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Some Fixed Point Theorems for C^*-Algebra-Valued b_2-Metric Spaces

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Abstract. The aim of this paper is to establish the structure of C^*-algebra-valued b_2-metric space and give some fixed point theorems for self-maps with contractive or expansive conditions on such spaces. As an application we investigate existence and uniqueness solution for a type of integral equation.

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1. Introduction

In recent times, one of the most attractive research topics in fixed point theory is to study these results in the setting of partially ordered metric spaces. After the appearance of the first works in this sense (cf. e.g. [7, 11, 14, 17] and the references therein), the literature on this topic has expanded significantly. The notion of C^*-algebra-valued metric spaces has been investigated by Ma et al., in [8]. They presented some fixed point theorems for self-maps with contractive or expansive conditions on such spaces. On the other hand in 1993, Czerwik [6] studied b-metric spaces as a generalization of metric spaces and

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proved the contraction mapping principle in these spaces that is an extension of the famous Banach contraction principle in metric spaces. After that series of articles about fixed point theorems in b-metric spaces started to appear (cf. e.g. [1, 2, 4, 5, 12, 13, 15, 16]). Very recently, the authors [10] proved some fixed point theorems by introducing the notion of b_2-metric spaces. Using the concepts of b_2-metric spaces and C^*-algebra-valued metric spaces, we define a new type of extended metric spaces. Then, we prove some fixed point theorems in this structure. As an application we investigate existence and uniqueness solution for a type of integral equation.
We provide some notations, definitions and auxiliary facts which will be used later in this paper.
Let $\mathbb{A}$ be a unital algebra with unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathbb{A}$, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. An assign to each $*$-algebra is $(\mathbb{A}, *)$. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ $(a \in \mathbb{A})$. For more details we refer the reader to [9].
Throughout this manuscript, $\mathbb{A}$ stands for a unital C^*-algebra with unit I. We say an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq 0$, if $x = x^*$ and $\sigma(x) \subseteq \mathbb{R}_+ = [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ as follows: $x \preceq y$ if and only if $y - x \succeq 0$ $(x, y \in \mathbb{A})$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq 0\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Remark 1.1. *When $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$, $x \preceq I$ if and only if $\|x\| \leqslant 1$ ([9]).*

Definition 1.2. ([8]) *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*
1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2)$d(x, y) = d(y, x)$ for all $x, y \in X$;
3)$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

Lemma 1.3. ([9]) *Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit I.*
1)If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$;
2)suppose that $a, b \in \mathbb{A}$ with $a, b \succeq 0$ and $ab = ba$, then $ab \succeq 0$;
3)by $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba,$ for all $b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0$ and $I - a \in \mathbb{A}'_+$is an invertible element, then $(I - a)^{-1}b \succeq (I - a)^{-1}c$.

Lemma 1.4. ([9]) *Let $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and*

x^*bx are positive elements and $x^*ax \preceq x^*bx$.

Definition 1.5. ([6]) *Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is a b-metric on X, if for all $x, y, z \in X$, the following conditions hold:*
(b_1) $d(x, y) = 0$if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.
In this case, the pair (X, d) is called a b-metric space.

Definition 1.6. ([10])*Let X be a nonempty set, $s \geqslant 1$ be a real number and let $d : X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:*
1)*for every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;*
2)*if at least two of three points x, y, z are the same, then $d(x, y, z) = 0$;*
3)*the symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$;*
4) *the rectangle inequality: $d(x, y, z) \leqslant s[d(x, y, t) + d(y, z, t) + d(z, x, t)]$ for all $x, y, z, t \in X$.*
Then d is called a b_2-metric on X and (X, d) is called a b_2-metric space with parameter s.

Now we introduce a new type of metric spaces as an extension of both concepts b_2-metric spaces and C^*-algebra-valued metric spaces.

2. Main Results

Definition 2.1. *Let X be a nonempty set, $s \geqslant 1$ be a real number, $\mathbb{A}$ be a C^*-algebra and $d : X \times X \times X \to \mathbb{A}$ be a map satisfying the following conditions:*
(bM1) for every pair of distinct elements $x, y \in X$, there exists $z \in X$ such that $d(x, y, z) \neq 0$;
(bM2) if at least two of three elements x, y, z are the same, then $d(x, y, z) = 0$;
(bM3) the symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$;
(bM4) the rectangle inequality: $d(x, y, z) \preceq s[d(t, x, y) + d(t, y, z) + d(t, x, z)]$ for all $x, y, z, t \in X$.
Then d is called a C^*-algebra-valued b_2-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued b_2-metric space with parameter s.

