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Optimal and Adaptive Control of an Epidemic Model of Influenza with Unknown Parameters

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Abstract. This paper deals with the nonlinear dynamics, chaos, optimal and adaptive control of an epidemic model for H1N1 influenza with unknown parameters. Two different control strategies are explored. First, we use the optimal control theory to reduce the infected individuals and the cost of vaccination. Then, we study the problem of optimal control of unstable steady-states of H1N1 influenza system using a nonlinear control approach. Finally, we propose the Lyapunov stability to control of the chaotic epidemic model of influenza with unknown parameters by a feedback control approach. Matlab `bvp4c` and `ode45` have been used for solving the autonomous chaotic systems and the extreme conditions obtained from the Pontryagin's maximum principle (PMP). Furthermore, numerical simulations are included to demonstrate the effectiveness of the proposed control strategies.

AMS Subject Classification: 49J15; 37B25; 37N25; 37N35
Keywords and Phrases: Optimal control, influenza, epidemic model, lyapunov function, pontryagin's maximum principle

1. Introduction

Influenza is an infectious viral disease, which is commonly known as the flu. At least four pandemic of influenza occurred in the 19th century, and three occurred in the 20th century. Symptoms include fever, body aches, especially

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joints and throat, headache, fatigue, cough, reddened eyes, irritated watering eyes, mouth, throat and nose. It also includes gastrointestinal symptoms such as diarrhoea and abdominal pain among children. In influenza, the incubation period is the time period between the entrance of the flu virus into the body and the clinical onset of the disease. Influenza has an incubation period of 1-4 days. The latent period is the time between clinical signs appearing and the onset of symptoms. Influenzas latent period is very short as compared to the incubation period with its length as one day. During this stage there is a very low level of infectivity. The latent period of an infectious period is followed by 2-10 days. During the infectious stage there is a very high level of infectivity. Infectious period can however be decreased by medications. Recovery is usually rapid, but some patients may have lingering depression and asthenia for several weeks. The recovery period starts as soon as the infectious period ends. For influenza, immunity is not permanent. Influenza virus changes dramatically as different strain and our immune system fails to recognize it quickly. After recovering, a person is usually partially susceptible to different strains of a virus within a few years [1, 2, 3, 4]. Influenza is an infectious disease caused by a virus commonly known as influenza virus and transmitted among humans mainly in three ways: (i) by direct contact with infected individuals; (ii) by contact with contaminated objects and (iii) by inhalation of aerosols that contain virus particles. There are millions of people who suffer or die annually from influenza worldwide. Although different control and prevention strategies are available to control influenza transmission, influenza has been a major cause of morbidity and mortality among humans all over the world. Comparative knowledge of the effectiveness and efficacy of different control strategies is necessary to design useful influenza control programs. The ability of mathematical modeling to predict the effectiveness of combined control strategies and positively influence public health policy is well established [5, 6]. Based on approaches using optimal control theory, mathematical modeling studies have been carried out to define optimal strategies involving various interventions such as limited vaccine supply [7] and social distancing [8]. Combined models of antiviral treatment and social distancing, or vaccination and antiviral treatment have also been proposed for influenza control and the application of optimal control theory. In addition, age-structured models of influenza transmission have indicated that optimal vaccine allocations differ markedly between age groups because both the risk of infection and its severity are dependent on age [9, 10].

Some authors discussed the problem of chaos and stability analysis of some biological models such as cancer and tumor model, genital herpes epidemic, chaotic and hyperchaotic systems and many other models, see, for example, [11, 12, 13, 14, 18]. In this paper, two control strategies are considered for optimal control of the diffusive epidemic model for H1N1 influenza. First, we

investigate the impact of a vaccination campaign on the spread of the influenza epidemy. The second proposed control strategy is to design the optimal control inputs u_1, u_2, u_3, and u_4 such that the state trajectories tend to the unstable equilibrium point $E_0(1, 0, 0, 0)$ in a given finite time interval $[0, t_f]$. Furthermore, an adaptive control law and a parameter estimation update law are introduced for the epidemic model for H1N1 influenza with completely unknown parameters. The global asymptotic stability of a diffusive epidemic model are derived.

The paper is organized as follows: In Section 2, we analyze the dynamics of the model and stability analysis. Section 3 is devoted to the optimal control of the system with time dependent controls. In Section 4, the dynamic estimators of uncertain parameters in the diffusive epidemic model system is investigated based on the Lyapunov stability theory from the conditions on the asymptotic stability of this system about its steady states. In Section 5, we summarize the main results obtained in this paper.

2. Model and Stability Analysis

The model for influenza [19] consists of a system of nonlinear ordinary differential equations, where population is divided into four subgroups: susceptible (those at risk of contracting the disease), exposed (those who are infected but not yet infectious), infective (those who are infectious and capable of transmitting the disease), and recovered (those who have not attained permanent immunity). It has been assumed that only susceptible populations are affected by the infectious populations. Since recovery does not give immunity, individuals move from the susceptible-exposed-infectious class to the susceptible class upon recovery when the temporary immunity disappears. The model consists of the following system of ordinary differential equations:

$$\dot{S} = -\beta \frac{IS}{N} - \mu S + rN + \delta R \tag{1}$$

$$\dot{E} = \beta \frac{IS}{N} - (\mu + \sigma + \kappa)E \tag{2}$$

$$\dot{I} = \sigma E - (\mu + \alpha + \gamma)I \tag{3}$$

$$\dot{R} = \kappa E + \gamma I - \mu R - \delta R \tag{4}$$

$$S + E + I + R = N, \tag{5}$$

where the variables S, E, I and R represent the proportion of the populations in each of the four categories: susceptible individuals, exposed individuals, infected individuals and recovered individuals, respectively. Here N represents

the total population. The parameters representation is as: β the transmission coefficient of the disease; μ, the natural mortality rate; r, the birth rate; σ^{-1}, the incubation period; κ and γ, the recovery rate for both exposed and infected populations; α, the disease induced morality rate and δ^{-1}, the loss of immunity period.

In terms of the dimensionless proportions of susceptible, exposed, infectious and recovered individuals it is assumed that $\mu = (r - \alpha\frac{I}{N}), s = \frac{S}{N}, e = \frac{E}{N}, i = \frac{I}{N}$ and $r_1 = \dfrac{R}{N}$. After some manipulations and replacing s by S, e by E, i by I and r_1 by R Eqs. (1)-(5) can be written as

$$\begin{aligned} \dot{S} &= -\beta IS + \alpha IS - rS + \delta R + r \\ \dot{E} &= \beta IS - (\sigma + \kappa + r)E + \alpha IE \\ \dot{I} &= \sigma E - (\alpha + \gamma + r)I + \alpha I^2 \\ \dot{R} &= \kappa E + \gamma I - rR + \alpha RI - \delta R \\ S + E + I + R &= 1. \end{aligned} \tag{6}$$

Table 1: Parameters used in the numerical solution.

parameter	biological meaning	value
β	transmission cofficient	0.514000000
σ^{-1}	mean duration of latency (days)	2.000000000
γ^{-1}	mean recovery time for clinically ill(days)	5.000000000
δ^{-1}	duration of immunity loss(days)	365.0000000
μ	natural mortality rate per day	5.500×10^{-8}
r	birth rate per day	7.140×10^{-5}
κ	recovery rate of latents per day	1.857×10^{-4}
α	flu induced, mortality rate per day	9.300×10^{-6}
ϵ	degree of seasonality	0.500000000

2.1 Equilibria and stability analysis

In what follows, we examine the behavior of the trajectories of the system (6) near the equilibrium points. We now investigate the linear stability analysis of disease-free equilibrium, $E_0 = (1, 0, 0, 0)$.

