Journal of Mathematical Extension
Vol. 11, No. 1, (2017), 85-98
ISSN: 1735-8299
URL: http://www.ijmex.com

A New Optimized Method for Solving Variable-Order Fractional Differential Equations

H. Hassani*
Shahrekord University

M. Sh. Dahaghin
Shahrekord University

M. H. Heydari
Fasa University

Abstract. In this paper, a new optimized method based on polynomials is proposed for solving variable-order fractional differential equations (V-FDEs) and systems of V-FDEs. To do this, a general polynomial of degree m with unknown coefficients is considered as an approximate solution for the problem under study. By using the initial conditions some of these unknown coefficients are obtained. Finally the rest of these unknown coefficients are obtained optimally by minimizing error of 2-norm of the approximate solution in a desired interval. In order to demonstrate the accuracy and efficiency of the proposed method some numerical examples are given.

AMS Subject Classification: 26A33; 65R10
Keywords and Phrases: Variable-order fractional differential equations (V-FDEs), optimized method, polynomials, free coefficients, fixed coefficients

Received: May 2016; Accepted: July 2016
*Corresponding author

1. Introduction

Fractional differential equations (FDEs) are generalized form integer order ones, which are obtained by replacing integer order derivatives by fractional order ones [1, 2]. FDEs have been successfully applied in various fields of physics and engineering such as biophysics, bioengineering, quantum mechanics, finance, control theory, image and signal processing, viscoelasticity and material sciences [3, 4]. The exact solutions of most FDEs can't be solved, so approximate and numerical techniques [5, 6, 7, 8], should be used. Several numerical and approximate methods such as variational iteration method [5], homotopy perturbation method [9], Adomian decomposition method [10], homotopy analysis method [11] and collocation methods in [12, 13], finite element method in [14], finite difference method [15, 16] and wavelets method [17, 18, 19, 20, 21] have been given in recent years to solve FDEs.
Variable-order fractional derivatives are an extension of constant-order fractional derivatives and have been introduced in several physical fields [17, 18, 19]. Many authors have introduced different definitions of variable-order differential operators, each of these with a specific meaning to suit desired goals. These definitions such as Riemann-Liouville, Grnwald, Caputo, Riesz [20, 21, 22], and some notes as Coimbra definition [23, 24]. Since the equations described by the variable-order derivatives are highly complex and also difficult to handle analytically, it is advisable to consider their numerical solutions. Although there exist enormous literatures on the numerical investigation for constant fractional order differential equations, the investigation of numerical methods of variable-order FDEs are quite limited. Several numerical methods have been proposed for V-FDEs in recent years, e.g. [25, 26, 27, 28, 29, 30, 31].
In this paper, we consider the general form of the V-FDEs as:

$$ {}_0^cD_t^{\alpha(t)}u(t) = f(t, u(t), {}_0^cD_t^{\alpha_1(t)}u(t), {}_0^cD_t^{\alpha_2(t)}u(t), \ldots, {}_0^cD_t^{\alpha_n(t)}u(t)), \quad (1) $$

on the interval $t \in [0, 1]$, subject to the initial conditions:

$$ u^{(i)}(0) = u_0^{(i)}, \qquad i = 0, 1, ..., q-1, \quad (2) $$

where q is the integer such that $q-1<\alpha(t)\leqslant q$, $0<\alpha_1(t)<\alpha_2(t)<\ldots<\alpha_n(t)<\alpha(t)$. Also, the real numbers $u_0^{(i)}$, $i=0,1,...,q-1$, are assumed to be given. Moreover, $D_t^{\alpha(t)}u(t)$ denotes the variable-order fractional derivative of order $\alpha(t)$ in the Caputo sense for $u(t)$, which is defined in [31, 32] by:

$$ {}_0^cD_t^{\alpha(t)}u(t)=\frac{1}{\Gamma(q-\alpha(t))}\int_0^t(t-s)^{q-\alpha(t)-1}\frac{d^qu(s)}{ds^q}ds,\quad t>0. \tag{3}$$