Remark 2.2. *Using condition (bM1) it readily verified that if for all $a \in X$, $d(x, y, a) = 0$, then $x = y$.*

Example 2.3. Let X be a set with the cardinal $card(X) \geqslant 4$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geqslant 3$ and $\mathbb{A}$ is a unital C^*-algebra. Let $s > 1$ be an arbitrary element in $\mathbb{R}$ and $0 \preceq A \preceq \frac{3}{2}I$. It is easy to verify that $d : X \times X \times X \to \mathbb{A}$ defined by

$$\mathrm{d}(x, y, z) = \begin{cases} 0, & \text{if at least two of three elements } x, y, z \text{ are the same,} \\ sA, & x, y, z \in X_1, \\ I, & \text{otherewise,} \end{cases}$$

is a C^*-algebra-valued b_2-metric on X.

Definition 2.4. *Let $\{x_n\}$ be a sequence in a C^*-algebra-valued b_2-metric space $(X, \mathbb{A}, d)$.*
1. $\{x_n\}$ is said to be a b_2-convergent to $x \in X$ with respect to $\mathbb{A}$, denoted by $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} \|d(x_n, x, a)\| = 0$.
2. $\{x_n\}$ is said to be a b_2-Cauchy sequence with respect to $\mathbb{A}$ in X, if for all $a \in X$, $\lim_{n,m\to\infty} \|d(x_n, x_m, a)\| = 0$.
3. $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b_2-metric space if every b_2-Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 2.5. Consider the set $X = C_{\mathbb{R}}(E)$ of all real valued continuous functions on E and $H = L^2(E)$, where E is a nonempty Lebesgue measurable compact set in $\mathbb{R}_+$. By $L(H)$ we denote the set of bounded linear operators on Hilbert space H. Clearly $L(H)$ is a C^*-algebra with the usual operator norm. Define $d : X \times X \times X \to L(H)$ by

$$d(f, g, h) = \pi_{\left[\max_{t\in E} \min\left\{|f(t) - g(t)|, |f(t) - h(t)|, |g(t) - h(t)|\right\}\right]^p},$$

where $\pi_\alpha : H \to H$, $(\alpha \in \mathbb{C})$, defined by $\pi_\alpha(\varphi) = \alpha.\varphi$ for each $\varphi \in H$, and $p \geqslant 1$. We know that, $d' : X \times X \times X \to [0, \infty)$ is defined by

$$d'(f, g, h) = \left[\max_{t\in E} \min\left\{|f(t) - g(t)|, |f(t) - h(t)|, |g(t) - h(t)|\right\}\right]^p$$

is a b_2-complete b_2-metric spaces with $s = 3^{p-1}$ [10].
Now, $d(f, g, h) = \pi_{d'(f,g,h)}$, for each f, g, h. Take

$$\alpha = 3^{p-1}[d'(f, g, k) + d'(f, h, k) + d'(g, h, k)] - d'(f, g, h).$$

Rectangle inequality for d' implies that $\alpha \geqslant 0$. Therefore $\langle \pi_\alpha \varphi, \varphi \rangle \geqslant 0$, where $\varphi \in H$ and $\langle ., . \rangle$ is the inner product of H, which means the self-adjoint operator π_α is positive. This shows that (bM4) is satisfied.

Definition 2.6. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b_2-metric space, $T : X \to X$ be a given mapping and $\alpha : X \times X \times X \to \mathbb{R}_+$ be a mapping, we say that T is α-admissible if for all $x, y, z \in X$*

$$\alpha(x, y, z) \geqslant 1 \quad \Rightarrow \alpha(Tx, Ty, z) \geqslant 1.$$

Definition 2.7. *A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies*

$$\alpha(x, y, a) \geqslant 1, \quad \alpha(y, z, a) \geqslant 1 \quad \Rightarrow \quad \alpha(x, z, a) \geqslant 1,$$

where $x, y, z, a \in X$.

The following crucial lemma is useful in proving our main results which state and prove according to Lemma 8 [3].

Lemma 2.8. *Let $T : X \to X$ be a triangular α-admissible mapping. Also there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geqslant 1$, $\alpha(Tx_0, x_0, a) \geqslant 1$ for all $a \in X$. If $x_n = T^n x_0$, then $\alpha(x_m, x_n, a) \geqslant 1$ for all $m, n \in \mathbb{N}$ and for all $a \in X$.*

Definition 2.9. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b_2-metric space, $\alpha : X \times X \times X \to \mathbb{R}_+$ is a mapping and $T : X \to X$ is a triangular α-admissible mapping. Then T is called α-contractive on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that fulfills the following condition:*

$$s\alpha(x, y, a)d(Tx, Ty, a) \preceq A^* d(x, y, a)A,$$

for all $x, y, a \in X$.