Proposition 2.1. The equilibrium point E_0 is unstable when $\alpha = 9.3 \times 10^{-6}$, $\beta = 0.514$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{365}$, $\kappa = 1.857e - 4$, $\sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$.

Proof. The Jacobian matrix of given by the system (6) is :

$$J = \begin{bmatrix} -r & 0 & (\alpha-\beta) & \delta \\ 0 & -(\sigma+\kappa+r) & \beta & 0 \\ 0 & \sigma & -(\alpha+\gamma+r) & 0 \\ 0 & \kappa & \gamma & -r-\delta \end{bmatrix}. \tag{7}$$

The Jacobian matrix at equilibrium $E_0(1,0,0,0)$ is obtained as:

$$J(E_0) = \begin{bmatrix} -0.0000714 & 0 & -0.514 & 0.0027 \\ 0 & -0.5001 & 0.514 & 0 \\ 0 & .5000 & -.2001 & 0 \\ 0 & .0002 & .2000 & -0.0028 \end{bmatrix}, \tag{8}$$

which has the eigenvalues:

$$\lambda_1 = -0.0001,\ \lambda_2 = -0.0028,\ \lambda_3 = 0.1786,\ \lambda_4 = -0.8788. \tag{9}$$

It is observed that the eigenvalue λ_3 is positive. According to the Lyapunov theorem, the equilibrium point E_0 is unstable.

The influenza system can exhibit limit cycles, quasi-periodic and chaotic attractors. Fig. 1 illustrate the oscillatory behavior of the system (6). In the following we display different limit cycles and attractors of the influenza model without control. Such limit cycles and attractors in Figs. 2 and 3 respectively agree well with the previous stability analysis that indicates that the system has a chaotic behavior. Therefore, it is useful to study the problem of optimal control for the interested model as will presented in the following section.

3. Optimal Control

In this section, we use the optimal control theory to analyze the behavior of the model (6). Two different control strategies are explored.

3.1 The first optimal control problem

We suggest $u(t)$ for control of the disease in infectious and exposed populations that can be vaccination and quarantine. The mathematical system with control is given by the nonlinear differential equations:

$$\begin{cases} \dot{S} = -(1-u)\beta IS + \alpha IS - rS + \delta R + r \\ \dot{E} = (1-u)\beta IS - (\sigma+\kappa+r)E + \alpha IE \\ \dot{I} = \sigma E - (\alpha+\gamma+r)I + \alpha I^2 \\ \dot{R} = \kappa E + \gamma I - rR + \alpha RI - \delta R. \end{cases} \tag{10}$$

Our goal is to reduce the infected individuals, exposed individuals and the cost of vaccination. Mathematically, the problem is to minimize the objective functional

$$J = \int_0^T [CI(t) + \tfrac{W}{2}u^2(t)]dt. \tag{11}$$

Our problem is to find optimal control $u(t)$, such that

$$J(u^*(t)) = \min_U J(u(t)), \tag{12}$$

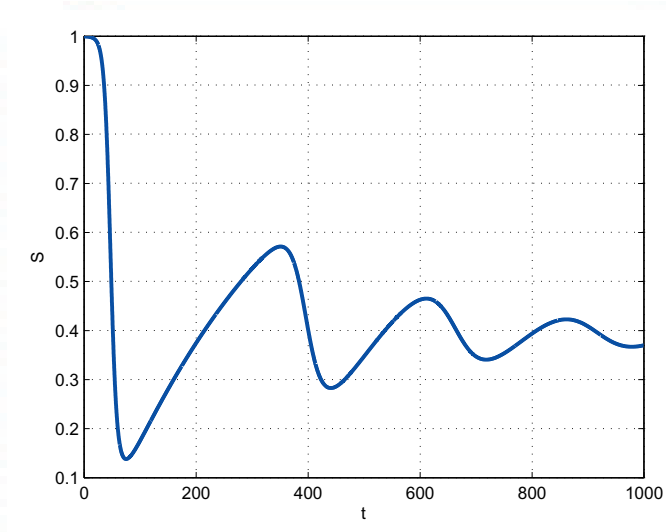
where U is the set of admissible controls defined by

$$U = \{u(t)|0 \leqslant u(t) \leqslant 1 \ , \ t \in [0,T]\}.$$

The necessary conditions that optimal solution must satisfy are derived from Pontryagin's Maximum Principle. The problem of minimizing Hamiltonian $\mathcal{H}$ given by

$$\begin{aligned}\mathcal{H} = CI + \frac{W_1}{2}u^2(t) + q_1(t)[-(1-u)\beta IS + \alpha IS - rS + \delta R + r] \\ + q_2(t)[(1-u)\beta IS - (\sigma+\kappa+r)E + \alpha IE] + q_3(t)[\sigma E - (\alpha+\gamma+r)I \\ + \alpha I^2] + q_4(t)[\kappa E + \gamma I - rR + \alpha RI - \delta R].\end{aligned} \tag{13}$$

By applying the Pontryagin's maximum principle [20] and the existence result of optimal control from [21], we obtain the following theorem:



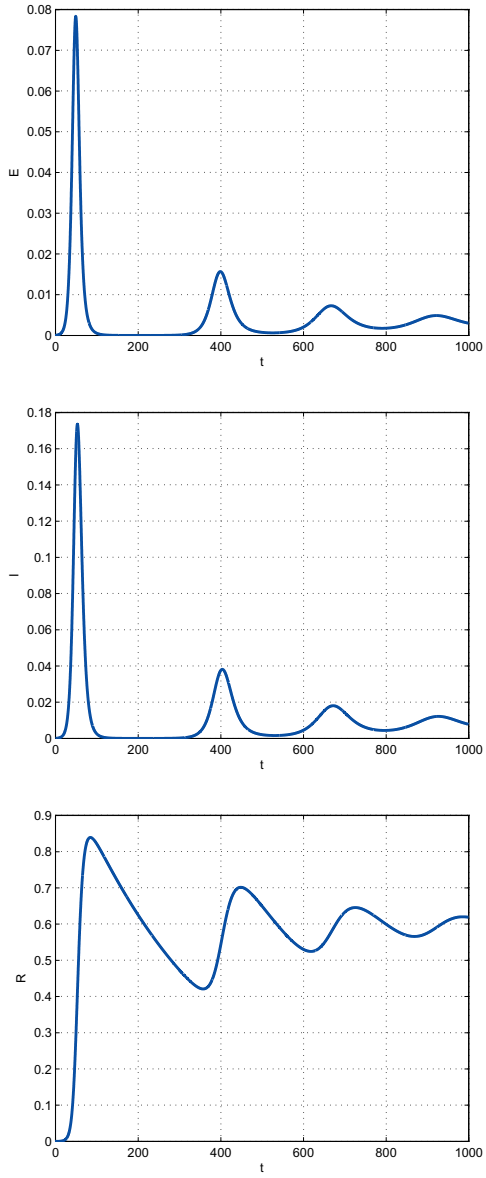


Figure 1. The densities of the Susceptible, Exposed, Infected and Recoverd populations for $\alpha = 9.3 \times 10^{-6}$, $\beta = 0.514$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{365}$, $\kappa = 1.857 \times 10^{-4}$, $\sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$ and the initial densities $S(0) = 0.9999, E(0) = 0, I(0) = 0.0001, R(0) = 0.$

