Journal of Mathematical Extension
Vol. 11, No. 3, (2017), 1-18
ISSN: 1735-8299
URL: http://www.ijmex.com

Equitable Coloring of Mycielskian of Some Graphs

J. Vernold Vivin*
University College of Engineering Nagercoil
(Anna University Constituent College)

K. Kaliraj
Ramanujan Institute for Advanced Study in Mathematics
University of Madras

Abstract. In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G) + 1$. This paper presents exact values of the equitable chromatic number $\chi_=$ for the Mycielski's graph of complete graphs $\mu(K_n)$, the Mycielski's graph of cycles $\mu(C_n)$, the Mycielski's graph of paths $\mu(P_n)$, the Mycielski's graph of Helm graphs $\mu(H_n)$ and the Mycielski's graph of Gear graphs $\mu(G_n)$.

AMS Subject Classification: 05C15; 05C75
Keywords and Phrases: Equitable coloring, Mycielski's graph, Helm graph, Gear graph

1. Introduction

The notion of equitable coloring was introduced by Meyer in 1973. This model of graph coloring has many applications. Everytime when we have to divide a system with binary conflicting relations into equal or almost

Received: July 2016; Accepted: October 2016
*Corresponding author

equal conflict-free subsystems we can model this situation by means of equitable graph coloring. This subject is widely discussed in literature [1, 3, 4, 6, 7, 11, 12, 16, 18].

One motivation for equitable coloring suggested by Meyer [13] concerns scheduling problems. In this application, the vertices of a graph represent a collection of tasks to be performed, and an edge connects two tasks that should not be performed at the same time. A coloring of this graph represents a partition of tasks into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of tasks in each time step, and this balancing is exactly what an equitable coloring achieves. Furmańczyk [7] mentions a specific application of this type of scheduling problem, namely assigning university courses to time slots in a way that spreads the courses evenly among the available time slots and avoids scheduling incompatible pairs of courses at the same time as each other, since then the usage of additional resources (e.g.rooms) is maximal.

The notion of equitable colorability was introduced by Meyer [13]. However, an earlier work of Hajnal and Szemerédi [9] showed that a graph G with maximal degree Δ is equitably k-colorable if $k \geqslant \Delta(G) + 1$. Recently, Kierstead et al. [10] have given an $O(\Delta|V(G)|^2)$-time algorithm for equitable $(\Delta + 1)$-coloring of a graph G. In 1973, Meyer [13] formulated the following conjecture:

Conjecture 1.1. [Equitable Coloring Conjecture (ECC)] For any connected graph G, other than complete graph or odd cycle, $\chi_=(G) \leqslant \Delta(G)$.

This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [11] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [16] considered a broader class of graphs, namely r-partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [17] and planar graphs with maximum degree at least 13 [18]. There are very few papers on the complexity of equitable coloring. First of all, a straightforward reduction from graph coloring to equitable coloring by adding sufficiently many isolated vertices to a graph, proves that it is NP-complete to test whether a graph has an equitable coloring with a given number of colors (greater than two). Secondly, Bodlaender and Fomin [1] showed that equitable coloring can be solved to optimality in polynomial time for trees (previously known due to Chen and Lih [4]) and outerplanar graphs. A polynomial time algorithm is also known for equitable coloring of split graphs [3]. Furmańczyk et al. [8] proved that the problem remains NP-complete for the corona of line graph of cubic graphs (i.e. $L(K_4) \circ K_2$).

2. Preliminaries

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leqslant 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable.* The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and denoted by $\chi_=(G)$.

For any integer $n \geqslant 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex x_0 to each of the $n-1$ vertices $\{x_1, x_2, \ldots, x_{n-1}\}$ of the cycle graph C_{n-1}. (see Figure. 1).

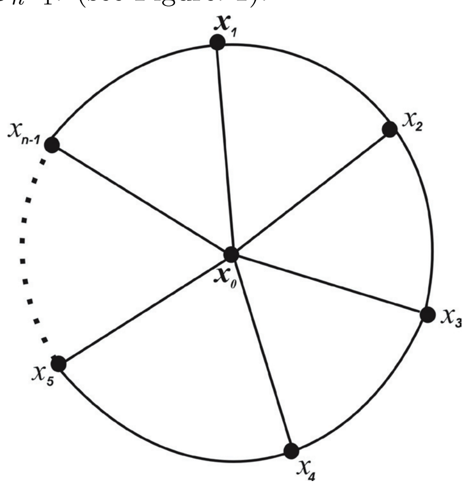

Figure 1. Wheel Graph W_n

The Helm graph H_n is the graph obtained from a wheel graph W_n by adjoining a pendant edge to each vertex of the $(n-1)$-cycle in W_n.(see Figure. 2).