Theorem 2.10. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b_2-metric space, $\alpha : X \times X \times X \to \mathbb{R}_+$ be a mapping and $T : X \to X$ be an α-contractive mapping satisfying the following conditions:*
(i) there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, a) \geqslant 1$ and $\alpha(x_0, Tx_0, a) \geqslant 1$,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_{n+1}, x_n, a) \geqslant 1$ and $\alpha(x_n, x_{n+1}, a) \geqslant 1$ as $n \to \infty$, then $\alpha(x_n, x, a) \geqslant 1$.
Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that condition (i) holds. Define the sequence $\{x_n\}$ in X as $x_n = Tx_{n-1} = T^n x_0,\ n = 1, 2, \ldots$. Since T is α-admissible and $\alpha(Tx_0, x_0, a) = \alpha(x_1, x_0, a) \geqslant 1$, then

$$\alpha(Tx_1, Tx_0, a) = \alpha(x_2, x_1, a) \geqslant 1, \tag{1}$$

for all $a \in X$. Repeatedly using (1) for all $a \in X$ we obtain

$$\alpha(x_{n+1}, x_n, a) \geqslant 1, \quad \forall n \in \mathbb{N}. \tag{2}$$

Let a be an arbitrary and fixed element of X. For convenience, by B_a we denote the element $d(x_1, x_0, a)$ in $\mathbb{A}$. We are going to show that $\{x_n\}$ is a b_2-Cauchy sequence with respect to $\mathbb{A}$. For, by the α-contraction of T we get

$$\begin{aligned}
d(x_{n+1}, x_n, a) &\preceq \alpha(x_n, x_{n-1}, a) d(Tx_n, Tx_{n-1}, a) \\
&\preceq \frac{1}{s} A^* d(x_n, x_{n-1}, a) A \\
&\preceq \frac{1}{s} A^* \alpha(x_{n-1}, x_{n-2}, a) d(Tx_{n-1}, Tx_{n-2}, a) A \\
&\preceq \frac{1}{s^2} (A^*)^2 d(x_{n-1}, x_{n-2}, a) A^2 \\
&\vdots \\
&\preceq \frac{1}{s^n} (A^*)^n d(x_1, x_0, a) A^n \\
&= \frac{1}{s^n} (A^*)^n B_a A^n.
\end{aligned}$$

Note that for each $n, m \in \mathbb{N}$, by Lemma 1 we have

$$\begin{aligned}
d(x_{m+1}, x_{n+1}, x_m) &\preceq \alpha(x_m, x_n, x_m) d(Tx_m, Tx_n, x_m) \\
&\preceq \frac{1}{s} A^* d(x_m, x_n, x_m) A = 0.
\end{aligned}$$

Next, let n and m be arbitrary elements in $\mathbb{N}$ with $n+1 > m$. So we have

$$\begin{aligned}
d(x_{n+1}, x_m, a) &\preceq sd(x_{m+1}, x_{n+1}, x_m) + sd(x_{m+1}, x_m, a) + sd(x_{m+1}, x_{n+1}, a) \\
&\preceq 0 + s(\frac{1}{s^m} (A^*)^m B_a A^m) + [s^2 d(x_{m+2}, x_{m+1}, x_{n+1}) \\
&\quad + s^2 d(x_{m+2}, x_{m+1}, a) + s^2 d(x_{m+2}, x_{n+1}, a)] \\
&\preceq s(\frac{1}{s^m} (A^*)^m B_a A^m) + s^2 (\frac{1}{s^{m+1}} (A^*)^{m+1} B_a A^{m+1}) \\
&\quad + s^2 d(x_{m+2}, x_{n+1}, a) \\
&\vdots \\
&\preceq s(\frac{1}{s^m} (A^*)^m B_a A^m) + s^2 (\frac{1}{s^{m+1}} (A^*)^{m+1} B_a A^{m+1}) \\
&\quad + \cdots + s^{n-m} d(x_n, x_{n-1}, a) + s^{n-m} d(x_n, x_{n+1}, a)
\end{aligned}$$

$$
\begin{aligned}
&\preceq \quad s(\frac{1}{s^m}(A^*)^m B_a A^m) + s^2(\frac{1}{s^{m+1}}(A^*)^{m+1} B_a A^{m+1}) \\
&\qquad + \cdots + s^{n-m}(\frac{1}{s^{n-1}}(A^*)^{n-1} B_a A^{n-1}) + s^{n-m}(\frac{1}{s^n}(A^*)^n B_a A^n) \\
&\preceq \quad \frac{1}{s^{m-1}}(A^*)^m B_a A^m + \frac{1}{s^{m-1}}(A^*)^{m+1} B_a A^{m+1} \\
&\qquad + \ldots + \frac{1}{s^{m-1}}(A^*)^{n-1} B_a A^{n-1} + \frac{1}{s^m}(A^*)^n B_a A^n \\
&\preceq \quad \sum_{k=m}^{n} (A^*)^k B_a A^k \\
&= \quad \sum_{k=m}^{n} (A^*)^k B_a^{\frac{1}{2}} B_a^{\frac{1}{2}} A^k \\
&= \quad \sum_{k=m}^{n} (B_a^{\frac{1}{2}} A^k)^* (B_a^{\frac{1}{2}} A^k) \\
&= \quad \sum_{k=m}^{n} |B_a^{\frac{1}{2}} A^k|^2 \\
&\preceq \quad \Big\| \sum_{k=m}^{n} |B_a^{\frac{1}{2}} A^k|^2 \Big\| I \\
&\preceq \quad \sum_{k=m}^{n} \|B_a^{\frac{1}{2}}\|^2 \|A^k\|^2 I \\
&\preceq \quad \|B_a^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|A\|^{2k} I \\
&\preceq \quad \|B_a^{\frac{1}{2}}\|^2 \frac{\|A\|^{2m}}{1-\|A\|^2} I \to 0 \ (as\ m \to \infty).
\end{aligned}
$$

