Journal of Mathematical Extension
Vol. 11, No. 2, (2017), 43-52
ISSN: 1735-8299
URL: http://www.ijmex.com

A Modification on Rubio's Theoretical Measure Method for Solving Classical Optimal Control Problems

A. Fakharzadeh J*
Shiraz University of Technology

M. Goodarzi
Shiraz University of Technology

Abstract. In this paper, a revised measure-theoretical approach is applied for solving some classical optimal control problems. Indeed, the problem is converted into an optimization problem in measure space and then it is transformed into a finite dimensional nonlinear programming problem by using approximation scheme. At last, the nearly optimal control and trajectory functions are determined from the solution of the nonlinear optimization problem. A numerical example is given to demonstrate the efficiency of this method.

AMS Subject Classification: 49J15; 49J20; 47N10
Keywords and Phrases: Optimal control, measure theory, nonlinear optimization, approximation, PSO algorithm

1. Introduction

Optimal Control (OC) theory is applied in different fields such as aerospace, mechanic, electronics and biomedical engineering [13]. Their problems are solved by direct and indirect methods, where indirect methods are usually based on transforming the OC problem into a boundary-value

Received: June 2016; Accepted: August 2016
*Corresponding author

problem [15]. In direct methods, the state and control functions are obtained by applying a suitable function transformation or approximation, where the necessary coefficients of this action are considered as a solution of a nonlinear optimization problem [10]. Both methods are sensitive to the initial guess and are unable to determine the global solution. The measure theoretical approach is a helpful method for a large class of applied mathematics problems, particularly in OC and optimal shape design problems. This approach was based on an idea of Young that Rubio represented it theoretically [16]. Then some researchers improved the method like Farahi et al. [8, 9], Kamyad et al. [11, 12], Effati [4, 5] and Fakharzadeh [6, 7]. Rubio in [16] showed that OC systems governed by ODE (and also PDE), can be solved by use of Radon measures in the following way: representing the OC problem in variational form, defining the related measure and transferring the problem into a measure theoretical one by extending the underlying space, discretization schemes and using two step of approximation. Indeed, the solution of the original problem is approximated by a nonlinear one and then, this new one is approximated by a finite linear programming (FLP) one. Here, we try to modify this method in away that solving the nonlinear problem directly, and show the way of extracting the optimal control and its related optimal trajectory from its solution.

2. Statement of The Classical Optimal Control Problem

A classical optimal control problem is defined as (P_1):

$$\begin{aligned} Min: I(x,u) &= \int_{t_a}^{t_b} f_0(t,x(t),u(t))dt \\ S.t.: \dot{x} &= g(t,x(t),u(t)),\ t \in (t_a,t_b); \\ x(t_a) &= x_a,\ x(t_b) = x_b. \end{aligned}$$

It is supposed that A and U are bounded and closed subsets in R^n and R^m, respectively that the trajectory and controls are taken their values on them, and $J = [t_a, t_b]$ is the time interval that $J^0 = (t_a, t_b)$. Moreover,

$f_0 : \Omega \to R$ and $g : \Omega \to R^n$ are continuous functions, where $\Omega = J \times A \times U$.

Definition 2.1. *A pair of trajectory and control $p = (x, u)$ is admissible if the following conditions hold:*

(i) The trajectory function x is absolutely continuous on J and staying in the bounded set A.

(ii) The control function u is Lebesgue-measurable on J and takes values in the bounded set U.

(iii) The pair (x, u) satisfies in the constraints of problem (P_1).

The set of all admissible pairs denoted by W and assume that it is nonempty.