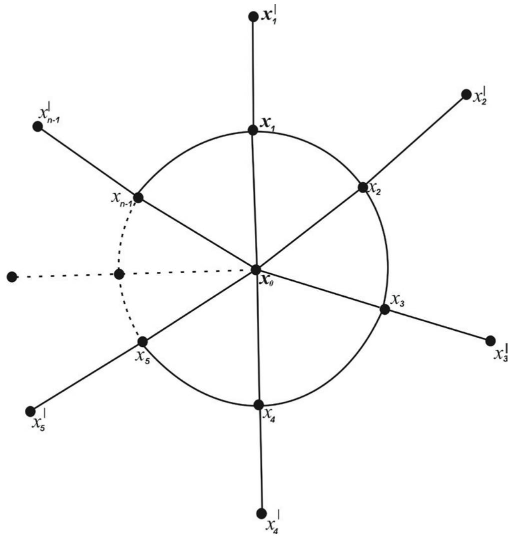

Figure 2. Helm Graph H_n

The Gear graph G_n is obtained from a wheel graph W_n by adding a vertex to each edge of the $(n-1)$-cycle in W_n. (see Figure.3).

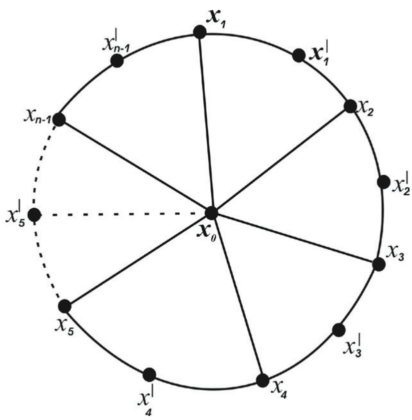

Figure 3. Gear Graph G_n

The open neighborhood of a vertex x in a graph G, denoted by $N_G(x)$, is the set of all vertices of G, which are adjacent to x. Also, $N_G[x] = N_G(x) \cup \{x\}$ is called the closed neighborhood of x in the graph G.

In this paper, by G we mean a connected graph. From a graph G, by Mycielski's construction [2, 14, 15], we get the Mycielskian $\mu(G)$ of G with $V(\mu(G)) = V \cup U \cup \{z\}$, where

$$V = V(G) = \{x_1, \ldots, x_n\}, \quad U = \{y_1, \ldots, y_n\}, \text{ and}$$
$$E(\mu(G)) = E(G) \cup \{y_i x : x \in N_G(x_i) \cup \{z\}, i = 1, \ldots, n\}.$$

The paper is organized as follows. We start the next section with a theorem concerning the equitable coloring of Mycielskian of complete graphs. In the subsequent Sections 4, 5, 6 and 7 results concerning the equitable colorability of Mycielskian of cycles, paths, Helm graphs and Gear graphs are discussed. In this way we establish a new class of graphs that can be colored optimally in polynomial time and confirm the ECC conjecture.

3. Equitable Coloring of Mycielskian of Complete Graphs

Theorem 3.1. *The equitable chromatic number of Mycielskian of complete graph* $\chi_=(\mu(K_n)) = n + 1$.

Proof. Let $V(K_n) = \{x_i : 1 \leqslant i \leqslant n\}$. By Mycielski's construction,

$$\begin{aligned} V(\mu(K_n)) &= V(K_n) \cup \{y_i : 1 \leqslant i \leqslant n\} \cup \{z\} \text{ and} \\ E(\mu(K_n)) &= E(K_n) \cup \{y_i x : x \in N_{K_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \end{aligned}$$

Since z is adjacent with each vertex of $\{y_i : 1 \leqslant i \leqslant n\}$, in which each vertex is adjacent with z. Also $\mu(K_n)$ contains a n-clique, $\chi(\mu(K_n)) \geqslant n$ and hence $\chi_=(\mu(K_n)) \geqslant n$.

Assume that $\chi_=(\mu(K_n)) = n$. (i.e), There exists a partition $P = \{V_1, V_2, \ldots, V_n\}$, since $|V(\mu(K_n))| = 2n+1$, each $V_i\,(1 \leqslant i \leqslant n)$ contains either 2 or 3 vertices. Since each x_i is adjacent with each $y_j\,(1 \leqslant j \leqslant n)$ where $i \neq j$, $x_i, y_i \in V_i\,(1 \leqslant i \leqslant n)$. Since z is adjacent to each $y_i (1 \leqslant i \leqslant n)$, $z \notin V_i$ for every $i, (1 \leqslant i \leqslant n)$

$$\begin{aligned} \text{i.e.,} |V_i| &= 2 \text{ for all } 1 \leqslant i \leqslant n \\ \Rightarrow \sum_{i=1}^{n} |V_i| &= 2n \neq 2n+1 \end{aligned}$$

This contradiction shows that $\chi_{=}(\mu(K_n)) \geqslant n+1$. Now we partition the vertex set $V(\mu(K_n))$ as follows,

$$\begin{aligned} V_i &= \{x_i, y_i : 1 \leqslant i \leqslant n\}, \\ V_{n+1} &= \{z\}. \end{aligned}$$

