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A Method for Construction of Wavelets in Hardy Space of Analytic Functions

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Abstract. The main aim of this paper is to construct an orthonormal wavelet on $H^2(\mathbb{D})$, the Hardy space of analytic functions on the open unit disc with square summable Taylor coefficients.

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1. Introduction

The development of wavelets can be linked to several separate trains of thought, starting with Haar's work in the early 20th century (see [6]). He showed that the appropriate translates and dilates of $Haar function$ form an orthonormal basis of $L^2([0,1])$. In [8], Mallat introduced the idea of a multiresolution analysis (MRA), where the general theory of finding a wavelet starting from an MRA scheme was developed. In [9], Meyer extended the concept of MRA to n-dimensions. Meyer constructed the first non-trivial wavelets, unlike the Haar wavelets.

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Wavelet turn out to have many advantages in studying various function spaces. They form an unconditional basis for variety of function spaces, e.g. $L^p(\mathbb{R})$, *Hardy* and *Sobolov spaces*. (see [1, 2, 7])
In [5], Goh gave a general approach of constructing orthonormal wavelet for a separable Hilbert space of complex-valued functions, and his construction is applied to several Hilbert spaces like the space of analytic functions on the unit disk.
In [10, 11], Pap generated a multiresolution in the *Hardy space* of analytic functions on the open unit disc and a multiresolution in the *Bergman space*. In [10], the method that was used is based on the analysis where instead of dilation and translations, it is used the *Blaschke group*.
The theory of *Hardy spaces* is very rich with many highly developed branches and originated in the context of complex function theory and Fourier analysis in the beginning of the twentieth century.
Let $\mathbb{D}$ denote the open unit disc in the complex plane, and the *Hardy space*, $H^2(\mathbb{D})$, be the set of all functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ holomorphic in $\mathbb{D}$ such that

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty,$$

where $\hat{f}(n)$ denotes the nth Taylor coefficient of f (see [12]). For systematic exposition of the subject see books by Duren [3] and Garnet [4].
We denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{N}_0$ the set of all nonnegative integers, by $\mathbb{N}_{\mathbb{M}}$ the set $\{0, 1, 2, \ldots, M-1\}$ for positive integer M and by $L^2[0,1]$ the Hilbert space of all square-integrable functions with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt,$$

for $f, g \in L^2[0,1]$.
Many examples of wavelets have been produced using the related concept of a *multiresolution analysis*.

Definition 1.1. *A sequence of closed subspaces* $\{V_j\}_{j \in \mathbb{N}_0}$ *in* $L^2[0,1]$ *is* *called a multiresolution analysis (MRA) for* $L^2[0,1]$ *if the following conditions are satisfied:*

i) $V_j \subseteq V_{j+1}$;

ii) $\overline{\bigcup_{j\in\mathbb{N}_0} V_j} = L^2[0,1]$;

iii) There exists $\phi \in L^2[0,1]$ *such that for every* $j \in \mathbb{N}_0$,

$$\{\phi_{j,l}(.) = 2^{\frac{j}{2}}\phi(2^j . - l) : l \in \mathbb{N}_{2^j}\}$$

is an orthonormal basis for V_j. *Herein,* ϕ *is called a scaling function of the MRA , and* ϕ *is said to generate the MRA.*

Definition 1.2. *Given an MRA* $\{V_j\}_{j\in\mathbb{N}_0}$, *and let for every* $j \in \mathbb{N}_0, W_j = V_{j+1} \ominus V_j$. *If there exists* $\psi \in W_0$, *such that*

$$\{\phi, \psi_{j,l}(.) = 2^{\frac{j}{2}}\psi(2^j . - l) : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$$

is an orthonormal basis for $L^2[0,1], \psi$ *is called an orthonormal wavelet of the MRA.*

Let T and D be the translation and dilation unitary operators in $B(L^2(\mathbb{R}))$ given by $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$. Then $\psi_{j,l}(t) = 2^{\frac{j}{2}}\psi(2^j t - l) = (D^j T^l \psi)(t)$ for all $j,l \in \mathbb{Z}$, and so we have ψ an orthonormal wavelet of the MRA if and only if $\{\phi, \psi_{j,l} : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2[0,1]$.