It is worth noting that there are two types of variable-order fractional differential definitions. One considers the derivative order without any memory related to past derivative order values, and the other with memory related to past derivative order values. In this study, we have adopted the definition of variable-order fractional derivatives in which the derivative order has no memory of past derivative order values. It should be also denoted that based on the definition expressed in Eq. (3) for any $q-1<\alpha(t)\leqslant q$, we have the following useful property [32]:

$$ {}_0^cD_t^{\alpha(t)}t^m=\begin{cases}\dfrac{\Gamma(m+1)}{\Gamma(m-\alpha(t)+1)}t^{m-\alpha(t)}, & q\leqslant m\in\mathbb{N},\\ 0, & o.\ w.\end{cases} \tag{4}$$

It is well-known that the Taylor series approximation for any analytic function $u(t)$ around $t=0$ is expressed as:

$$u(t)=\sum_{i=0}^{\infty}\frac{u^{(i)}(0)}{i!}\,t^i, \tag{5}$$

where $u^{(i)}(0)$ is the ith derivative of u evaluated at $t=0$.
It is worth to mention that in practical uses only a number of finite terms of the above series is considered as an approximation of the Taylor series.
The main aim of this paper is to propose an efficient and accurate optimized method based on the polynomials for solving V-FDEs. In particular the efficiency and reliability of the proposed technique will be demonstrated through extensive numerical analysis, considering several

examples with known exact solution.
The structure of the remainder of this paper is as follows: In Section 2, the proposed method is described for solving the problem under study. In Section 3, some numerical examples are given. Finally, a conclusion is drawn in Section 4.

2. Description of the Proposed Method

In this section, we apply a new optimized method based on polynomials to find approximate solutions for V-FDEs in Eq. (1).

2.1 Function approximation

Let $X = L^2[0,1]$, and assume that $P_m(t) = [1 \quad t \quad t^2 \quad \ldots \quad t^m]^T$, $Y_m = span\{1, t, t^2, \ldots, t^m\}$ and $\tilde{u}(t)$ be an arbitrary element in X. Since Y_m is a finite dimensional vector subspace of X, $\tilde{u}(t)$ has a unique best approximation out of Y_m such as $u_0(t) \in Y_m$, that is

$$\forall \ \hat{u}(t) \in Y_m, \qquad \| \tilde{u}(t) - u_0(t) \| \leqslant \| \tilde{u}(t) - \hat{u}(t) \| .$$

Since $u_0(t) \in Y_m$, there exist the unique coefficients $a_0,\ a_1, \ldots,\ a_m$, such that

$$\tilde{u}(t) \simeq u_0(t) = \sum_{i=0}^{m} a_i \ t^i = \ A^T \ P_m(t), \tag{6}$$

and

$$A^T = [a_0 \quad a_1 \quad \ldots \quad a_m].$$

Let the function $\tilde{u}(t)$ defined in Eq. (6) be an approximate solution for Eq. (1). By using the initial conditions in Eq. (2), we have

$$a_i = \frac{1}{i!}\tilde{u}^{(i)}(0), \qquad i = 0, 1, \ldots, q-1. \tag{7}$$

The coefficients $a_0, a_1, \ldots, a_{q-1}$ are chosen as the fixed coefficients. Now by replacing $\tilde{u}(t)$ and corresponding derivatives in Eq. (1), we define the residual function:

$$g(t, a_q, a_{q+1}, \ldots, a_m) = \\ {}_0^cD_t^{\alpha(t)}\tilde{u}(t) - f(t, \tilde{u}(t), {}_0^cD_t^{\alpha_1(t)}\tilde{u}(t), {}_0^cD_t^{\alpha_2(t)}\tilde{u}(t), \ldots, {}_0^cD_t^{\alpha_n(t)}\tilde{u}(t)), \tag{8}$$

and then the error function:

$$\mathbf{e}(a_q, a_{q+1}, \ldots, a_m) = \int_0^1 g^2(t, a_q, a_{q+1}, \ldots, a_m)dt. \tag{9}$$