Clearly $V_i\,(1 \leqslant i \leqslant n)$ and V_{n+1} are independent sets, since $|V_i| = 2(1 \leqslant i \leqslant n)$ and $|V_{n+1}| = 1$ satisfying the condition $||V_i| - |V_j|| \leqslant 1$, for any $i \neq j$, $\chi_{=}(\mu(K_n)) \leqslant n+1$. Hence $\chi_{=}(\mu(K_n)) = n+1$. □

4. Equitable Coloring of Mycielskian of Cycles

Theorem 4.1. *The equitable chromatic number of Mycielskian of cycle*

$$\chi_{=}(\mu(C_n)) = \begin{cases} 3 & n = 4, 6, 8 \\ 4 & \text{if } n \geqslant 10,\ n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $V(C_n) = \{x_i : 1 \leqslant i \leqslant n\}$ be the set of vertices of C_n taken in cyclic order. By Mycielski's construction,

$$\begin{aligned} V(\mu(C_n)) &= V(C_n) \cup \{y_i : 1 \leqslant i \leqslant n\} \cup \{z\} \text{ and} \\ E(\mu(C_n)) &= E(C_n) \cup \{y_i x : x \in N_{C_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \end{aligned}$$

Now we partition the vertex set of $V(\mu(C_n))$ as follows:
Case (i)a For $n = 4$:

$$\begin{aligned} V_1 &= \{x_1, x_3, z\}, \\ V_2 &= \{x_2, x_4, y_4\}, \\ V_3 &= \{y_1, y_2, y_3\}. \end{aligned}$$

Case (i)b For $n = 6$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, z\}, \\ V_2 &= \{x_2, x_4, x_6, y_4, y_6\}, \\ V_3 &= \{y_1, y_2, y_3, y_5\}. \end{aligned}$$

Case (i)c For $n = 8$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, x_7, z\}, \\ V_2 &= \{x_2, x_4, x_6, x_8, y_4, y_6\}, \\ V_3 &= \{y_1, y_2, y_3, y_5, y_7, y_8\}. \end{aligned}$$

It is clear that $V_i \cap V_j = \emptyset$ for $i \neq j$. Since there exists a 5-cycle $x_1x_2y_3zy_2x_1$, $\chi(\mu(C_n)) \geqslant 3$ and hence $\chi_=(\mu(C_n)) = 3$.

Case (ii) If $n \geqslant 10$, n is even and n is odd.

Since there exists a 5-cycle as shown in the previous case, it is clear that $\chi_=(\mu(C_n)) \geqslant 3$.

We know that $|V(\mu(C_n))| = 2n+1$. Out of these $2n+1$, the vertices of $\left\{x_{2i} : 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\right\} \cup \{z\}$ can be assigned with one color. i.e., $\left\lfloor \dfrac{n}{2} \right\rfloor + 1$ vertices can be colored with one color. Hence remaining number of vertices to be colored is $(2n+1) - \left(\left\lfloor \dfrac{n}{2} \right\rfloor + 1\right) = \left\lfloor \dfrac{3n}{2} \right\rfloor$.

If $\chi_=(\mu(C_n)) = 3$, these $\left\lfloor \dfrac{3n}{2} \right\rfloor$ vertices should be divided into two sets, each containing $\left\lfloor \dfrac{3n}{4} \right\rfloor$ and $\left\lfloor \dfrac{3n}{4} \right\rfloor + 1$ vertices.

Since $\left\lfloor \dfrac{3n}{4} \right\rfloor \neq \left\lfloor \dfrac{n}{2} \right\rfloor + 1$ and $\left\lfloor \dfrac{3n}{4} \right\rfloor + 1 \neq \left\lfloor \dfrac{n}{2} \right\rfloor + 1$ for even $n \geqslant 10$ and odd n, we conclude that the partition of $V(\mu(C_n))$ into three sets satisfying $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$ is not possible and hence $\chi_=(\mu(C_n)) \geqslant 4$, if $n \geqslant 10$ and n is odd.

Now the following partition gives the equitable coloring of $\mu(C_n)$ for even $n \geqslant 10$ and odd n.

If $n \geqslant 10$, n is even:

$$\begin{aligned} V_1 &= \left\{x_{2i-1} : 1 \leqslant i \leqslant \frac{n}{2}\right\} \cup \{z\}, \\ V_2 &= \left\{x_{2i} : 1 \leqslant i \leqslant \frac{n}{2}\right\}, \\ V_3 &= \left\{y_{2i-1} : 1 \leqslant i \leqslant \frac{n}{2}\right\}, \\ V_4 &= \left\{y_{2i} : 1 \leqslant i \leqslant \frac{n}{2}\right\}. \end{aligned}$$

If n is odd:

$$\begin{aligned} V_1 &= \left\{x_{2i-1} : 1 \leqslant i \leqslant \frac{n-1}{2}\right\} \cup \{z\}, \\ V_2 &= \left\{x_{2i} : 1 \leqslant i \leqslant \frac{n-1}{2}\right\} \cup \{y_{n-1}\}, \\ V_3 &= \left\{y_{2i-1} : 2 \leqslant i \leqslant \frac{n+1}{2}\right\} \cup \{x_n\}, \\ V_4 &= \left\{y_{2i} : 1 \leqslant i \leqslant \frac{n-3}{2}\right\} \cup \{y_1\}. \end{aligned}$$