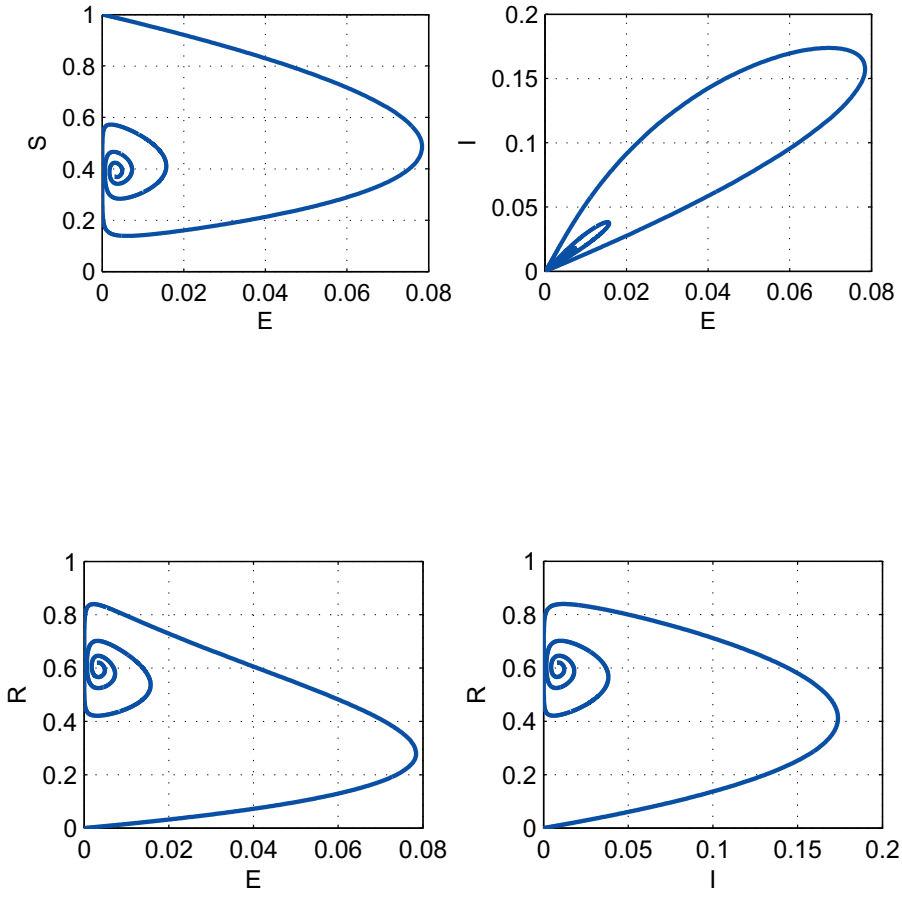


Figure 2. The limit cycles for $\alpha = 9.3 \times 10^{-6}$, $\beta = 0.514$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{365}$, $\kappa = 1.857 \times 10^{-4}$, $\sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$ and the initial densities $S(0) = 0.9999, E(0) = 0, I(0) = 0.0001, R(0) = 0$

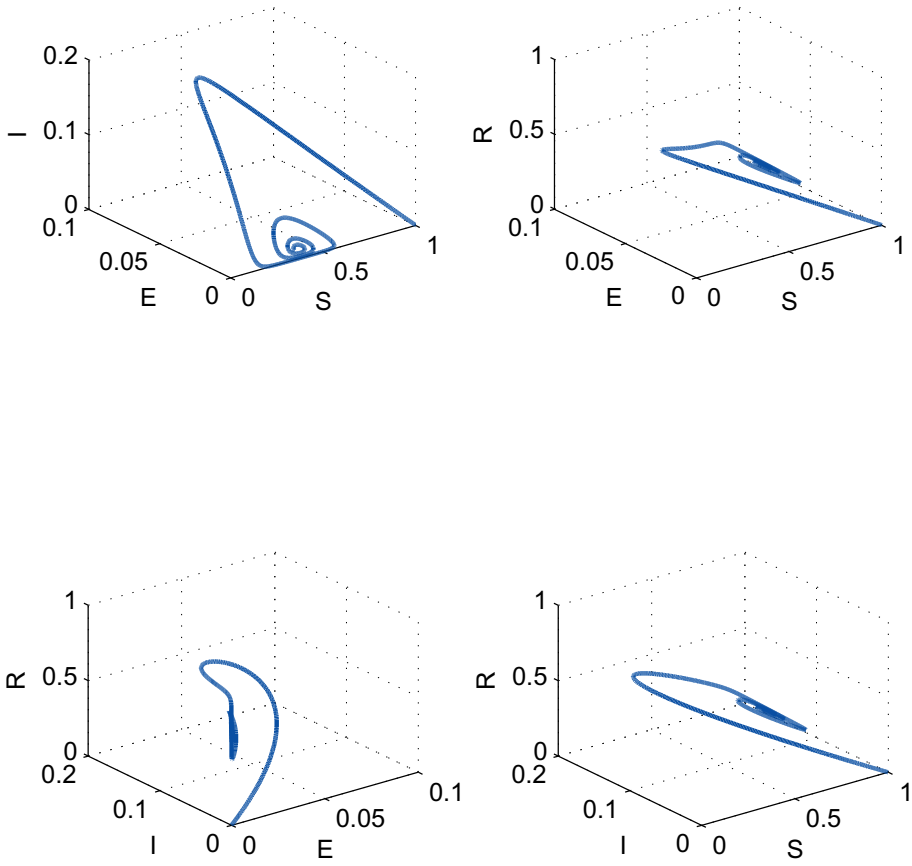


Figure 3. Influenza 3D dimensional phase plot which represent the attractors for $\alpha = 9.3 \times 10^{-6}$, $\beta = 0.514$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{365}$, $\kappa = 1.857 \times 10^{-4}$, $\sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$ and the initial densities $S(0) = 0.9999, E(0) = 0, I(0) = 0.0001, R(0) = 0$