Thus, we have proved that $\{x_n\}$ is a b_2-Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Therefore

$$
\begin{aligned}
0 &\preceq d(Tx, x, a) \preceq sd(Tx_n, Tx, x) + sd(Tx_n, x, a) + s\alpha(x_n, x, a)d(Tx_n, Tx, a) \\
&\preceq sd(x_{n+1}, Tx, x) + sd(x_{n+1}, x, a) + A^* d(x_n, x, a)A \to 0 \ (as\ n \to \infty),
\end{aligned}
$$

hence $Tx = x$, i.e., x is a fixed point of T. $\square$

The usability of this result is demonstrated by the following example.

Example 2.11. Let $X = \{a, b, c, d\}$, $\mathbb{A}$ be an arbitrary C^*-algebra and $A \in \mathbb{A}$ with $\sigma(A) \subseteq (\frac{1}{\sqrt{6}}, 1)$. Define $d : X \times X \times X \to \mathbb{A}$ as follows

$$d(a,b,c)=0,\ d(a,b,d)=4A,\ d(b,c,d)=2A,\ d(a,c,d)=A.$$

Clearly, $(X,\mathbb{A},d)$ is a complete C^*-algebra valued b_2-metric space with $s=\frac{4}{3}$. Consider $T:X\to X$ defined by $Ta=a$, $Tb=c$, $Tc=d$ and $Td=c$, also $\alpha:X\times X\times X\to\mathbb{R}_+$ defined by

$$\alpha(x,y,z)=\begin{cases}1, & x=y=a,\\ \frac{1}{2}, & x\neq y \text{ and } x,y\in\{a,b\},\\ 0, & \text{otherwise.}\end{cases}$$

Then T is an α-contractive mapping and all conditions of Theorem 2.10 are satisfied, so T has a fixed point.

Theorem 2.12. *Let $(X,\mathbb{A},d)$ be a complete C^*-algebra-valued b_2-metric space. Suppose $\alpha:X\times X\times X\to\mathbb{R}_+$ and $T:X\to X$ are two mappings satisfying the following conditions:*
(i) T is triangular α-admissible mapping and there exists $x_0\in X$ such that $\alpha(Tx_0,x_0,a)\geqslant 1$ and $\alpha(x_0,Tx_0,a)\geqslant 1$,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n\to x$, $\alpha(x_{n+1},x_n,a)\geqslant 1$ and $\alpha(x_n,x_{n+1},a)\geqslant 1$ as $n\to\infty$, then $\alpha(x_n,x,a)\geqslant 1$.
(iii) for all $x,y,a\in X$, the following inequality holds:

$$s^2\alpha(x,y,a)d(Tx,Ty,a)\preceq A[d(Tx,y,a)+d(Ty,x,a)], \tag{3}$$

where $A\in\mathbb{A}'_+$ and $\|A\|<\frac{1}{2}$.
Then T has a fixed point in X.

Proof. First note that $A\in\mathbb{A}'_+$, implies that $A[d(Tx,y,a)+d(Ty,x,a)]$ is a positive element of $\mathbb{A}$.
Take $x_0\in X$ and set $x_n=Tx_{n-1}=T^nx_0\quad n=1,2,\ldots$. From condition (i) we get $\alpha(x_{n-1},x_n,a)\geqslant 1$ for all $a\in X$ so we have

$$\begin{aligned}d(Tx_{n-1},Tx_n,Tx_{n-2}) &\preceq \alpha(x_{n-1},x_n,x_{n-1})d(Tx_{n-1},Tx_n,x_{n-1})\\ &\preceq \frac{1}{s^2}A[d(Tx_{n-1},x_n,x_{n-1})+d(Tx_n,x_{n-1},x_{n-1})]\\ &= \frac{1}{s^2}A[d(x_n,x_n,x_{n-1})+d(Tx_n,x_{n-1},x_{n-1})]=0.\end{aligned}$$