This implies $\chi_{=}(\mu(C_n)) = 4$. □

5. Equitable Coloring of Mycielskian of Paths

Theorem 5.1. *The equitable chromatic number of Mycielskian of path*

$$\chi_{=}(\mu(P_n)) = \begin{cases} 2 & n = 1, \\ 3 & \text{if } n \leqslant 11,\ n \neq 10, \\ 4 & \text{if } n \geqslant 10,\ n \neq 11. \end{cases}$$

Proof. Let $V(P_n) = \{x_i : 1 \leqslant i \leqslant n\}$ be the set of vertices of P_n. By Mycielski's construction,

$$\begin{aligned} V(\mu(P_n)) &= V(P_n) \cup \{y_i : 1 \leqslant i \leqslant n\} \cup \{z\} \text{ and} \\ E(\mu(P_n)) &= E(P_n) \cup \{y_i x : x \in N_{P_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \end{aligned}$$

Now we partition the vertex set of $V(\mu(P_n))$ as follows:

Case (i) For $n = 1$

$$\begin{aligned} V_1 &= \{x_1, z\}, \\ V_2 &= \{y_1\}. \end{aligned}$$

It is obvious.

Case (ii) By this partitions $\chi_=(\mu(P_n)) = 3$

1. For $n = 2$:

$$\begin{aligned} V_1 &= \{x_1, z\}, \\ V_2 &= \{x_2, y_2\}, \\ V_3 &= \{y_1\}. \end{aligned}$$

2. For $n = 3$:

$$\begin{aligned} V_1 &= \{x_1, x_3, z\}, \\ V_2 &= \{x_2, y_2\}, \\ V_3 &= \{y_1, y_3\}. \end{aligned}$$

3. For $n = 4$:

$$\begin{aligned} V_1 &= \{x_1, x_3, z\}, \\ V_2 &= \{x_2, x_4, y_2\}, \\ V_3 &= \{y_1, y_3, y_4\}. \end{aligned}$$

4. For $n = 5$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, z\}, \\ V_2 &= \{x_2, x_4, y_2, y_4\}, \\ V_3 &= \{y_1, y_3, y_5\}. \end{aligned}$$

5. For $n = 6$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, z\}, \\ V_2 &= \{x_2, x_4, x_6, y_2, y_4\}, \\ V_3 &= \{y_1, y_3, y_5, y_6\}. \end{aligned}$$

6. For $n = 7$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, x_7, z\}, \\ V_2 &= \{x_2, x_4, x_6, y_2, y_4\}, \\ V_3 &= \{y_1, y_3, y_5, y_6, y_7\}. \end{aligned}$$

7. For $n = 8$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, x_7, z\}, \\ V_2 &= \{x_2, x_4, x_6, x_8, y_2, y_4\}, \\ V_3 &= \{y_1, y_3, y_5, y_6, y_7, y_8\}. \end{aligned}$$

8. For $n = 9$:

$$\begin{aligned} V_1 &= \{x_1, x_3, x_5, x_7, x_9, z\}, \\ V_2 &= \{x_2, x_4, x_6, x_8, y_2, y_4\}, \\ V_3 &= \{y_1, y_3, y_5, y_6, y_7, y_8, y_9\}. \end{aligned}$$

9. For $n = 11$:

$$\begin{aligned} V_1 &= \left\{x_{2i-1} : 1 \leqslant i \leqslant \left\lceil \frac{n}{2} \right\rceil\right\} \cup \{z\}, \\ V_2 &= \left\{x_{2i} : 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor\right\} \cup \{y_2, y_4, y_6\}, \\ V_3 &= \{y_i : 7 \leqslant i \leqslant n\} \cup \{y_1, y_3, y_5\}. \end{aligned}$$

It is clear that $V_i \cap V_j = \emptyset$ for $i \neq j$. Since there exists a 5-cycle $x_1x_2y_3zy_2x_1$, $\chi(\mu(P_n)) \geqslant 3$ and hence $\chi_=(\mu(P_n)) = 3$.

Case (iii) By this partition $\chi_=(\mu(P_n)) = 4$ for $n \geqslant 10$, $n \neq 11$.

Since there exists a 5-cycle $x_1x_2y_3zy_2x_1$, it is clear that $\chi_=(\mu(P_n)) \geqslant 3$. We know that $|V(\mu(P_n))| = 2n+1$. Out of these $2n+1$, the vertices of $\left\{x_{2i} : 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\right\} \cup \{z\}$ can be assigned with one color. i.e., $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices can be colored with one color. Hence remaining number of vertices to be colored is $(2n+1) - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left\lfloor \frac{3n}{2} \right\rfloor$.