Now to obtain an approximate solution for Eq. (1), we choose the free coefficients, $a_q, a_{q+1}, \ldots, a_m$ optimally. to do this, we solve the following system of algebraic equations:

$$\frac{\partial \mathbf{e}(a_q, a_{q+1}, \ldots, a_m)}{\partial a_i} = 0 \ , \qquad i = q, q+1, \ldots, m \ .$$

Theorem 2.1.1. *Let X be a normed space, (Y_n) be a sequence in X such that $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq X$ and $\overline{\cup Y_n} = X$. If $x \in X$ and y_n the best approximation of x in Y_n i.e. $d(x, y_n) = dist(x, Y_n)$, then $y_n \to x$.*

Proof. Since (Y_n) is an increasing sequence of sets, therefore

$$\|y_n - x\| = dist(x, Y_n)$$

is an decreasing sequence of sets in $[0, \infty)$ and this sequence has a limit. For example $\|y_n - x\| \to \alpha$ for some $\alpha \geqslant 0$. If $\alpha > 0$, then

$$dist(x, Y_n) = \|y_n - x\| \geqslant \|y_{n+1} - x\| \geqslant \cdots \geqslant \alpha > 0,$$

and

$$\forall n \in \mathbb{N}, \qquad dist(x, Y_n) \geqslant \alpha,$$

then

$$\forall n \in \mathbb{N}, \qquad \forall y \in Y_n, \qquad dist(x, Y_n) \geqslant \alpha.$$

Therefore

$$B(x, \frac{\alpha}{2}) \cap (\cup Y_n) = \emptyset.$$

So, $\cup Y_n$is not dense in X which contradicts to what we assume. This contradiction shows $\alpha = 0$. Thus $\|y_n - x\| \to \alpha = 0$ and then $y_n \to x$. $\square$

3. Numerical Results

The purpose of this section is to show that the proposed method designed in this paper provides good approximations for V-FDEs. It is worth mentioning that all numeric computation is performed by MAPLE software with enough decimal digits.

Example 3.1. Consider the following V-FDE:

$$ {}_{0}^{c}D_{t}^{\alpha(t)}u(t) + u(t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} - \frac{t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} + t^2 - t, \tag{10} $$

where $0 < \alpha(t) \leqslant 1$ and the initial condition is $u(0) = 0$. It can be verified that the exact solution for this problem is $u(t) = t^2 - t$. This problem solved by the proposed method for $\alpha(t) = 1 - 0.5e^{-t}$. We estimate $u(t)$ by truncation Eq. (6) after the five terms as:

$$ \tilde{u}(t) = A^T P_4(t), \tag{11} $$

where $A^T = [a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4]$ and $P_4(t) = [1 \;\; t \;\; t^2 \;\; t^3 \;\; t^4]^T$.
Coefficient a_0 is chosen as the fixed coefficient, and by the initial condition we have, $a_0 = 0$. Coefficients a_1, a_2, a_3 and a_4 are chosen as the free coefficients. Therefore the approximate solution for Eq. (10) is given as:

$$ \tilde{u}(t) = A^T P_4(t) = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4. \tag{12} $$

Substitute Eq. (12) into Eq. (10) and define the residual function:

$$ g(t, a_1, a_2, a_3, a_4) = {}_{0}^{c}D_{t}^{\alpha(t)}\tilde{u}(t) + \tilde{u}(t) - \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \frac{t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} - t^2 + t, $$

and then the error function:

$$ \mathbf{e}(a_1, a_2, a_3, a_4) = \int_0^1 g^2(t, a_1, a_2, a_3, a_4)\,dt. $$

The values for the free coefficients are obtained by minimizing $\mathbf{e}(a_1, a_2, a_3, a_4)$ as:

$$ \frac{\partial \mathbf{e}(a_1, a_2, a_3, a_4)}{\partial a_i} = 0 \;, \quad i = 1, 2, 3, 4. $$

By solving the above system of algebraic equations the free coefficients are obtained as:

$$a_1 = -1, \qquad a_2 = 1, \qquad a_3 = 0, \qquad a_4 = 0,$$

and therefore, we gain the exact solution.