Theorem 3.1. *There exist optimal control $u^*(t)$ corresponding solutions S^*, E^*, I^* and R^* that minimizes $J(u(t))$ over U. In order for the above statement to be true, it is necessary that there exist continuous functions $q_i(t),\ i = 1, ..., 4$ such that*

$$\begin{cases} \dot{q_1}(t) = ((1-u)\beta I - \alpha I + r)q_1 - ((1-u)\beta I q_2 \\ \dot{q_2}(t) = ((\sigma + \kappa + r) - \alpha I)q_2 - \sigma q_3 - \kappa q_4 \\ \dot{q_3}(t) = -C + ((1-u)\beta S - \alpha S)q_1 - ((1-u)\beta S + \alpha E)q_2 \\ \qquad\quad + (-2\alpha I + (\alpha + \gamma + r)q_3 - (\gamma + \alpha R)q_4 \\ \dot{q_4}(t) = -\delta q_1 + (r - \alpha I + \delta)q_4, \end{cases} \tag{14}$$

with the transversality conditions,

$$q_i(T) = 0 \quad , i = 1, 2, 3, 4. \tag{15}$$

Furthermore, the optimal control $u^*(t)$ given by

$$u^* = \min\left\{\max\left\{0, \frac{\beta IS(q_2 - q_1)}{W_1}\right\}, 1\right\}. \tag{16}$$

Proof. The existence of optimal control can be proved by using the results from [21] (see Theorem 2.1). The adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle :

$\frac{dq_1(t)}{dt} = -\frac{\partial \mathcal{H}}{\partial S}$, $\frac{dq_2(t)}{dt} = -\frac{\partial \mathcal{H}}{\partial E}$, $\frac{dq_3(t)}{dt} = -\frac{\partial \mathcal{H}}{\partial I}$, $\frac{dq_4(t)}{dt} = -\frac{\partial \mathcal{H}}{\partial R}$,

with $q_i(T) = 0$ for $i = 1, \cdots, 4$ and evaluated at the optimal control and corresponding states, which results in the adjoint System (14). The Hamiltonian $\mathcal{H}$ is minimized with respect to the control at the optimal control, so we differentiate $\mathcal{H}$ with respect to $u(t)$ on the set U giving the following optimality condition:

$$\frac{\partial \mathcal{H}}{\partial u} = W_1 u + \beta IS(q_1 - q_2) = 0. \tag{17}$$

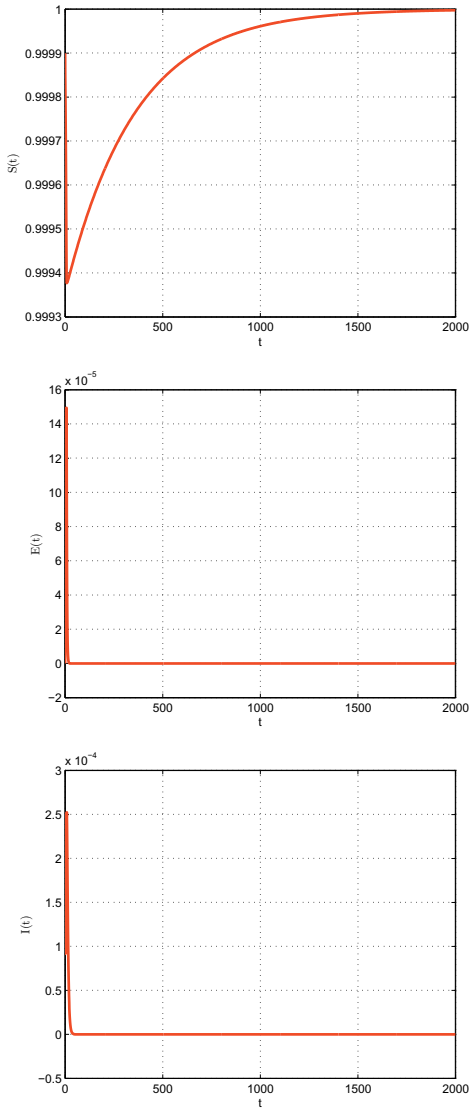
And therefore

$$u^* = \frac{\beta IS(q_2 - q_1)}{W_1}. \tag{18}$$

By standard variation arguments with the control bounds, we obtain the propertie:

$$u^* = \min\left\{\max\left\{0, \frac{\beta IS(q_2 - q_1)}{W_1}\right\}, 1\right\}. \quad \square \tag{19}$$

Fig. 4 shows a time optimal educational schedule for $T = 2000$ days. As a result of the decrease of the number of infectious, the susceptible population will increase and the recovered population will decrease. Fig. 5 shows the optimal treatment policy.



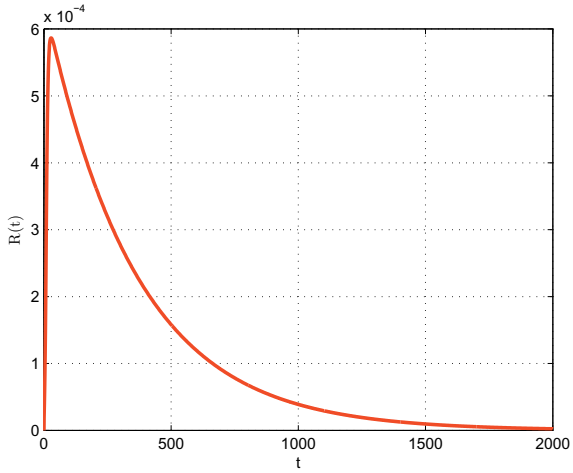


Figure 4. The optimal trajectories computed for Strategy 1 and using only one control function

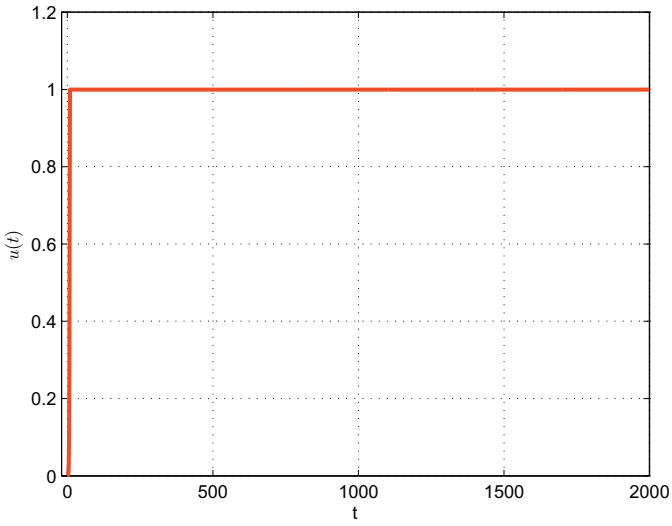


Figure 5. The optimal control functions as a function of time computed for Strategy 1.