If $\chi_{=}(\mu(P_n)) = 3$, these $\left\lfloor \frac{3n}{2} \right\rfloor$ vertices should be divided into two sets, each containing $\left\lfloor \frac{3n}{4} \right\rfloor$ and $\left\lfloor \frac{3n}{4} \right\rfloor + 1$ vertices.

Since $\left\lfloor \frac{3n}{4} \right\rfloor \neq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $\left\lfloor \frac{3n}{4} \right\rfloor + 1 \neq \left\lfloor \frac{n}{2} \right\rfloor + 1$ for $n \geqslant 10$, $n \neq 11$, we conclude that the partition of $V\left(\mu\left(P_n\right)\right)$ into three sets satisfying $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$ is not possible and hence $\chi_{=}(\mu(P_n)) \geqslant 4$, for $n \geqslant 10$, $n \neq 11$.

Now the following partition gives the equitable coloring of $\mu\left(P_n\right)$ for $n \geqslant 10$, $n \neq 11$.

$$\begin{aligned}
V_1 &= \left\{x_{2i} : 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor\right\} \cup \{z\}, \\
V_2 &= \left\{x_{2i-1} : 1 \leqslant i \leqslant \left\lceil \frac{n}{2} \right\rceil\right\}, \\
V_3 &= \left\{y_{2i-1} : 1 \leqslant i \leqslant \left\lceil \frac{n}{2} \right\rceil\right\}, \\
V_4 &= \left\{y_{2i} : 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor\right\}.
\end{aligned}$$

This implies $\chi_{=}\left(\mu\left(P_n\right)\right) = 4$. □

6. Equitable Coloring of Mycielskian of Helm Graphs

Theorem 6.1. *The equitable chromatic number of Mycielski's graph of Helm graph*

$$\chi_{=}(\mu(H_n)) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let

$$\begin{aligned}
V\left(H_n\right) &= \{x_0\} \cup \{x_i : 1 \leqslant i \leqslant n\} \cup \left\{x_i' : 1 \leqslant i \leqslant n\right\} \\
\text{and } E\left(H_n\right) &= \{x_0 x_i : 1 \leqslant i \leqslant n\} \cup \{x_i x_{i+1} : 1 \leqslant i \leqslant n-1\} \\
&\quad \cup \{x_n x_1\} \cup \left\{x_i x_i' : 1 \leqslant i \leqslant n\right\}.
\end{aligned}$$

By Mycielski's construction,

$$V\left(\mu\left(H_n\right)\right) \;=\; V\left(H_n\right) \cup \{y_i : 0 \leqslant i \leqslant n\} \cup \left\{y_i' : 1 \leqslant i \leqslant n\right\} \cup \{z\}.$$

In $\mu\left(H_n\right)$, y_0 is adjacent with each of $\{x_i : 1 \leqslant i \leqslant n\}$, each y_i is adjacent with the neighbour of $x_i\,(0 \leqslant i \leqslant n)$, z is adjacent with each of $\{y_i : 0 \leqslant i \leqslant n\}$ and $\{y_i' : 1 \leqslant i \leqslant n\}$. Also each y_i' is adjacent with $x_i\,(1 \leqslant i \leqslant n)$
Since x_0 and y_0 are non adjacent vertices, we use same color to color x_0 and y_0 let it be 1. Since y_0 is adjacent with each vertex of $\{x_i : 1 \leqslant i \leqslant n\}$, the color 1 cannot be assigned to any vertex $\{x_i : 1 \leqslant i \leqslant n\}$.

Case (i) If n is even:

Since the set of vertices $\{x_i : 1 \leqslant i \leqslant n\}$ induce a cycle of length n in $\mu\left(H_n\right)$, we use two colors 2 and 3 to color all the vertices of this cycle. Also same color cannot be assigned to color the vertices of $\{y_i : 1 \leqslant i \leqslant n\}$, since z is adjacent with each vertex of $\{y_i : 0 \leqslant i \leqslant n\}$, z cannot be assigned with any of the three colors and let it be assigned with color 4. Hence $\chi_=(\mu(H_n)) \geqslant 4$.

We use the following partition to color $\mu\left(H_n\right)$ equitably with 4 given colors.

$$\begin{aligned}
V_1 &= \left\{x_0, y_0, y_i' : 3 \leqslant i \leqslant n\right\},\\
V_2 &= \left\{x_{2i-1}, y_{2i-1} : 1 \leqslant i \leqslant \frac{n}{2}\right\} \cup \left\{y_2'\right\},\\
V_3 &= \left\{x_{2i}, y_{2i} : 1 \leqslant i \leqslant \frac{n}{2}\right\} \cup \left\{y_1'\right\},\\
V_4 &= \left\{z, x_i' : 1 \leqslant i \leqslant n\right\}.
\end{aligned}$$

Clearly $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$.
Hence $\chi_=(\mu(H_n)) \leqslant 4$ and so $\chi_=\left(\mu\left(H_n\right)\right) = 4$.