Let a be an arbitrary fixed element in X. We denote by B_a the element $d(x_1,x_0,a)$ in $\mathbb{A}$. By (bM4) and using (3) we obtain

$$\begin{aligned}
d(x_{n+1}, x_n, a) &\preceq \alpha(x_n, x_{n-1}, a) d(Tx_n, Tx_{n-1}, a) \\
&\preceq \frac{1}{s^2} A[d(Tx_n, x_{n-1}, a) + d(Tx_{n-1}, x_n, a)] \\
&= \frac{1}{s^2} A[d(Tx_n, Tx_{n-2}, a) + d(x_n, x_n, a)] \\
&\preceq \frac{1}{s^2} A[sd(Tx_{n-1}, Tx_n, Tx_{n-2}) + \\
&\quad sd(Tx_{n-1}, Tx_{n-2}, a) + sd(Tx_{n-1}, Tx_n, a)] \\
&= \frac{1}{s} Ad(x_n, x_{n-1}, a) + \frac{1}{s} Ad(x_n, x_{n+1}, a) \\
&\preceq \frac{1}{s} Ad(x_n, x_{n-1}, a) + Ad(x_{n+1}, x_n, a).
\end{aligned}$$

Thus we have

$$(I - A) d(x_{n+1}, x_n, a) \preceq \frac{1}{s} Ad(x_n, x_{n-1}, a),$$

and using Lemma 1.3 we get

$$\begin{aligned}
d(x_{n+1}, x_n, a) &\preceq \frac{1}{s} A(I - A)^{-1} d(x_n, x_{n-1}, a) \\
&= \frac{1}{s} td(x_n, x_{n-1}, a) \\
&\vdots \\
&\preceq \frac{1}{s^n} t^n d(x_1, x_0, a) = \frac{1}{s^n} t^n B_a,
\end{aligned}$$

where $t = A(I - A)^{-1}$. For $n + 1 \geqslant m$ by induction and Lemma 1, we conclude that

$$\begin{aligned}
d(x_{m+1}, x_{n+1}, x_m) &\preceq \alpha(x_m, x_n, x_m) d(Tx_m, Tx_n, x_m) \\
&\preceq \frac{1}{s^2} A[d(Tx_m, x_n, x_m) + d(Tx_n, x_m, x_m)] \\
&= \frac{1}{s^2} Ad(Tx_m, x_n, x_m) \\
&\preceq \frac{1}{s^2} A\alpha(x_m, x_{n-1}, x_m) d(Tx_m, Tx_{n-1}, x_m) \\
&\preceq \frac{1}{s^4} A^2[d(Tx_m, x_{n-1}, x_m) + d(Tx_{n-1}, x_m, x_m] \\
&\vdots \\
&\preceq \frac{1}{s^{2n-2m}} A^{n-m} d(Tx_m, x_{m+1}, x_m) \\
&= \frac{1}{s^{2n-2m}} A^{n-m} d(Tx_m, Tx_m, x_m) = 0.
\end{aligned}$$

For $n+1 \geqslant m$, we deduce

$$
\begin{aligned}
d(x_{n+1}, x_m, a) &\preceq sd(x_{m+1}, x_{n+1}, x_m) + sd(x_{m+1}, x_m, a) + sd(x_{m+1}, x_{n+1}, a) \\
&\preceq 0 + s(\frac{t^m}{s^m} B_a) + [s^2 d(x_{m+2}, x_{m+1}, x_{n+1}) + \\
&\quad s^2 d(x_{m+2}, x_{m+1}, a) + s^2 d(x_{m+2}, x_{n+1}, a)] \\
&\preceq s(\frac{t^m}{s^m} B_a) + s^2(\frac{t^{m+1}}{s^{m+1}} B_a) + s^2 d(x_{m+2}, x_{n+1}, a) \\
&\vdots \\
&\preceq s(\frac{t^m}{s^m} B_a) + s^2(\frac{t^{m+1}}{s^{m+1}} B_a) + \cdots + \\
&\quad s^{n-m} d(x_{n-1}, x_n, a) + s^{n-m} d(x_n, x_{n+1}, a) \\
&\preceq s(\frac{t^m}{s^m} B_a) + s^2(\frac{t^{m+1}}{s^{m+1}} B_a) + \cdots + s^{n-m}(\frac{t^{n-1}}{s^{n-1}} B_a) + s^{n-m}(\frac{t^n}{s^n} B_a) \\
&\preceq \frac{1}{s^{m-1}}(t^m + t^{m+1} + \cdots + t^n) B_a \\
&\preceq (t^m + t^{m+1} + \cdots + t^n) B_a \\
&= \sum_{k=m}^{n} t^k B_a \\
&= \sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B_a^{\frac{1}{2}} B_a^{\frac{1}{2}} \\
&= \sum_{k=m}^{n} B_a^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B_a^{\frac{1}{2}} \\
&= \sum_{k=m}^{n} (t^{\frac{k}{2}} B_a^{\frac{1}{2}})^* (t^{\frac{k}{2}} B_a^{\frac{1}{2}}) \\
&= \sum_{k=m}^{n} |t^{\frac{k}{2}} B_a^{\frac{1}{2}}|^2 \\
&\preceq \Big\| \sum_{k=m}^{n} |t^{\frac{k}{2}} B_a^{\frac{1}{2}}|^2 \Big\| I \\
&\preceq \sum_{k=m}^{n} \|B_a^{\frac{1}{2}}\|^2 \|t^{\frac{k}{2}}\|^2 I \\
&\preceq \|B_a^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|t\|^k I \\
&\preceq \|B_a^{\frac{1}{2}}\|^2 \frac{\|t\|^m}{1 - \|t\|} I \to 0 \quad (as\ m \to \infty).
\end{aligned}
$$