If we cosider $\phi = \chi_{[0,1]}(t)$ and $\psi(t) = \chi_{[0,\frac{1}{2}]}(t) - \chi_{[\frac{1}{2},1]}(t)$ then the system $\{\phi, \psi_{j,l} : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2[0,1]$. (see [13])

In section 2 of the present paper, by using the definition of multiresolution analysis and orthonormal wavelet on $L^2[0,1]$, and this fact that $L^2([0,1])$ and $H^2(\mathbb{D})$ are isomorphic, we generalize these concepts to $H^2(\mathbb{D})$. Also by virtue of Haar wavelet on $L^2[0,1]$ we give an example of wavelet on Hardy space.

2. Wavelet on Hardy Space

In this section we extend the concept of MRA from $L^2[0,1]$ to $H^2(\mathbb{D})$. In our situation, the main definition is as follows:

Definition 2.1. *A multiresolution analysis (MRA) on the Hardy space consists of a sequence $\{V_j\}_{j\in\mathbb{N}_0}$ of closed subspaces of $H^2(\mathbb{D})$ satisfying*

i) $V_j \subseteq V_{j+1}$;

ii) $\overline{\bigcup_{j\in\mathbb{N}_0} V_j} = H^2(\mathbb{D})$;

iii) *There exists an isometric isomorphism S from $L^2[0,1]$ onto $H^2(\mathbb{D})$ and $\phi \in L^2[0,1]$ such that for every $j \in \mathbb{N}_0$,*

$$\{S\phi_{j,l}(.) = 2^{\frac{j}{2}} S\phi(2^j . - l) : l \in \mathbb{N}_{2^j}\}$$

is an orthonormal basis for V_j. Herein, $S\phi$ is called a scaling function of the MRA, and $S\phi$ is said to generate the MRA .

Definition 2.2. *If S is an isometric isomorphism from $L^2[0,1]$ onto $H^2(\mathbb{D})$, an orthonormal wavelet on $H^2(\mathbb{D})$ is a unit vector $S\psi \in L^2[0,1]$ with the property that the set*

$$\{S\phi, S\psi_{j,l}(.) = 2^{\frac{j}{2}} S\psi(2^j . - l) : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$$

is an orthonormal basis for $H^2(\mathbb{D})$, where $S\phi$ is a scaling function of the MRA on $H^2(D)$.

Theorem 2.3. *Let S be an isometric isomorphism from $L^2[0,1]$ onto $H^2(\mathbb{D})$, $\{V'_j\}_{j\in\mathbb{N}_0}$ be an MRA on $L^2[0,1]$ with the scaling function ϕ and orthonormal wavelet ψ. Put $V_j = S(V'_j)$. Then $\{V_j\}_{j\in\mathbb{N}_0}$ is an MRA on $H^2(\mathbb{D})$ and $\{S\phi, S\psi_{j,l} : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$ is an orthonormal basis for $H^2(\mathbb{D})$. Furthermore $S\psi$ is an orthonormal wavelet on $H^2(\mathbb{D})$.*

Proof. We have ψ is an orthonormal wavelet of the MRA on $L^2[0,1]$ if and only if $\{\phi, \psi_{j,l} : l \in \mathbb{N}_{2^j}, j \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2[0,1]$. The proof is completed by using the following facts. Since S is an isometric isomorphism, for each $j, l, m, n \in \mathbb{N}_0$, we have

$$\langle S\psi_{j,l}, S\psi_{m,n}\rangle = \langle \psi_{j,l}, \psi_{m,n}\rangle \ \text{ and } \ \langle S\psi_{j,l}, S\phi\rangle = \langle \psi_{j,l}, \phi\rangle.$$