3. Modifying Classical Control Problem Into a Nonclassical One

Rubio demonstrated some characteristics of the pairs in W as below [16]:

- Let B be an open ball in R^{n+1} containing $J \times A$; the space of real-valued continuously differentiable functions on B that they and their first derivatives are bounded on B is denoted by $C'(B)$. Let $\phi \in C'(B)$, define $\phi^g \in C(\Omega)$ as: $\phi^g(t, x, u) = \phi_x(t, x)g(t, x, u) + \phi_t(t, x)$; therefore one can result that:

$$\int_J \phi^g(t, x, u)dt = \phi(t_b, x_b) - \phi(t_a, x_a) \equiv \Delta\phi, \ \forall\phi \in C'(B).$$

- Let $D(J^\circ)$ be the space of infinitely differentiable functions with compact support in J°. For $\psi \in D(J^\circ)$, we define: $\psi_j(t, x, u) = x_j\psi'(t) + g_j(t, x, u)\psi(t), \ j = 1, 2, ..., n$. Then we have:

$$\int_J \psi_j dt = 0, \ \forall\psi \in D(J^\circ).$$

Similarly have: $\int_J f(t,x,u)dt = a_f,\ \forall f \in C_1(\Omega)$, where $C_1(\Omega)$ is the space of all functions which depend only on the variable t and a_f is the Lebesgue integral of f over J.
Now, consider the following functional for each admissible pair p:

$$\Lambda_p : F \in C(\Omega) \to \int_J F(t,x(t),u(t))dt \in R.$$

Λ_p is a positive continuous linear functional on $C(\Omega)$. If we denote the space of all positive Radon measures on Ω by $M^+(\Omega)$, by using the Riesz's representation theorem [17], there is an one-to-one correspondence that assign each $p \in W$ to unique measure $\mu_p \in M^+(\Omega)$ such that:

$$\Lambda_p(F) = \int_\Omega F d\mu \equiv \mu_p(F), \qquad \forall F \in C(\Omega).$$

To ensure that problem (P_1) has a solution, Rubio extended the underlying space and considered all the measures that are satisfies the related constraints of P_1 (not only those that indeed by Riesz's theorem). So, the problem (P_1) can be changed into the following nonclassical measure-theoretical one (P_2):

$$(P_2) \qquad \begin{array}{ll} \underset{\mu \in M^+(\Omega)}{Inf} & \mu(f_0) \\ S.to: & \mu(\phi^g) = \Delta_\phi, \quad \forall \phi \in C'(B); \\ & \mu(\psi_j) = 0, \qquad \forall \psi \in D(J^\circ); \\ & \mu(f) = a_f, \qquad \forall f \in C_1(\Omega). \end{array}$$

Let Q be the solution space of the problem (P_2). By help of $weak^*$ topology, after extending the underlying spaces, Rubio then showed that the new problem has solution. To identify it, he applied some step of approximations.

4. Approximation Scheme

First, we consider minimization of $\mu \to \mu(f_0)$ not over the set Q, but over a subset of it that satisfied only a finite number of the constraints of (P_2) that called it $Q(M_1, M_2, M_3)$. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces and then, considering a finite number M_1, M_2 and M_3 of them [16] as:

$$(P_3) \qquad \begin{aligned} &\underset{Q(M_1,M_2,M_3)}{Inf} \quad I(P) = \mu(f_0) \\ &S.to: \ \mu(\phi_i^g) = \Delta_\phi, \quad i = 1,\, 2,\, ..., M_1; \\ &\qquad \mu(\chi_h) = 0, \qquad h = 1, 2, ..., M_2; \\ &\qquad \mu(f_s) = a_{f_s}, \quad s = 1, 2, ..., M_3, \end{aligned}$$

where χ_h is the sequence of functions of the type ψ_j. If M_1, M_2 and M_3 tend to infinity, the optimal solution of (P_3) will converge to the optimal solution of problem (P_2). But even (P_3) is linear, the solution space of (P_3) is still infinite. According to theorem A.5 of [16], we apply the second stage of approximation by writing an optimal measure μ^* in the form:

$$\mu^* = \sum_{j=1}^{M_1+M_2+M_3} \alpha_j^* \delta(q_j^*) \tag{1}$$

with α_j^* and $q_j^* \in \Omega$, $j = 1, 2, \ldots, M_1 + M_2 + M_3$; here $\delta(q) \in M^+(\Omega)$ is the unitary atomic measure supported by singleton sets $\{q\}$ that $q \in \Omega$ and characterized by $\delta(q)(F) = F(q)$, $\forall F \in C(\Omega)$, $q \in \Omega$. So, the problem (P_3) is equivalent to a nonlinear optimization problem (P_4) that its unknowns are the coefficients α_j^* and supports $\{q_j^*\}$, $j = 1, 2, \ldots, M_1 + M_2 + M_3$:

$$(P_4) \qquad \begin{aligned} &Min \quad \sum_{j=1}^{M_1+M_2+M_3} \alpha_j f_0(q_j^*) \\ &S.to: \sum_{j=1}^{M_1+M_2+M_3} \alpha_j \phi_i^g(q_j^*) = \Delta_{\phi_i}, \quad i = 1,\, 2,\, ...,\, M_1; \\ &\qquad \sum_{j=1}^{M_1+M_2+M_3} \alpha_j \chi_h(q_j^*) = 0\ , \quad h = 1, 2, ..., M_2\, ; \\ &\qquad \sum_{j=1}^{M_1+M_2+M_3} \alpha_j f_s(q_j^*) = a_{f_s}\ , \quad s = 1, 2, ..., M_3\, ; \\ &\qquad \alpha_j \geqslant 0, \qquad j = 1,\, 2,\, ...,\, M_1 + M_2 + M_3. \end{aligned}$$

Rubio in [16] converted the above problem into a finite-dimensional linear problem by introducing support points in a countable dense subset of Ω by applying discretization scheme, which also contains an other step of approximation as well. Here, we solve directly the nonlinear optimization problem (P_4) without use of discretization and so on.

5. Optimization Algorithms

When M_1, M_2 and M_3 tend to infinity, the above nonlinear problem (NLP) is a large scale problem; therefore, the metaheuristic algorithms can be used for solving it. Among of them we apply the PSO (Particle Swarm Optimization) and TLBO (Teaching-Learning Based Optimization) algorithms; PSO is a metaheuristic algorithm that it needs few assumptions about the problem which being optimized and can search in a very large spaces of candidate solutions. Indeed PSO algorithm is widely used and rapidly developed for its easy implementation and few particles required to be tuned [2, 3]. TLBO algorithm is also a population based optimization method that uses a population of solutions to proceed the global solution [1, 14]; it is important to remind that the procedure to construct a piecewise constant control and trajectory functions from the obtained solution set $\left\{\alpha_j^* > 0,\ j = 1, 2, ..., M_1 + M_2 + M_3\right\}$ by solving (P_4), approximates the action of the optimal measure in (1). This fact is based on the presented analysis in [16], since there is no need to construct any partition on $J \times A$.

6. Numerical Example

For demonstrating and comparing the efficiency of our method, we consider a numerical example which is also solved in [16].

Example 6.1. Consider the following optimal control problem:

$$\begin{array}{l} Min\ \ \int_J x(t)^2 dt \\ S.to:\ \dot{x} = u; \\ x(0) = 0,\ x(1) = 0.5, \end{array}$$

Table 1: The optimal value of the cost function and number of iterations

Method	Optimal value of the cost function	Number of iterations
Analytically	0.041667	....
Revised Simplex for FLP	0.041758	25
TLBO for NLP	0.041745	20
PSO for NLP	0.041723	20

where $J = [0,1]$, $x(t) \in A = [0,1]$ and $U = [0,1]$. In [16], the trajectory and control functions was obtained by solving the corresponding finite-dimensional linear programming problem to (P_4). As Rubio in [16], we selected $M_1 = 2$, $M_2 = 8$ and $M_3 = 10$ in the nonlinear problem (P_4) and solving it by using PSO and TLBO algorithms. The obtained nearly optimal trajectory and control functions for the corresponding linear and nonlinear problems are shown in Figures 1, 2 and 3, respectively. The optimal value of the cost function and the number of iterations with different procedures, have been shown in Table 1, which indicates that the obtained result by TLBO is better than FLP and the obtained result by PSO is better than both.