This implies that $\{x_n\}$ is a b_2-Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, i.e., $\lim_{n\to\infty} Tx_{n-1} = x$. We observe that

$$\begin{aligned}
d(Tx,x,a) &\preceq sd(Tx_n,Tx,x)+sd(Tx_n,x,a)+sd(Tx_n,Tx,a)\\
&\preceq sd(x_{n+1},Tx,x)+sd(x_{n+1},x,a)+s\alpha(x_n,x,a)d(Tx_n,Tx,a)\\
&\preceq sd(x_{n+1},Tx,x)+sd(x_{n+1},x,a)+\frac{1}{s}A[d(Tx_n,x,a)+d(Tx,x_n,a)]\\
&= sd(x_{n+1},Tx,x)+sd(x_{n+1},x,a)+\frac{1}{s}Ad(x_{n+1},x,a)+\frac{1}{s}A[sd(x,Tx,x_n)\\
&\quad +sd(x,x_n,a)+sd(x,Tx,a)].
\end{aligned}$$

Consequently,
$(I-A)d(Tx,x,a) \preceq sd(x_{n+1},Tx,x)+sd(x_{n+1},x,a)+\frac{1}{s}Ad(x_{n+1},x,a)+Ad(x,Tx,x_n)+Ad(x,x_n,a)$,
and so

$$\begin{aligned}
\|d(Tx,x,a)\| &\leqslant \|(I-A)^{-1}\|\Big[s\|d(x_{n+1},Tx,x)\|+s\|d(x_{n+1},x,a)\|\\
&\quad +\frac{1}{s}\|A\|\|d(x_{n+1},x,a)\|+\|A\|\|d(x,Tx,x_n)\|\\
&\quad +\|A\|\|d(x,x_n,a)\|\Big] \to 0 \ (as\ n\to\infty).
\end{aligned}$$

Thus $Tx = x$ i.e., x is a fixed point of T. $\square$

Example 2.13. Let $X = \{(x,0) : x \in [0,+\infty)\} \cup \{(0,2)\} \subseteq \mathbb{R}^2$ and $\mathbb{A}$ be an arbitrary C^*-algebra. Suppose that $A \in \mathbb{A}_+$ with $\|A\| < \frac{1}{2}$ and $\sigma(A) \subseteq [\frac{2}{5}, \frac{1}{2})$. Define $d : X \times X \times X \to \mathbb{A}$ by $d((x,0),(y,0),(z,0)) = 0$, for each $(x,0),(y,0),(z,0) \in X$, otherwise $(x-y)^2A$. Also, define $T : X \to X$ by $T(x,0) = (\frac{x}{3},0)$, $T(0,2) = (0,0)$ and $\alpha : X \times X \times X \to \mathbb{R}_+$ by

$$\alpha(x,y,z) = \begin{cases} 1, & y = (0,0),\\ 0, & \text{otherwise.}\end{cases}$$

The only nontrivial case for checking the contractive condition (3) is when $x = (x,0),\ y = (0,0),\ z = (0,2)$. In this case we have

$$\frac{4}{9}x^2A = s^2\alpha((x,0),(0,0),(0,2))d(T(x,0),T(0,0),(0,2))$$

$$\preceq A[d(T(x,0),(0,0),(0,2)) + d(T(0,0),T(x,0),(0,2))] = \frac{10}{9}x^2A^2,$$

which holds by our assumptions. Considering $x_0 = (0,0)$, all conditions of Theorem 2.12 hold and so T has a fixed point.