Because S is onto, for any $f \in H^2(\mathbb{D})$ there is $g \in L^2[0,1]$ such that $f = S(g)$, and so

$\langle S\psi_{j,l}, f\rangle = \langle S\psi_{j,l}, S(g)\rangle = \langle \psi_{j,l}, g\rangle$ and $\langle S\phi, f\rangle = \langle S\phi, S(g)\rangle = \langle \phi, g\rangle$. $\square$

Now let $\varphi : \mathbb{Z} \longrightarrow \mathbb{N}_0$ be defined as

$$\varphi(n) = \begin{cases} 2n & \text{if } n \geqslant 0; \\ -(2n+1) & \text{if } n < 0. \end{cases}$$

Since $\{e_k(t) = \chi_{[0,1]}e^{2\pi ikt} : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2[0,1]$, for given $f \in L^2[0,1]$ we have $f(t) = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k(t)$ a.e. Define an operator $S : L^2[0,1] \longrightarrow H^2(\mathbb{D})$ by

$$S(f) = S(\sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k) = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle z^{\varphi(k)}.$$

Since $\{z^n : n \in \mathbb{N}_0\}$ is an orthonormal basis for $H^2(\mathbb{D})$ and φ is bijection, S is an isometric isomorphism.

Lemma 2.4. *If* $0 \leqslant a \leqslant b \leqslant 1$,

$$S(\chi_{[a,b]}) = -\frac{1}{2\pi i}\left[z^{-1}\ln(\frac{1-e^{2\pi ib}z^2}{1-e^{2\pi ia}z^2}) - \ln(\frac{1-e^{-2\pi ib}z^2}{1-e^{-2\pi ia}z^2})\right] + b - a.$$

Proof. For $k \neq 0$ we have

$$\langle \chi_{[a,b]}, e_k\rangle = \int_a^b e^{-2\pi ikt}dt = \frac{e^{-2\pi ikb} - e^{-2\pi ika}}{-2\pi ik}.$$

So

$$\begin{aligned} F(t) &= \sum_{k=-\infty, k\neq 0}^{\infty} \frac{e^{-2\pi ikt}z^{\varphi(k)}}{k} \\ &= \sum_{k=-\infty}^{-1} \frac{e^{-2\pi ikt}z^{\varphi(k)}}{k} + \sum_{k=1}^{\infty} \frac{e^{-2\pi ikt}z^{\varphi(k)}}{k} \\ &= \sum_{k=-\infty}^{-1} \frac{e^{-2\pi ikt}z^{-(2k+1)}}{k} + \sum_{k=1}^{\infty} \frac{e^{-2\pi ikt}z^{2k}}{k} \\ &= -z^{-1}\sum_{k=1}^{\infty} \frac{e^{2\pi ikt}z^{2k}}{k} + \sum_{k=1}^{\infty} \frac{e^{-2\pi ikt}z^{2k}}{k} \\ &= z^{-1}\ln(1-e^{2\pi it}z^2) - \ln(1-e^{-2\pi it}z^2). \end{aligned}$$

So we have

$$\begin{aligned} S(\chi_{[a,b]}) &= -\frac{1}{2\pi i}(F(b)-F(a))+b-a \\ &= -\frac{1}{2\pi i}\left[z^{-1}\ln(\frac{1-e^{2\pi ib}z^2}{1-e^{2\pi ia}z^2})-\ln(\frac{1-e^{-2\pi ib}z^2}{1-e^{-2\pi ia}z^2})\right]+b-a. \quad \square \end{aligned}$$