3.2 The second optimal control problem

The main objective of this section is to arrive at a suitable mathematical formulation of the optimal control problem for the epidemic model for H1N1 influenza. Optimal control is defined by the admissible u_1, u_2, u_3 and u_4 which minimize the total cost given by

$$J = \frac{1}{2}\int_0^{t_f} (\alpha_1(S - S_f)^2 + \alpha_2(E - E_f)^2 + \alpha_3(I - I_f)^2 + \alpha_4(R - R_f)^2 + \beta_1 {u_1}^2 + \beta_2 {u_2}^2 + \beta_3 {u_3}^2 + \beta_4 {u_4}^2)dt, \tag{20}$$

subject to:

$$\begin{cases} \dot{S} = -\beta IS + \alpha IS - rS + \delta R + r + u_1(t), \\ \dot{E} = \beta IS - (\sigma + \kappa + r)E + \alpha IE + u_2(t), \\ \dot{I} = \sigma E - (\alpha + \gamma + r)I + \alpha I^2 + u_3(t), \\ \dot{R} = \kappa E + \gamma I - rR + \alpha RI - \delta R + u_4(t), \end{cases} \tag{21}$$

where α_j and β_j $(j = 1, 2, 3, 4)$ are positive constants, u_1, u_2, u_3 and u_4 are the control inputs, which will be satisfied the optimality conditions, obtained via the PMP. The proposed control strategy is to design the optimal control inputs u_1, u_2, u_3 and u_4 such that the state trajectories tend to an unstable equilibrium point in a given finite time interval $[0, t_f]$. Hence, the boundary conditions are considered as:

$$\begin{cases} S(0) = S_0, S(t_f) = S_f, \\ E(0) = E_0, E(t_f) = E_f, \\ I(0) = I_0, I(t_f) = I_f, \\ R(0) = R_0, R(t_f) = R_f, \end{cases} \tag{22}$$

where S_f, E_f, I_f and R_f denotes the coordinates of equilibrium point $E_0 = (1, 0, 0, 0)$.

To solve the above optimal control problem, we will derive the optimality conditions as a nonlinear two-point boundary value problem (TPBVP) via the PMP. In the following, we shall find it convenient to use the function $\mathcal{H}$, called the Hamiltonian, defined as:

$$\begin{aligned} \mathcal{H} = &-\frac{1}{2}(\alpha_1(S - S_f)^2 + \alpha_2(E - E_f)^2 + \alpha_3(I - I_f)^2 + \alpha_4(R - R_f)^2 \\ &+ \beta_1 {u_1}^2 + \beta_2 {u_2}^2 + \beta_3 {u_3}^2 + \beta_4 {u_4}^2) + \lambda_1[-\beta IS + \alpha IS - rS \\ &+ \delta R + r + u_1] + \lambda_2[\beta IS - (\sigma + \kappa + r)E + \alpha IE + u_2] \\ &+ \lambda_3[\sigma E - (\alpha + \gamma + r)I + \alpha I^2 + u_3] \\ &+ \lambda_4[\kappa E + \gamma I - rR + \alpha RI - \delta R + u_4], \end{aligned} \tag{23}$$

where λ_1, λ_2,λ_3 and λ_4 are the co-state variables. Using this notation, the optimality conditions can be written as follows:

$$\begin{cases} \dot{S} = \frac{\partial \mathcal{H}}{\partial \lambda_1}, \\ \dot{E} = \frac{\partial \mathcal{H}}{\partial \lambda_2}, \\ \dot{I} = \frac{\partial \mathcal{H}}{\partial \lambda_3}, \\ \dot{R} = \frac{\partial \mathcal{H}}{\partial \lambda_4} \end{cases} \tag{24}$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial \mathcal{H}}{\partial S}, \\ \dot{\lambda}_2 = -\frac{\partial \mathcal{H}}{\partial E}, \\ \dot{\lambda}_3 = -\frac{\partial \mathcal{H}}{\partial I}, \\ \dot{\lambda}_4 = -\frac{\partial \mathcal{H}}{\partial R}, \end{cases} \tag{25}$$

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial u_1} = 0, \\ \frac{\partial \mathcal{H}}{\partial u_2} = 0, \\ \frac{\partial \mathcal{H}}{\partial u_3} = 0, \\ \frac{\partial \mathcal{H}}{\partial u_4} = 0, \end{cases} \tag{26}$$

Substituting the Hamiltonian function $\mathcal{H}$ from (23) into (25), the co-state equations can be derived in the form:

$$\begin{aligned} \dot{\lambda}_1 &= \alpha_1(S - S_f) - \lambda_1(\alpha I - r - \beta I) - \lambda_2 \beta I, \\ \dot{\lambda}_2 &= \alpha_2(E - E_f) - \lambda_2[\alpha I - (\sigma + \kappa + r)] - \lambda_3 \sigma - \lambda_4 \kappa, \\ \dot{\lambda}_3 &= \alpha_3(I - I_f) - \lambda_1(\alpha S - \beta S) - \lambda_2(\beta S + \alpha E) - \lambda_3[2\alpha I - (\alpha + \gamma + r)] \\ &\quad - \lambda_4(\gamma + \alpha R), \\ \dot{\lambda_4} &= \alpha_4(R - R_f) - \lambda_1 \delta - \lambda_4(\alpha I - r - \delta), \end{aligned} \tag{27}$$

and to obtain the optimal control laws, we have

$$\begin{cases} \beta_1 u_1 - \lambda_1 = 0, \\ \beta_2 u_2 - \lambda_2 = 0, \\ \beta_3 u_3 - \lambda_3 = 0, \\ \beta_4 u_4 - \lambda_4 = 0. \end{cases} \tag{28}$$

Let us solve Eq. (28) to obtain the expressions for ${u_1}^*(t), {u_2}^*(t), {u_3}^*(t)$, and

$u_4{}^*(t)$; that is:

$$\begin{cases} u_1{}^* = \frac{\lambda_1}{\beta_1}, \\ u_2{}^* = \frac{\lambda_2}{\beta_2}, \\ u_3{}^* = \frac{\lambda_3}{\beta_3}, \\ u_4{}^* = \frac{\lambda_4}{\beta_4}. \end{cases} \tag{29}$$

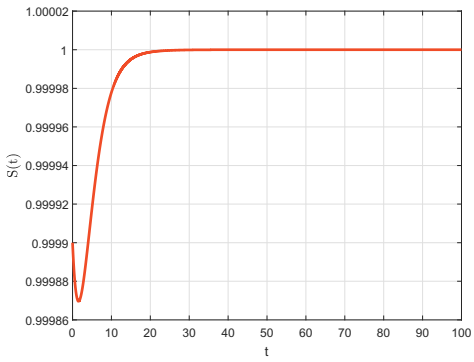
If these expressions are substituted into (21), we have a set of first order nonlinear ODEs as:

$$\begin{cases} \dot{S} = -\beta IS + \alpha IS - rS + \delta R + r - \frac{\lambda_1}{\beta_1}, \\ \dot{E} = \beta IS - (\sigma + \kappa + r)E + \alpha IE - \frac{\lambda_2}{\beta_2}, \\ \dot{I} = \sigma E - (\alpha + \gamma + r)I + \alpha I^2 \frac{\lambda_3}{\beta_3}, \\ \dot{R} = \kappa E + \gamma I - rR + \alpha RI - \delta R - \frac{\lambda_4}{\beta_4}, \\ \dot{\lambda}_1 = \alpha_1(S - S_{fi}) - \lambda_1(\alpha I - r - \beta I) - \lambda_2 \beta I, \\ \dot{\lambda}_2 = \alpha_2(E - E_{fi}) - \lambda_2[\alpha I - (\sigma + \kappa + r)] - \lambda_3 \sigma - \lambda_4 \kappa, \\ \dot{\lambda}_3 = \alpha_3(I - I_{fi}) - \lambda_1(\alpha S - \beta S) - \lambda_2(\beta S + \alpha E) \\ \qquad - \lambda_3[2\alpha I - (\alpha + \gamma + r)] - \lambda_4(\gamma + \alpha R), \\ \dot{\lambda}_4 = \alpha_4(R - R_{fi}) - \lambda_1 \delta - \lambda_4(\alpha I - r - \delta), \end{cases} \tag{30}$$

The boundary conditions for these equations are given by Eq. (22). Notice that, as expected, we are confronted by a nonlinear TPBVP. By solving this problem, we can obtain the optimal control law and the optimal state trajectories.