Example 3.2. Consider the following V-FDE:

$$\begin{aligned} {}_0^cD_t^{\alpha(t)}u(t) + \sin t\ {}_0^cD_t^{\beta(t)}u(t) + \cos t\ u(t) \\ = \frac{6t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))} + \frac{6\sin t\ t^{3-\beta(t)}}{\Gamma(4-\beta(t))} + t^3\cos t, \end{aligned} \tag{13}$$

where the initial conditions are $u(0) = u^{'}(0) = 0$, and $1 < \alpha(t) \leqslant 2,\ 0 < \beta(t) \leqslant 1$. The exact solution for this problem is $u(t) = t^3$. This problem is also solved by the proposed method for $\alpha(t) = 2 - \sin^2(t)$ and $\beta(t) = 1 - \frac{e^{-t^3}}{6}$. We consider $m = 4$ in Eq. (6) for approximate solution of Eq. (13) as:

$$\tilde{u}(t) = A^T P_4(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4.$$

By the initial conditions, we obtain the fixed coefficients:

$$a_0 = \tilde{u}(0) = 0 \quad , \quad a_1 = \tilde{u}^{'}(0) = 0.$$

Coefficients a_2, a_3 and a_4 are chosen as the free coefficients. By applying the same process in Example 3., the free coefficients are obtained as followig:

$$a_2 = 0, \qquad a_3 = 1, \qquad a_4 = 0,$$

and therefore we gain the exact solution.

Example 3.3. Consider the following nonlinear problem:

$$\begin{aligned} {}_0^cD_t^{\alpha(t)}u(t) + e^t\ {}_0^cD_t^{\beta(t)}u(t) + \frac{2}{2t-1}\ {}_0^cD_t^{\gamma(t)}u(t) + \sqrt{t}\ (u(t))^2 \\ = \frac{2e^t t^{2-\beta(t)}}{\Gamma(3-\beta(t))} + \frac{4t^{2-\gamma(t)}}{(2t-1)\Gamma(3-\gamma(t))} + t^{\frac{9}{2}}, \end{aligned}$$

where $2 < \alpha(t) \leqslant 3$, $1 < \beta(t) \leqslant 2$ and $0 < \gamma(t) \leqslant 1$, with the initial conditions $u(0) = u'(0) = 0$, $u''(0) = 2$. The exact solution for this problem is $u(t) = t^2$. We also solve this problem by the proposed method for $\alpha(t) = 3 - \frac{1}{3}e^{-t}$, $\beta(t) = 2 - \cos^2 t$ and $\gamma(t) = 1 - \frac{1}{2}\cos t$. Consider the truncation of Eq. (6) with first four terms as:

$$\tilde{u}(t) = A^T P_3(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

By the initial conditions, we obtain the fixed coefficients:

$$a_0 = \tilde{u}(0) = 0 \quad , \quad a_1 = \tilde{u}^{'}(0) = 0 \quad , \quad a_2 = \frac{\tilde{u}^{''}(0)}{2} = 1.$$

Coefficient a_3 is chosen as the free coefficient. By applying the same process in example 1, the free coefficient is obtained as:

$$a_3 = 0.$$

Thus, we get $\tilde{u}(t) = t^2$, which is the exact solution.

Example 3.4. Consider the following nonlinear V-FDE:

$${}_{0}^{c}D_{t}^{\alpha(t)} u(t) + \sin(t)(u(t))^2 = f(t),$$

where $f(t) = \frac{\Gamma(\frac{9}{2})\ t^{\frac{7}{2}-\alpha(t)}}{\Gamma(\frac{9}{2}-\alpha(t))} + \sin(t)\ t^7$, $0 < \alpha(t) \leqslant 1$ and the initial condition is $u(0) = 0$. The exact solution for this problem is $u(t) = t^{\frac{7}{2}}$. This problem solved by the proposed method for $\alpha(t) = 1 - 0.5e^{-t}$. To solve this problem, we estimate $u(t)$ by consider Eq. (6) with $m = 6$ as:

