpha_0 \in (0,\ 1)$ *and* $x' \in F(f)$*(set of fixed point of* f*) such that for any* $r > 0, \exists M(r)$ *such that*

$$m(x_n - x',\ x_n - x',\ r) \geqslant \alpha_0 \quad \forall n \geqslant M(r).$$

Definition 4.4. *[10] Let* $(V,\ m)$ *be a fuzzy inner product space satisfying* $FIP-7$ *and* N *be its induced fuzzy norm. Then*

1. (V, m) *is said to be fuzzy Hilbert space if* (V, N) *is complete fuzzy normed linear space.*
2. (V, m) *is said to be* $\alpha-$ *complete fuzzy inner product space if* (V, N) *is* $\alpha-$ *complete fuzzy normed linear space.*
3. $C \subseteq V$ *is said to be l-fuzzy closed if l-fuzzy closed w.r.t* N*.*
4. $C \subseteq V$ *is said to be bounded if* C *is bounded w.r.t.* N *norm.*
5. $C \subseteq V$ *is said to be l-fuzzy bounded if* C *is l-fuzzy bounded w.r.t.* N *norm.*
6. (V, m) *is said to be l-fuzzy complete inner product space if* (V, N) *is l-fuzzy complete fuzzy normed linear space.*

Theorem 4.5. *(Browder-Petryshyin) Let* (V, N) *fuzzy normed space satisfying* $(N6)$ *and* C *be an l-fuzzy closed, convex and bounded set in* H*. If* $f : C \to C$ *is a fuzzy non-expansive mapping on* C*, then* f *has a fixed point in* C*.*

Proof. Since (V, N) is a l-fuzzy complete fuzzy normed space satisfying $(N6)$, then we get inner product space satisfying $(FIP-7),(FIP-8)$ induced by (V, N). If (V, N) be an l-fuzzy complete fuzzy normed space then so (V, m). Thus we have by Theorem 4.1, f has a fixed point in C. □

Theorem 4.6. *Let (V, N) be an l-fuzzy complete normed space satisfying $(N6)$ and C be an l-fuzzy closed, convex and bounded set in V. Let $N(x, .)$ is upper semicontinuous for each $x \in V$. If $f : C \to C$ is a fuzzy non-expansive mapping on C, then $F(f_r)$, the set of fixed point of f_r is nonempty, where $f_r(x) = rx + (1-r)f(x);\ r \in (0,\ 1)$.*
Moreover f_s have the same fixed points as of f.

Proof. Since (V, N) is a fuzzy normed space satisfying $(N6)$, then we get inner product space satisfying $(FIP-7), (FIP-8)$ induced by (V, N). Since $N(x, .)$ is upper semicontinuous then induced fuzzy inner product $m(x, x, .) = N(x, .)$ is upper semicontinuous. Now the state follows from Therom 4.2. □

Definition 4.7. *[10] Let $(V,\ m)$ be an l-fuzzy complete inner product space satisfying $(FIP-7)$. The space V is said to have fixed point property(f.p.p.) if for any sectional fuzzy continuous function $f : V \to V$, there exist x_0 such that $f(x_0) = x_0$.*

Theorem 4.8. *[10] Let K be an l-fuzzy closed , convex set in a real l-fuzzy complete inner product space satisfying $(FIP-7), (FIP-8)$. Then K is fuzzy bounded if K has f.p.p. for any fuzzy non-expansive mapping.*

Theorem 4.9. *Let K be an l-fuzzy closed, convex set in a real l-fuzzy complete linear normed space satisfying $(N6)$. Then K is fuzzy bounded if K has f.p.p. for any fuzzy non-expansive mapping.*

Proof. Since the linear normed space is complete and satisfying $N6$, then, by Theorem 3.1, we get complete inner product satisfying $(FIP-7), (FIP-8)$. K be an l-fuzzy closed, convex set in this induced fuzzy normed space. Now the state follows from Theorem 4.8. □

We rewrite Theorem 4.3 as follows. Proof is similar to the previous.

Theorem 4.10 *(D. de Figueiredo). Let $(V,\ N)$ be an l-fuzzy complete linear normed space satisfying $(N6)$ and C be an l-fuzzy closed, convex and fuzzy bounded subset of V containing 0. If $f : C \to C$ is a fuzzy non-expansive mapping, then for any x_0 in C the sequence $\{x_n\}$, with*

$$x_n = f_n^{n^2} x_{n-1}, n = 1, 2, 3, ...$$

and $f_n x = \frac{n}{n+1} f x$ there exists $\alpha_0 \in (0,\ 1)$ and $x' \in F(f)$(set of fixed point of f) such that for any $r > 0, \exists M(r)$ such that

$$N(x_n - x',\ r) \geqslant \alpha_0 \quad \forall n \geqslant M(r).$$

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