In the next section, we will discuss the numerical solution of the above-mentioned nonlinear TPBVP using the MATLAB in-built solver `bvp4c`, which is a finite difference code to solve TPBVPs.

In the following numerical simulations, the MATLAB's `bvp4c` in-built solver is used to solve the systems. Here, the initial values for the state variables are taken as $S(0) = 0.9999, E(0) = 0, I(0) = 0.0001, R(0) = 0$, and $\alpha_1 = 0.3,\ \alpha_2 = 5.5,\ \alpha_3 = 0.1,\ \alpha_4 = 1.001,\ \beta_1 = 50,\ \beta_2 = 3,\ \beta_3 = 5,\ \beta_4 = 10$. The behaviors of the state and control variables are displayed in Figs. 6 and 7.



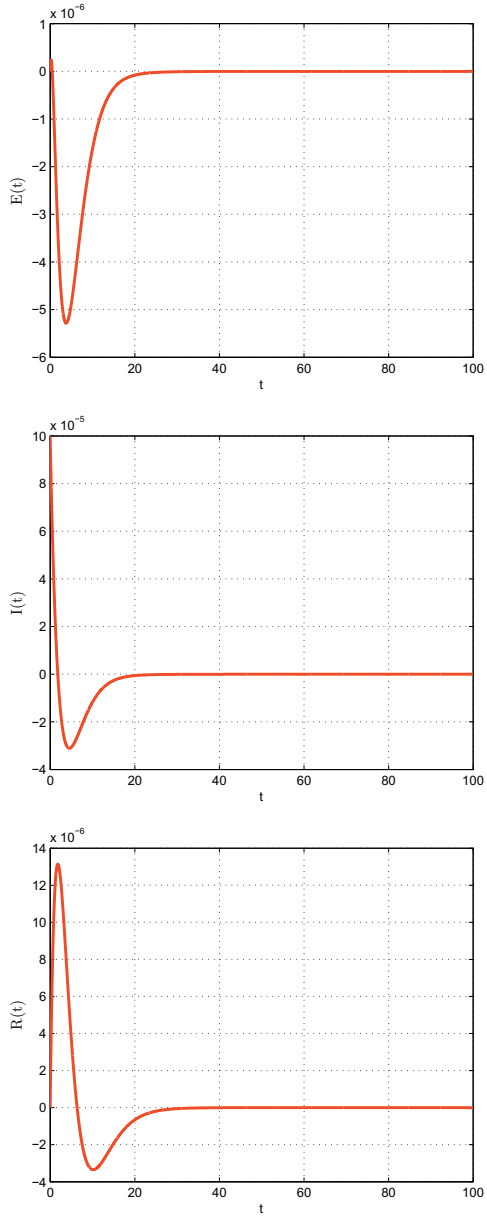


Figure 6. Time history of the state functions for equilibrium point E_0.

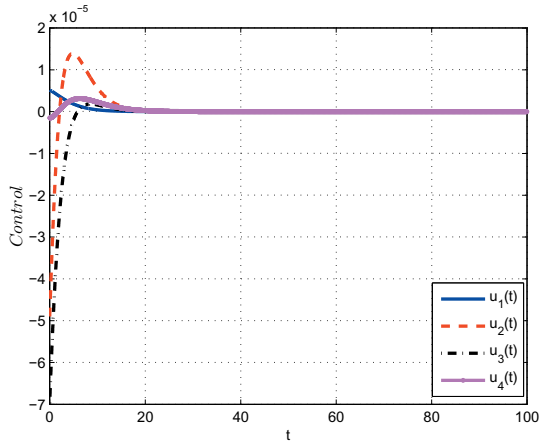


Figure 7. Time history of the parameter estimates for equilibrium point E_0.

4. Adaptive Control of the Chaotic and Hyper-Chaotic System

This section is concerned to study the adaptive control of the epidemic model for H1N1 influenza. In order to study the adaptive control of the epidemic model for H1N1 influenza using nonlinear feedback control approach, we start by assuming that the system (6) can be written in the following suitable form

$$\begin{cases} \dot{S} = -\beta IS + \alpha IS - rS + \delta R + r + v_1, \\ \dot{E} = \beta IS - (\sigma + \kappa + r)E + \alpha IE + v_2, \\ \dot{I} = \sigma E - (\alpha + \gamma + r)I + \alpha I^2 + v_3, \\ \dot{R} = \kappa E + \gamma I - rR + \alpha RI - \delta R + v_4, \end{cases} \tag{31}$$

where S, E, I and R are the states of the system, $\alpha, \beta, \gamma, \delta, \mu, \sigma, \kappa$ and r are unknown parameters of the system, and v_1, v_2, v_3 and v_4 are the adaptive controllers to be designed.

Theorem 4.1. *The controlled system (31) will asymptotically and globally converge to the unstable equilibrium point $E_0 = (1, 0, 0, 0)$, under the adaptive controller:*

$$\boldsymbol{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \beta_1 IS - \alpha_1 IS + r_1 S - \delta_1 R - r_1 - m_1(S - S_f) \\ -\beta_1 IS + (\sigma_1 + \kappa_1 + r_1)E - \alpha_1 IE - m_2(E - E_f) \\ -\sigma_1 E + (\alpha_1 + \gamma_1 + r_1)I - \alpha_1 I^2 - m_3(I - I_f) \\ -\kappa_1 E - \gamma_1 I + r_1 R - \alpha_1 RI + \delta_1 R - m_4(R - R_f) \end{bmatrix}, \tag{32}$$

and the following parameter estimation update law

$$\begin{cases} \dot{\alpha}_1 = IE(E-E_f) + IS(S-S_f) + I(I-I_f) + I^2(I-I_f) \\ \qquad + IR(R-R_f) + m_5(\alpha-\alpha_1), \\ \dot{\beta}_1 = IS(E-E_f) - IS(S-S_f) + m_6(\beta-\beta_1), \\ \dot{\delta}_1 = R(S-S_f) - R(R-R_f) + m_7(\delta-\delta_1), \\ \dot{\gamma}_1 = I(R-R_f) - I(I-I_f) + m_8(\gamma-\gamma_1), \\ \dot{\kappa}_1 = E(R-R_f) - E(E-E_f) + m_9(\kappa-\kappa_1), \\ \dot{\sigma}_1 = E(I-I_f) - E(E-E_f) + m_{10}(\sigma-\sigma_1), \\ \dot{r}_1 = (S-S_f) - S(S-S_f) - E(E-E_f) - I(I-I_f) \\ \qquad - R(R-R_f) + m_{11}(r-r_1), \end{cases} \tag{33}$$

where $\alpha_1, \beta_1, \gamma_1, \delta_1, \mu_1, \sigma_1, \kappa_1$ *and* r_1 *are estimate values of uncertain parameters* $\alpha, \beta, \gamma, \delta, \mu, \sigma, \kappa$ *and* r *and* $m_i (i = 1, \cdots, 11)$ *are positive constants, respectively.*