Case (ii) If n is odd:

Since the vertices $\{x_i : 1 \leqslant i \leqslant n\}$ induce a cycle of odd length n, in $\mu\left(H_n\right)$, we use three colors (2,3 and 4) to color the vertices of this cycle. Now z is colored with a new color 5. Hence $\chi_=(\mu(H_n)) \geqslant 5$.

We use the following partition to color the vertices of $\mu(H_n)$ equitable in the following cases,

Case (ii)a

1. For $n = 3$:

$$\begin{aligned}
V_1 &= \left\{x_0, y_0, x_3'\right\},\\
V_2 &= \left\{x_1, y_1, y_3'\right\},\\
V_3 &= \left\{x_2, y_2, x_1'\right\},\\
V_4 &= \left\{x_3, y_3, y_2'\right\},\\
V_5 &= \left\{x_1', x_2', z\right\}.
\end{aligned}$$

2. For $n = 6k - 3$, $k \geqslant 2$:

$$\begin{aligned}
V_1 &= \{x_0, y_0\} \cup \left\{x_i' : \frac{2n+6}{3} \leqslant i \leqslant n\right\} \cup \left\{y_i' : \frac{n-3}{3} \leqslant i \leqslant n\right\},\\
V_2 &= \left\{x_{3i-2}, y_{3i-2} : 1 \leqslant i \leqslant \frac{n}{3}\right\} \cup \left\{y_{3i}' : 1 \leqslant i \leqslant \frac{n-3}{3}\right\},\\
V_3 &= \left\{x_{3i-1}, y_{3i-1} : 1 \leqslant i \leqslant \frac{n}{3}\right\} \cup \left\{y_{3i-2}' : 1 \leqslant i \leqslant \frac{n-3}{3}\right\},\\
V_4 &= \left\{x_{3i}, y_{3i} : 1 \leqslant i \leqslant \frac{n}{3}\right\} \cup \left\{y_{3i-1}' : 1 \leqslant i \leqslant \frac{n-3}{3}\right\},\\
V_5 &= \left\{x_i' : 1 \leqslant i \leqslant \frac{2n+3}{3}\right\} \cup \{z\}.
\end{aligned}$$

Case (ii)b

1. For $n = 5$:

$$\begin{aligned}
V_1 &= \left\{x_0, y_0, x_4', x_5'\right\},\\
V_2 &= \left\{x_1, y_1, x_4, y_4, y_3'\right\},\\
V_3 &= \left\{x_2, y_2, x_5, y_5, y_1'\right\},\\
V_4 &= \left\{x_3, y_3, y_2', y_4', y_5'\right\},\\
V_5 &= \left\{x_1', x_2', x_3', z\right\}.
\end{aligned}$$

2. For $n = 6k-1$, $k \geqslant 2$:

$$\begin{aligned}
V_1 &= \{x_0, y_0\} \cup \left\{x'_i : \frac{2n+2}{3} \leqslant i \leqslant n\right\} \cup \left\{y'_i : \frac{n+7}{2} \leqslant i \leqslant n\right\},\\
V_2 &= \left\{x_{3i-2}, y_{3i-2} : 1 \leqslant i \leqslant \frac{n+1}{3}\right\} \cup \left\{y'_{3i} : 1 \leqslant i \leqslant \frac{n+1}{6}\right\},\\
V_3 &= \left\{x_{3i-1}, y_{3i-1} : 1 \leqslant i \leqslant \frac{n+1}{3}\right\} \cup \left\{y'_{3i-2} : 1 \leqslant i \leqslant \frac{n+1}{6}\right\},\\
V_4 &= \left\{x_{3i}, y_{3i} : 1 \leqslant i \leqslant \frac{n-2}{3}\right\} \cup \left\{y'_{3i-1} : 1 \leqslant i \leqslant \frac{n+1}{6}\right\},\\
V_5 &= \left\{x'_i : 1 \leqslant i \leqslant \frac{2n-1}{3}\right\} \cup \{z\}.
\end{aligned}$$

Case (ii)c

1. For $n = 7$:

$$\begin{aligned}
V_1 &= \{x_0, y_0, x'_6, x'_7, y'_1, y'_4\},\\
V_2 &= \{x_1, y_1, x_4, y_4, y'_3, y'_6, y'_7\},\\
V_3 &= \{x_2, y_2, x_5, y_5, x_7, y_7\},\\
V_4 &= \{x_3, y_3, x_6, y_6, y'_2, y'_5\},\\
V_5 &= \{x'_1, x'_2, x'_3, x'_4, x'_5, z\}.
\end{aligned}$$