Theorem 2.14. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued b_2-metric space. Suppose $\alpha : X \times X \times X \to \mathbb{R}_+$ and $T : X \to X$ are mappings satisfying the following conditions:*
(i) T is triangular α-admissible mapping and there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, a) \geqslant 1$ and $\alpha(x_0, Tx_0, a) \geqslant 1$,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_{n+1}, x_n, a) \geqslant 1$ and $\alpha(x_n, x_{n+1}, a) \geqslant 1$ as $n \to \infty$, then $\alpha(x_n, x, a) \geqslant 1$.
(iii) for all $x, y, a \in X$, the following inequality holds:

$$s\alpha(x, y, a)d(Tx, Ty, a) \preceq A[d(Tx, x, a) + d(Ty, y, a)], \tag{4}$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$.
Then T has a fixed point in X.

Proof. Notice that $A \in \mathbb{A}'_+$ implies that $A[d(Tx, x, a)+d(Ty, y, a)]$ is a positive element of $\mathbb{A}$. Choose $x_0 \in X$, set $x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \dots$. Let a be an arbitrary element of X. We denote by B_a the element $d(x_1, x_0, a)$ in $\mathbb{A}$. Then by (2) and (4), it follows that

$$\begin{aligned} d(x_{n+1}, x_n, a) &\preceq \alpha(x_n, x_{n-1}, a)d(Tx_n, Tx_{n-1}, a) \\ &\preceq \frac{1}{s} A[d(Tx_n, x_n, a) + d(Tx_{n-1}, x_{n-1}, a)] \\ &= \frac{1}{s} A[d(x_{n+1}, x_n, a) + d(x_n, x_{n-1}, a)] \\ &\preceq Ad(x_{n+1}, x_n, a) + \frac{1}{s} Ad(x_n, x_{n-1}, a). \end{aligned}$$

Therefore $(I - A)d(x_{n+1}, x_n, a) \preceq \frac{1}{s} Ad(x_n, x_{n-1}, a)$. Applying Lemma 1.3, we get

$$\begin{aligned} d(x_{n+1}, x_n, a) &\preceq \frac{1}{s} A(I - A)^{-1} d(x_n, x_{n-1}, a) = \frac{1}{s} td(x_n, x_{n-1}, a) \\ &\vdots \\ &\preceq \frac{1}{s^n} t^n d(x_1, x_0, a) = \frac{1}{s^n} t^n B_a, \end{aligned}$$

where $t = A(I - A)^{-1}$.
For $n, m \in \mathbb{N}$ with $n + 1 > m$, we have

$$\begin{aligned} \|d(x_{n+1}, x_m, a)\| &\leqslant \|\alpha(x_n, x_{m-1}, a)d(Tx_n, Tx_{m-1}, a)\| \\ &\leqslant \frac{1}{s} \|A[d(Tx_n, x_n, a) + d(Tx_{m-1}, x_{m-1}, a)]\| \\ &= \frac{1}{s} \|A[d(x_{n+1}, x_n, a) + d(x_m, x_{m-1}, a)]\| \\ &\leqslant \frac{1}{s} \|A\| \big[(\| \frac{1}{s^n} t^n B_a\| + \| \frac{1}{s^{m-1}} t^{m-1} B_a) \| \\ &\leqslant \|A\| \|t\|^n \|B_a\| + \|A\| \|t\|^{m-1} \|B_a\| \to 0 \ \ (as \ \ n, m \to \infty). \end{aligned}$$

By the completeness of $(X,\mathbb{A},d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, i.e., $\lim_{n\to\infty} Tx_{n-1} = x$. We show that x is a fixed point of T. For, by (ii) and (4) we have

$$\begin{aligned} d(Tx,x,a) &\preceq sd(Tx_n,Tx,x) + sd(Tx_n,x,a) + s\alpha(x_n,x,a)d(Tx_n,Tx,a) \\ &\preceq sd(x_{n+1},Tx,x) + sd(x_{n+1},x,a) + A[d(Tx_n,x_n,a) + d(Tx,x,a)]. \end{aligned}$$

Therefore we deduce

$$(I-A)d(Tx,x,a) \preceq sd(x_{n+1},Tx,x) + sd(x_{n+1},x,a) + Ad(x_{n+1},x_n,a),$$

and so,

$$\|d(Tx,x,a)\| \leqslant \|(I-A)^{-1}\| \Big[s\|d(x_{n+1},Tx,x)\| + s\|d(x_{n+1},x,a)\| \\ + \|A\|\|d(x_{n+1},x_n,a)\| \Big].$$

The right hand side of the above inequality is equal to zero as n tends to infinity. Thus $Tx = x$. Hence T has a fixed point. $\square$

Proposition 2.15. *In Theorem 2.10, Theorem 2.12 and Theorem 2.14, if $\alpha(x,y,a) \geqslant 1$ for each $x,y,a \in X$, then T has a unique fixed point in X.*

We terminate this section with a fact concerning expansion mappings in C^*-algebra-valued b_2-metric spaces.

Definition 2.16. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued b_2-metric space. We call a mapping $T : X \to X$ is an expansion mapping on X, if it satisfies the following conditions:*
(E1) $T(X) = X$;
(E2) $d(Tx,Ty,a) \succeq sA^*d(x,y,a)A$, for each $x,y,a \in X$,
where A is an invertible element in $\mathbb{A}$ such that $\|A^{-1}\| < 1$.