Proof. By substituting (32) into (31), we get the closed-loop system as

$$\begin{cases} \dot{S} = -(\beta-\beta_1)IS + (\alpha-\alpha_1)IS - (r-r_1)S + (\delta-\delta_1)R + (r-r_1) \\ \qquad - m_1(S-S_f), \\ \dot{E} = (\beta-\beta_1)IS - [(\sigma-\sigma_1) + (\kappa-\kappa_1) + (r-r_1)]E + (\alpha-\alpha_1)IE \\ \qquad - m_2(E-E_f), \\ \dot{I} = (\sigma-\sigma_1)E - [((\alpha-\alpha_1) + (\gamma-\gamma_1) + (r-r_1)]I + (\alpha-\alpha_1)I^2 \\ \qquad - m_3(I-I_f), \\ \dot{R} = (\kappa-\kappa_1)E + (\gamma-\gamma_1)I - (r-r_1)R + (\alpha-\alpha_1)RI - (\delta-\delta_1)R \\ \qquad - m_4(R-R_f). \end{cases} \tag{34}$$

For the derivation of the update law for adjusting the parameter estimates, the Lyapunov approach is used. We consider the following quadratic Lyapunov function

$$\begin{aligned} V(S,E,I,R,\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta},\tilde{\kappa},\tilde{\sigma},\tilde{r}) = \frac{1}{2}[(S-S_f)^2 + (E-E_f)^2 + (I-I_f)^2 \\ + (R-R_f)^2 + \tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{\gamma}^2 + \tilde{\delta}^2 + \tilde{\kappa}^2 + \tilde{\sigma}^2 + \tilde{r}^2], \end{aligned} \tag{35}$$

where $\tilde{\alpha} = \alpha - \alpha_1, \tilde{\beta} = \beta - \beta_1, \tilde{\gamma} = \gamma - \gamma_1, \tilde{\delta} = \delta - \delta_1, \tilde{\kappa} = \kappa - \kappa_1, \tilde{\sigma} = \sigma - \sigma_1$ and $\tilde{r} = r - r_1$.

The time derivative of the Lyapunov function V along with the trajectories of

(34), we obtain

$$\dot{V} = (S - S_f)\dot{S} + (E - E_f)\dot{E} + (I - I_f)\dot{I} + (R - R_f)\dot{R} + \tilde{\alpha}\dot{\tilde{\alpha}} + \tilde{\beta}\dot{\tilde{\beta}} + \tilde{\gamma}\dot{\tilde{\gamma}} + \tilde{\delta}\dot{\tilde{\delta}} + \tilde{\kappa}\dot{\tilde{\kappa}} + \tilde{\sigma}\dot{\tilde{\sigma}} + \tilde{r}\dot{\tilde{r}}, \tag{36}$$

then,

$$\begin{aligned}\dot{V} = {} & [(\alpha - \alpha_1)IS - (\beta - \beta_1)IS - (r - r_1)S + (\delta - \delta_1)R + (r - r_1) \\ & - m_1(S - S_f)](S - S_f) \\ & + [(\beta - \beta_1)IS - ((\sigma - \sigma_1) + (\kappa - \kappa_1) + (r - r_1))E + (\alpha - \alpha_1)IE \\ & - m_2(E - E_f)](E - E_f) \\ & + [(\sigma - \sigma_1)E - ((\alpha - \alpha_1) + (\gamma - \gamma_1) + (r - r_1))I + (\alpha - \alpha_1)I^2 \\ & - m_3(I - I_f)](I - I_f) \\ & + [(\kappa - \kappa_1)E + (\gamma - \gamma_1)I - (r - r_1)R + (\alpha - \alpha_1)RI - (\delta - \delta_1)R \\ & - m_4(R - R_f)](R - R_f) \\ & - (\dot{\alpha_1})(\alpha - \alpha_1) - (\dot{\beta_1})(\beta - \beta_1) - (\dot{\delta_1})(\delta - \delta_1) - (\dot{\gamma_1})(\gamma - \gamma_1) \\ & - (\dot{\kappa_1})(\kappa - \kappa_1) - (\dot{\sigma_1})(\sigma - \sigma_1) - (r_1)(r - r_1),\end{aligned} \tag{37}$$

hence, we have

$$\begin{aligned}\dot{V} = {} & -m_1(S - S_f)^2 - m_2(E - E_f)^2 - m_3(I - I_f)^2 - m_4(R - R_f)^2 \\ & + (\alpha - \alpha_1)[(S - S_f)IS + (E - E_f)IE - (I - I_f)I + (I - I_f)I^2 \\ & + (R - R_f)IR - \dot{\alpha_1}] + (\beta - \beta_1)[(E - E_f)IS - (S - S_f)IS - \dot{\beta_1}] \\ & + (\delta - \delta_1)[(S - S_f)R - (R - R_f)R - \dot{\delta_1}] + (\gamma - \gamma_1) \\ & [(R - R_f)I - (I - I_f)I - \dot{\gamma_1}] + (\kappa - \kappa_1)[(R - R_f)E - (E - E_f)E - \dot{\kappa_1}] \\ & + (\sigma - \sigma_1)[(I - I_f)E - E(E - E_f) - \sigma_1] + (r - r_1) \\ & [((S - S_f) - (S - S_f)S - (E - E_f)E - (I - I_f)I - (R - R_f)R - \dot{r_1}].\end{aligned} \tag{38}$$

Substituting (33) into (38), the time derivative of the Lyapunov function becomes