Lemma 2.5. For $\psi_1(t)=\chi_{[0,\frac{1}{2}]}(t)-\chi_{[\frac{1}{2},1]}(t)$, $\phi_1(t)=\chi_{[0,1]}(t)$ we have $S(\psi_1)=\frac{1}{\pi iz}(z-1)\ln(\frac{1+z^2}{1-z^2})$ and $S(\phi_1)=1$ and the system

$$\left\{1,-\frac{2^{\frac{j}{2}-1}}{\pi i}\left[z^{-1}\ln\left(\frac{1-e^{2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{2\pi i(\frac{l}{2^j})}z^2}\right)-\ln\left(\frac{1-e^{-2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{-2\pi i(\frac{l}{2^j})}z^2}\right)\right]\right.$$
$$-\frac{2^{\frac{j}{2}-1}}{\pi i}\left[z^{-1}\ln\left(\frac{1-e^{2\pi i(\frac{l}{2^j}+\frac{1}{2^j})}z^2}{1-e^{2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}\right)-\ln\left(\frac{1-e^{-2\pi i(\frac{l}{2^j}+\frac{1}{2^j})}z^2}{1-e^{-2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}\right)\right]$$
$$\left.: l\in\mathbb{N}_{2^j}, j\in\mathbb{N}_0\right\}$$

is an *orthonormal basis* for $H^2(\mathbb{D})$.

Proof. By Lemma 2.4 we have

$$\begin{aligned} S(\psi_1) &= S(\chi_{[0,\frac{1}{2}]})-S(\chi_{[\frac{1}{2},1]}) \\ &= -\frac{1}{2\pi i}\left[z^{-1}\ln\left(\frac{\frac{1-e^{\pi i}z^2}{1-z^2}}{\frac{1-e^{2\pi i}z^2}{1-e^{\pi i}z^2}}\right)-\ln\left(\frac{\frac{1-e^{-\pi i}z^2}{1-z^2}}{\frac{1-e^{-2\pi i}z^2}{1-e^{-\pi i}z^2}}\right)\right] \\ &= -\frac{1}{2\pi i}\left[z^{-1}\ln\frac{(1-z^2e^{-\pi i})^2}{(1-z^2)(1-e^{2\pi i}z^2)}-\ln\frac{(1-z^2e^{-\pi i})^2}{(1-z^2)(1-e^{-2\pi i}z^2)}\right] \\ &= \frac{1}{\pi iz}(z-1)\ln\left(\frac{1+z^2}{1-z^2}\right). \end{aligned}$$

Since $l\in\mathbb{N}_{2^j}, j\in\mathbb{N}_0$ we have

$$\begin{aligned} S(\psi_{j,l}) &= 2^{\frac{j}{2}}(S(\chi_{[\frac{l}{2^j},\frac{l}{2^j}+\frac{1}{2^{j+1}}]})-S(\chi_{[\frac{l}{2^j}+\frac{1}{2^{j+1}},\frac{l}{2^j}+\frac{1}{2^j}]})) \\ &= -\frac{2^{\frac{j}{2}-1}}{\pi i}\left[z^{-1}\ln\left(\frac{1-e^{2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{2\pi i(\frac{l}{2^j})}z^2}\right)-\ln\left(\frac{1-e^{-2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{-2\pi i(\frac{l}{2^j})}z^2}\right)\right. \\ &= -\frac{2^{\frac{j}{2}-1}}{\pi i}\left[z^{-1}\ln\left(\frac{1-e^{2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{2\pi i(\frac{l}{2^j})}z^2}\right)-\ln\left(\frac{1-e^{-2\pi i(\frac{l}{2^j}+\frac{1}{2^{j+1}})}z^2}{1-e^{-2\pi i(\frac{l}{2^j})}z^2}\right)\right. \end{aligned}$$

By using Lemma 2.5 we know $\phi(z) = 1$ is a *scaling function* on *Hardy space* and $\psi(z) = \frac{1}{\pi i z}(z-1)\ln(\frac{1+z^2}{1-z^2})$ is an *orthonormal wavelet* on $H^2(\mathbb{D})$.

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