2. For $n = 6k+1$, $k \geqslant 2$:

$$\begin{aligned}
V_1 &= \{x_0, y_0\} \cup \left\{x'_i : \frac{5n+1}{6} \leqslant i \leqslant n\right\} \cup \left\{y'_i : \frac{n+11}{3} \leqslant i \leqslant n\right\},\\
V_2 &= \left\{x_{3i-2}, y_{3i-2} : 1 \leqslant i \leqslant \frac{n+2}{3}\right\} \cup \left\{y'_{3i+2} : 1 \leqslant i \leqslant \frac{n-7}{6}\right\},\\
V_3 &= \left\{x_{3i-1}, y_{3i-1} : 1 \leqslant i \leqslant \frac{n+2}{3}\right\} \cup \left\{y'_{3i+3} : 1 \leqslant i \leqslant \frac{n-7}{6}\right\},\\
&\cup \{y'_1, y'_3\}\\
V_4 &= \left\{x_{3i}, y_{3i} : 1 \leqslant i \leqslant \frac{n-1}{3}\right\} \cup \left\{y'_{3i+4} : 1 \leqslant i \leqslant \frac{n-7}{6}\right\},\\
&\cup \{y'_2, y'_4\}\\
V_5 &= \left\{x'_i : 1 \leqslant i \leqslant \frac{5n-5}{6}\right\} \cup \{z\}.
\end{aligned}$$

Clearly in all the cases $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$.
Hence $\chi_=(\mu(H_n)) \leqslant 5$ and so $\chi_= (\mu(H_n)) = 5$ for n is odd. $\square$

7. Equitable Coloring of Mycielskian of Gear Graphs

Theorem 7.1. *The equitable chromatic number of Mycielski's graph of Gear graph*

$$\chi_=(\mu(G_n)) = \begin{cases} 3 & \textit{if } 3 \leqslant n \leqslant 5 \\ 4 & \textit{if } n \geqslant 6 \end{cases}$$

Proof. Let

$$\begin{aligned} V(G_n) &= \{x_0\} \cup \{x_i : 1 \leqslant i \leqslant n\} \cup \left\{x'_i : 1 \leqslant i \leqslant n\right\} \\ \text{and } E(G_n) &= \{x_0 x_i : 1 \leqslant i \leqslant n\} \cup \left\{x_i x'_i : 1 \leqslant i \leqslant n\right\} \\ &\cup \left\{x'_i x_{i+1} : 1 \leqslant i \leqslant n-1\right\} \cup \left\{x'_n x_1\right\}. \end{aligned}$$

By Mycielski's construction,

$$V(\mu(G_n)) = V(G_n) \cup \{y_i : 1 \leqslant i \leqslant n\} \cup \left\{y'_i : 1 \leqslant i \leqslant n\right\} \cup \{z\}.$$

In $\mu(H_n)$, y_0 is adjacent with each vertex of $\{x_i : 1 \leqslant i \leqslant n\}$, each y_i is adjacent with the neighbour of $x_i\,(1 \leqslant i \leqslant n)$ and each y'_i is adjacent with the neighbours of $x'_i\,(1 \leqslant i \leqslant n)$, z is adjacent with each vertex of $\{y_i : 1 \leqslant i \leqslant n\} \cup \{y'_i : 1 \leqslant i \leqslant n\} \cup \{y_0\}$.

Case (i) For $n = 3, 4, 5$:

Since the set of vertices $\{x_i : 1 \leqslant i \leqslant n\} \cup \{x'_i : 1 \leqslant i \leqslant n\}$ induce an even cycle in $\mu(G_n)$, it requires atleast two colors to color the vertices of this cycle (say 1 and 2). If we assign the same color 1 and 2 to color the vertices of $\{y_i : 1 \leqslant i \leqslant n\} \cup \{y'_i : 1 \leqslant i \leqslant n\}$ then z will receive a new color other than 1 and 2. Hence $\chi_=(\mu(G_n)) \geqslant 3$.

1. For $n = 3$:

$$\begin{aligned} V_1 &= \left\{x_0, x_1', x_2', x_3', z\right\}, \\ V_2 &= \{x_1, x_2, x_3, y_1, y_3\}, \\ V_3 &= \left\{y_0, y_2, y_1', y_2', y_3'\right\}. \end{aligned}$$

2. For $n = 4$:

$$\begin{aligned} V_1 &= \left\{x_0, x_1', x_2', x_3', x_4', z\right\}, \\ V_2 &= \{x_1, x_2, x_3, x_4, y_1, y_2\}, \\ V_3 &= \left\{y_0, y_3, y_4, y_1', y_2', y_3', y_4'\right\}. \end{aligned}$$

3. For $n = 5$:

$$\begin{aligned} V_1 &= \left\{x_0, x_1', x_2', x_3', x_4', x_5', z\right\}, \\ V_2 &= \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3\}, \\ V_3 &= \left\{y_0, y_4, y_5, y_1', y_2', y_3', y_4', y_5'\right\}. \end{aligned}$$

Clearly $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$.

Hence $\chi_=(\mu(G_n)) \leqslant 3$ and hence $\chi_= (\mu (G_n)) = 3$.

Case (ii) For $n \geqslant 6$:

We know that $|V(\mu(G_n))| = 4n + 3$. Out of these $4n + 3$, the vertices of $\{x_i' : 1 \leqslant i \leqslant n\} \cup \{x_0\} \cup \{z\}$ can be assigned with one color. i.e., $n + 2$ vertices can be colored with one color. Hence remaining number of vertices to be colored is $(4n + 3) - (n + 2) = 3n + 1$.