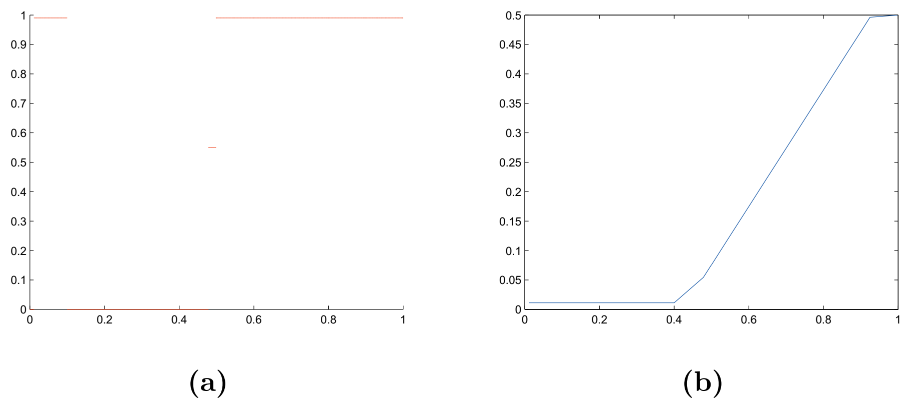
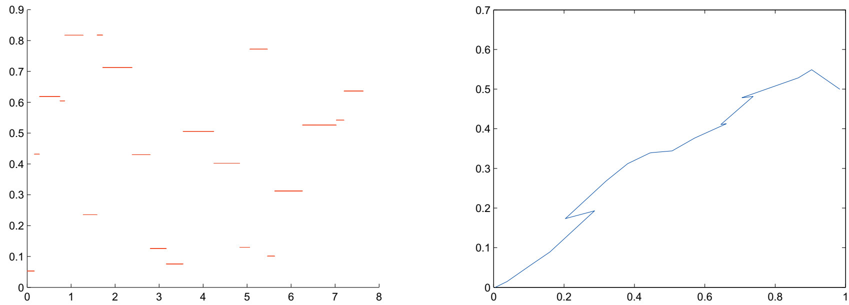
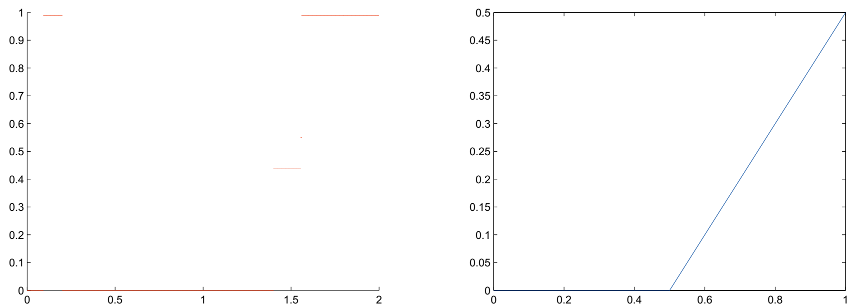


Figure 1. (a) Nearly optimal control and (b) trajectory functions obtained from FLP



(a) **(b)**

Figure 2. (a) Nearly optimal control and (b) trajectory functions from NLP by TLBO algorithm



(a) **(b)**

Figure 3. (a) Nearly optimal control and (b) trajectory functions from NLP by PSO algorithm

7. Conclusion

By using the measure-theoretical approach, the classical optimal control problem was transformed to a problem in measure space that is an infinite-dimensional linear programming problem. By omitting an approximation step in compare with [16], in this paper we present a modified method which that the solution of the problem can be obtained from the solution of a finite-dimensional nonlinear programming problem; in this regard, the suitable metaheuristic and related heuristic optimization algorithms can be used and a step of approximation can be omitted. Therefore, the accuracy of the results are better and the new obtained optimal approximated solutions (optimal value and optimal trajectory and control functions) are nearer to the actual one in compare with the previous method.