Theorem 2.17. *Let $(X,\mathbb{A},d)$ be a complete C^*-algebra-valued b_2-metric space, $T : X \to X$ be an expansion mapping on X. Then T has a unique fixed point in X.*

Proof. We first show that T is injective. If there exist $x,y \in X$ such that $Tx = Ty$, then for each $a \in X$ we have

$$0 = d(Tx,Ty,a) \succeq sA^*d(x,y,a)A.$$

Since $A^*d(x,y,a)A \in \mathbb{A}_+$, thus $A^*d(x,y,a)A = 0$. Now invertibility of A implies that $d(x,y,a) = 0$ for each $a \in X$. Applying Remark 2.2, we have

$x = y$. Therefore T is injective and so by (E1) it is invertible.
Next, we will show T has a unique fixed point in X. Substitute x, y with $T^{-1}x, T^{-1}y$, respectively in (E2), then

$$d(x, y, a) \succeq sA^* d(T^{-1}x, T^{-1}y, a)A.$$

This means

$$(A^*)^{-1} d(x, y, a)A^{-1} \succeq sd(T^{-1}x, T^{-1}y, a),$$

and thus

$$(A^{-1})^* d(x, y, a)A^{-1} \succeq sd(T^{-1}x, T^{-1}y, a).$$

Then by Theorem 2.10 and Proposition 2.15 T^{-1} has a unique fixed point x. On the other hand, the equality $T^{-1}x = x$ gives us $Tx = x$, and so x is a unique fixed point of T. $\square$

3. Application

As an application of our results, we will consider the following integral equation:

$$x(t) = \int_E K(t, s, x(s))ds + g(t). \tag{5}$$

Let E, X, $L(H)$ and π_α be as in Example 2.5. We will assume that the following conditions are satisfied:
(i) $\alpha : X \times X \times X \to \mathbb{R}^+$be such that $\alpha(x, y, z) \geqslant 1$ for all $x, y, z \in X$
(ii) $K : E \times E \times \mathbb{R} \to \mathbb{R}$ *and* $g \in X$;
(iii) for some $A \in L(H)$ such that $\|A\| < 1$, $p \geqslant 1$ and $x, y, a \in X$,

$$\begin{aligned}
3^{p-1}\pi_{\left[\max\limits_{t \in E}\min\{\right.} & \left| \int_E K(t, s, x(s))ds - \int_E K(t, s, y(s))ds \right|, \\
& \left| g(t) + \int_E K(t, s, x(s))ds - a(t) \right|, \\
& \left. \left| g(t) + \int_E K(t, s, y(s))ds - a(t) \right| \}\right]^p \\
\preceq \ & A^* \pi_{\left[\max\limits_{t \in E}\min\{|x(t) - y(t)|, |x(t) - a(t)|, |y(t) - a(t)|\}\right]^p} A.
\end{aligned}$$

Theorem 3.1. *Under assumptions (i)-(iii) equation (5) has a unique solution in X.*

Proof. Define $d : X \times X \times X \to L(H)$ by

$$d(x, y, z) = \pi_{\left[\max\limits_{t \in E}\min\left\{|x(t) - y(t)|, |x(t) - z(t)|, |y(t) - z(t)|\right\}\right]^p}.$$

Then by Example 2.5, $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b_2-metric space with $s = 3^{p-1}$. Also we define the mapping $T : X \to X$ by

$$Tx(t) = \int_E K(t, s, x(s))ds + g(t).$$

We will show that T is a contractive mapping. By using assumptions we obtain that

$$\begin{aligned} sd(Tx, Ty, a) &= 3^{p-1}\alpha(x, y, a)d(Tx, Ty, a) \\ &= 3^{p-1}\pi\big[\max_{t\in E}\min\big\{|Tx(t) - Ty(t)|, |Tx(t) - a(t)|, |Ty(t) - a(t)|\big\}\big]^p \\ &= 3^{p-1}\pi\big[\max_{t\in E}\min\big\{|\int_E K(t, s, x(s))ds - \int_E K(t, s, y(s))ds|, |g(t) + \\ &\quad \int_E K(t, s, x(s))ds - a(t)| + |g(t) + \int_E K(t, s, y(s))ds - a(t)|\big\}\big]^p \\ &\preceq A^*\big[\max_{t\in E}\min\big\{\pi_{|x(t)-y(t)|}, \pi_{|x(t)-a(t)|}, \pi_{|y(t)-a(t)|}\big\}\big]^p A \\ &= A^* d(x, y, a)A. \end{aligned}$$

Thus we conclude that T has a fixed point, and integral equation (5) has a unique solution in X. □

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