If $\chi_=(\mu(G_n)) = 3$, these $3n + 1$ vertices should be divided into two sets, each containing $\frac{3n}{2}$ and $\frac{3n+1}{2}$ vertices.

Since $\frac{3n}{2} \neq n + 2$ and $\frac{3n+1}{2} \neq n + 2$ for $n \geqslant 6$, we conclude that the partition of $V(\mu(G_n))$ into three sets satisfying $||V_i| - |V_j|| \leqslant 1$ for

$i \neq j$ is not possible and hence $\chi_=(\mu(G_n)) \geqslant 4$.

$$\begin{aligned} V_1 &= \left\{x_i' : 1 \leqslant i \leqslant n\right\} \cup \{x_0\}, \\ V_2 &= \{x_i : 1 \leqslant i \leqslant n\} \cup \{z\}, \\ V_3 &= \left\{y_i' : 1 \leqslant i \leqslant n\right\} \cup \{y_0\}, \\ V_4 &= \{y_i : 1 \leqslant i \leqslant n\}. \end{aligned}$$

Clearly $||V_i| - |V_j|| \leqslant 1$ for $i \neq j$.
Hence $\chi_=(\mu(G_n)) \leqslant 4$ and so $\chi_= (\mu (G_n)) = 4$. □

References

[1] H. L. Bodleander and F. V. Fomin, Equitable colorings of bounded treewidth graphs, *Theoret. Comput. Sci.*, 349(1)(2005), 22–30.

[2] G. J. Chang, L. Huang and X. Zhu, Circular chromatic numbers of Mycielski's graphs, *Discrete Math.*, 205(1–3)(1999), 23–37.

[3] B. L. Chen, M. T. Ko and K. W. Lih, Equitable and m-bounded coloring of split graphs, in: *Combinatorics and Computer Science (Brest, 1995)* LCNS 1120, Springer (1996), 1–5.

[4] B. L. Chen and K.W. Lih, Equitable coloring of trees, *J. Combin. Theory Ser. B*, 61(1994), 83–87.

[5] B. L. Chen, K.W. Lih and P.L. Wu, Equitable coloring and the maximum degree, *European J. Combin.* 15(5)(1994), 443–447.

[6] H. Furmańczyk, *Equitable coloring of graphs*, in: Graph Colorings, (M. Kubale, ed.) American Mathematical Society Providence, Rhode Island (2004).

[7] H. Furmańczyk, Equitable coloring of graph products, *Opuscula Math.*, 26(1)(2006), 31–44.

[8] H. Furmańczyk, K. Kaliraj, M. Kubale and V. J. Vernold, Equitable coloring of corona products of graphs, *Adv. Appl. Discrete Math.*, 11(2)(2013), 103–120.

[9] A. Hajnal and E. Szemerédi, *Proof of a conjecture of Erdös*, in: Combinatorial Theory and Its Applications, II 601–623, Colloq. Math. Soc. János Bolyai, Vol. 4, North-Holland, Amsterdam, (1970).

[10] H.A. Kiestead, A.V. Kostochka, M. Mydlarz and E.Szemerédi, A fast algorithm for equitable coloring, *Combinatoria* 30(2) (2010), 217–224.

[11] K. W. Lih and P. L. Wu, On equitable coloring of bipartite graphs, *Discrete Math.*, 151(1–3)(1996), 155–160.

[12] W. H. Lin and G. J. Chang, Equitable colorings of Cartesian products of graphs, *Discrete Appl. Math.*, 160 (2012), 239–247.

[13] W. Meyer, Equitable coloring, *Amer. Math. Monthly*, 80(1973), 920–922.

[14] J. Miškuf, R. Škrekovski, and M. Tancer, Backbone Colorings and generalized Mycielski Graphs, *SIAM J. Discrete Math.*, 23(2)(2009), 1063–1070.

[15] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.*, 3(1955), 161–162.

[16] W. Wang and K. Zhang, Equitable colorings of line graphs and complete r-partite graphs, *Systems Sci. Math. Sci.*, 13(2)(2000), 190–194.

[17] H.P. Yap and Y. Zhang, The Equitable Δ-Coloring Conjecture holds for outerplanar graphs, *Bull. Inst. Math. Acad. Sinica*, 25(2) (1997), 143–149.

[18] H. P. Yap and Y. Zhang, Equitable colourings of planar graphs, *J. Combin. Math. Combin. Comput.*, 27(1998), 97–105.

Joseph Vernold Vivin
Department of Mathematics
Assistant Professor of Mathematics
University College of Engineering Nagercoil
(Anna University Constituent College)
Nagercoil 629 004, Tamil Nadu, India
E-mail: vernoldvivin@yahoo.in

Kalimuthu Kaliraj
Department of Mathematics
Assistant Professor of Mathematics
Ramanujan Institute for Advanced Study in Mathematics,
University of Madras
Chepauk, Chennai 600 005, Tamil Nadu, India
E-mail: sk.kaliraj@gmail.com