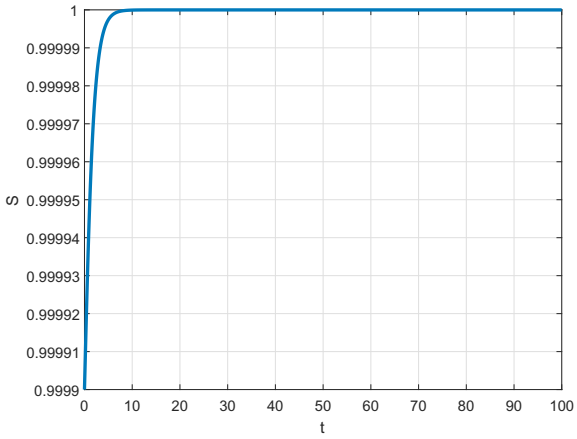
$$\begin{aligned}\dot{V} = {} & -m_1(S - S_f)^2 - m_2(E - E_f)^2 - m_3(I - I_f)^2 - m_4(R - R_f)^2 \\ & - m_5(\alpha - \alpha_1)^2 - m_6(\beta - \beta_1)^2 - m_7(\delta - \delta_1)^2 - m_8(\gamma - \gamma_1)^2 - m_9(\kappa - \kappa_1)^2 \\ & - m_{10}(\sigma - \sigma_1)^2 - m_{11}(r - r_1)^2 < 0.\end{aligned} \tag{39}$$

Since the Lyapunov function V is positive denite and its derivative $\dot{V}$ is negative denite in the neighborhood of the zero solution for system (31), according

to the Lyapunov stability theory, the equilibrium solution of the controlled system (31) is asymptotically stable, namely, the controlled system (31) can asymptotically converge to its equilibrium points with the adaptive control law (32) and the parameter estimation update law (33). This completes the proof.

For the numerical simulations, we solve the controlled novel chaotic system (31) with the adaptive control law (32) and the parameter update law (33). In the following numerical simulations, the MATLAB's `ode45` in-built solver is used to solve the systems. The initial values and system parameters are selected as $S(0) = 0.9999,\ E(0) = 0,\ I(0) = 0.0001,\ R(0) = 0,\ \alpha = 9.3 \times 10^{-6},\ \beta = 0.514,\ \gamma = \frac{1}{5},\ \delta = \frac{1}{365},\ \kappa = 1.857 \times 10^{-4},\ \sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$. For the adaptive and update laws, we take $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ m_4 = 1,\ m_5 = 1,\ m_6 = 1,\ m_7 = 8,\ m_8 = 1,\ m_9 = 1,\ m_{10} = 1,\ m_{11} = 1$.

Suppose that the initial values of the parameter estimates are chosen as $\alpha_1, \beta_1, \gamma_1, \delta_1, \mu_1, \sigma_1, \kappa_1$ and r_1. Figs. 8 and 9, show that the controlled chaotic system (31) converges to $E_0 = (1, 0, 0, 0)$ asymptotically with time. Also, these figures show that the parameter estimates $\alpha_1, \beta_1, \gamma_1, \delta_1, \mu_1, \sigma_1, \kappa_1$ and r_1 converge to the system parameter values $\alpha = 9.3 \times 10^{-6}$, $\beta = 0.514$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{365}$, $\kappa = 1.857 \times 10^{-4}$, $\sigma = \frac{1}{2}$ and $r = 7.14 \times 10^{-5}$ asymptotically with time.



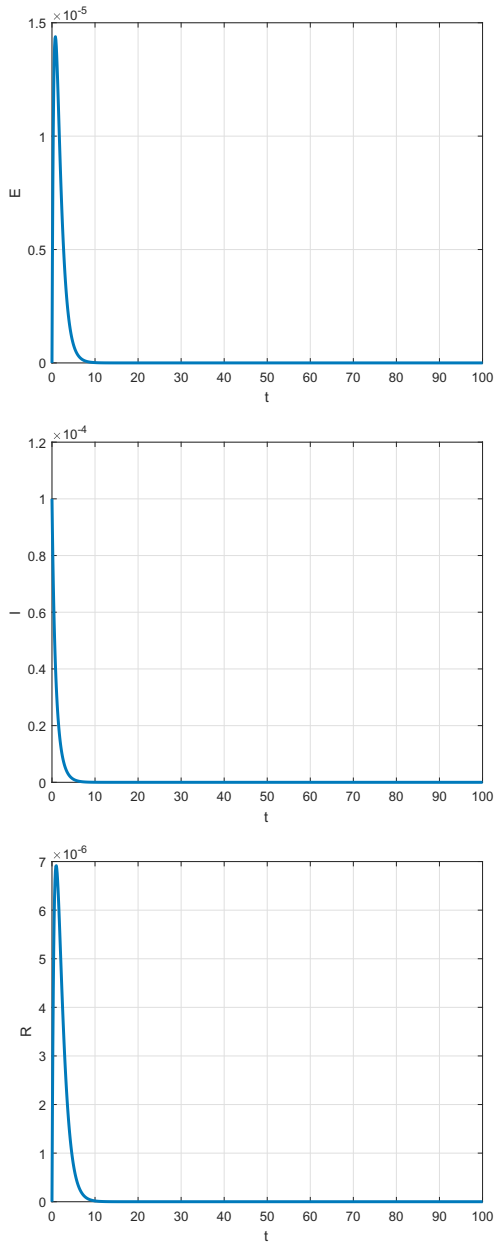


Figure 8. Time history of the state functions for equilibrium point E_0.

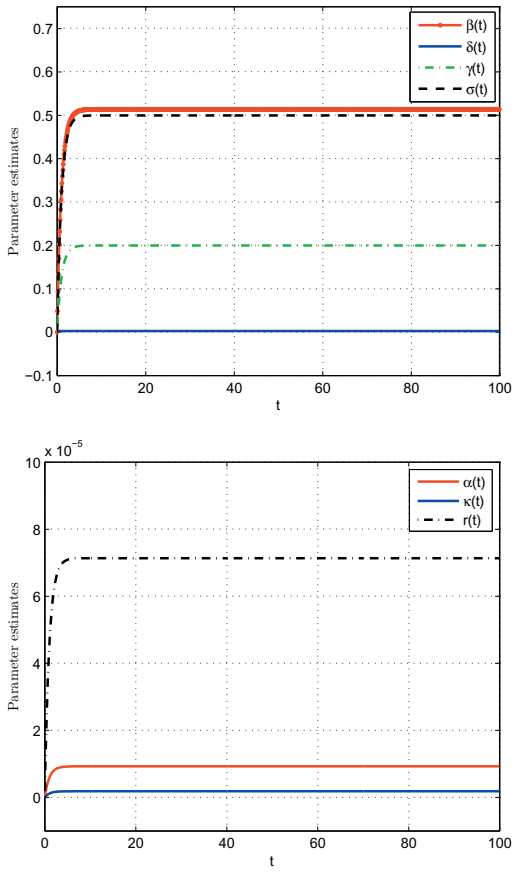


Figure 9. Time history of the parameter estimates for equilibrium point E_0.

5. Conclusion

We study the problem of optimal and adaptive control of a epidemic model of influenza with unknown parameters. The chaotic behavior of continuous time influenza model is investigated. We investigate the impact of a vaccination campaign on the spread of the influenza model. Also, based on the Pontryagin Minimum Principle (PMP), this system is stabilized to its equilibrium points. The stability and instability of the steady-states of this system are studied using the linear stability approach. In addition, we proposed Lyapunov stability to

control the chaotic system by a feedback control approach. In fact, we used the feedback control approach for estimating the system of unknown parameters.

Finally, extensive numerical examples and simulation are introduced.

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