References

[1] U. Aich and S. Banerjee, Application of teaching learning based optimization procedure for the development of SVM learned EDM process and its pseudo pareto optimization, *Applied Soft Computing,* 39 (2016), 64-83.

[2] A. Azimifar and S. Payan, Enhancement of heat transfer of confined enclosures with free convection using blocks with PSO algorithm, *Applied Thermal Engineering,* 101 (2016), 79-91.

[3] Q. Bai, Analysis of particle swarm optimization algorithm, *Computer and Information Science,* 3 (1) (2010), 180-184.

[4] S. Effati, M. Janfada, and M. Esmaeili, Solving the optimal control problem of the prabolic PDEs in exploitation of oil, *Journal of Mathematical Analysis and Applications,* 340(2008), 606-620.

[5] S. Effati, A. Nazemi, and H. Shabani, Time optimal control problem of the heat equation with thermal source, *IMA Journal of Mathematical Control and Information,* 31 (2014), 385-402.

[6] A. Fakharzadeh, S. A. Ghasemiyan, and A. J. Badiozzaman, Survey for abilities of wavelets in solving optimal control problems by embedding methods, *Int. J. Contemp. Math. Sciences.,* 14(2008), 683-694.

[7] A. Fakharzadeh and N. Jafarpoor Firozabadi, A new discretization method for solving an optimal vibrating controlled system on an arbitrary domain, *Journal of Vibration and Control,* 19(14), 2092-2099.

[8] M. H. Farahi, J. E. Rubio, and D. A. Wilson, The global control of a nonlinear wave equa tion, *International J. of Control.,* 65(1)(1996),1-15.

[9] M. H. Farahi, J. E. Rubio, and D. A. Wilson, The optimal control of the linear wave equation, *International J. of Control.,* 63(1)(1996), 833-848.

[10] C. R Hargraves and S. W. Paris, Direct Trajectory Optimization Using Nonlinear Programming and Collocation, *Journal of Guidance, Control, and Dynamics,* 10 (1987), 338-342.

[11] A. V. Kamyad, J. E. Rubio, and D. A. Wilson, An Optimal control problem for the multidimensional dif- fusion equation with a generalized control variable, *J. of Optimization Theory and Applications,* 75 (1992), 101-132.

[12] A. V. Kamyad, J. E. Rubio, and D. A. Wilson, Optimal control of multidimensional diffusion equation, *J. of Optimization Theory and Applications,* 70 (1991),191-209.

[13] F. Lewis and V. L. Syrmos, *Applied Optimal Control and Estimation: Digital Design & Implementation,* Prentice Hall, (1992).

[14] R. V. Rao, V. J. Savsani, and D. P. Vakharia, Teaching-Learning-Based Optimization: A novel method for constrained mechanical design optimization problems, *Computer-Aided Design,* 43 (2011), 303-315.

[15] I. M. Ross,*A Primer on Pontryagin's Principal in Optimal Control,* Collegiate publishers, (2009).

[16] J. E. Rubio,*Control and Optimization: the Linear Treatment of Nonlinear Problems,* Manchester, U. K., Manchester University Press.(1986).

[17] W. Rudin, *Functional Analysis,* 2nd ed., McGraw-Hill, New York, (1991).

Alireza Fakharzadeh J
Department of Mathematics
Associate Professor of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: a_fakharzadeh@sutech.ac.ir

Mina Goodarzi
Department of Mathematics
Ph. D Student of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: m.goodarzi@sutech.ac.ir