$$\tilde{u}(t) = A^T\ P_6(t),$$

where $A^T = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6]$ and $P_6(t) = [1\ t\ t^2\ t^3\ t^4\ t^5\ t^6]^T$. The coefficient a_0 is chosen as the fixed coefficient, and by the initial condition we have $a_0 = 0$. The coefficients $a_1, a_2, a_3, a_4, a_5, a_6$ are chosen as the free coefficients. If we use the same process in Example 3., the free coefficients $a_1, a_2, a_3, a_4, a_5, a_6$ are obtained as follows:

$$a_1 = 0.001145942683, \qquad a_2 = -0.027441178290, \qquad a_3 = 0.396466775615,$$
$$a_4 = 0.843570591951, \qquad a_5 = -0.273010798021, \qquad a_6 = 0.059278716669.$$

From Theorem 2.1, $\tilde{u}(t)$ tends to exact solution of the problem under study.
The graph of the absolute error for this problem by the presented method for $m = 6$ is shown in Fig. 1. From Fig. 1, it can be seen that the proposed method provides a good approximate solution for this problem.

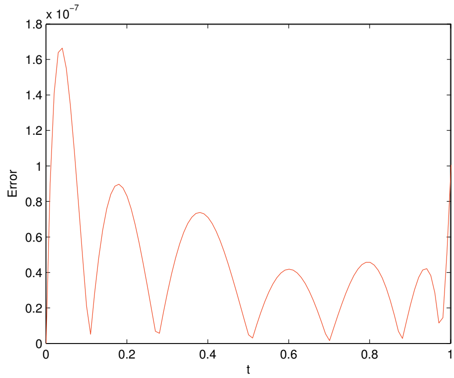

Figure 1. The graph of the absolute error for Example 3.4.

Example 3.5. Consider the following system of V-FDEs:

$$\begin{cases} {}_0^cD_t^{\alpha(t)}u(t) + v(t) = \dfrac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + t^3 \\ {}_0^cD_t^{\beta(t)}v(t) - t^2u(t) = \dfrac{6t^{3-\beta(t)}}{\Gamma(4-\alpha(t))} - t^4 \end{cases} \tag{14}$$

where $0 < \alpha(t), \beta(t) \leqslant 1$, with the initial conditions $u(0) = v(0) = 0$. The exact solutions are $u(t) = t^2$ and $v(t) = t^3$. We also have solved this system by the proposed method for $\alpha(t) = 1 - \cos^2(t)$ and $\beta(t) = 1 - \frac{1}{5}e^{-t}$. Consider the truncation of Eq. (6) with first four terms for approximate solution of Eq. (14) as:

$$\tilde{u}(t) = A^T P_3(t), \qquad \tilde{v}(t) = B^T Q_3(t),$$

where $A^T = [a_0 \quad a_1 \quad a_2 \quad a_3]$, $P_3(t) = [1 \quad t \quad t^2 \quad t^3]^T$, $B^T = [b_0 \quad b_1 \quad b_2 \quad b_3]$ and $Q_3(t) = [1 \quad t \quad t^2 \quad t^3]^T$.
By applying the initial conditions, we obtain the fixed coefficients as:

$$a_0 = \tilde{u}(0) = 0 \quad , \quad b_0 = \tilde{v}(0) = 0.$$

Coefficients a_1, a_2, a_3, b_1, b_2, b_3 are chosen as the free coefficients. Define the error function:

$$\mathbf{e}(a_1, a_2, a_3, b_1, b_2, b_3) =$$
$$\int_0^1 \Big(g_1^2(t, a_1, a_2, a_3, b_1, b_2, b_3) + g_2^2(t, a_1, a_2, a_3, b_1, b_2, b_3) \Big) dt,$$

where

$$\begin{cases} g_1(t, a_1, a_2, a_3, b_1, b_2, b_3) = {}_0^cD_t^{\alpha(t)}\tilde{u}(t) + \tilde{v}(t) - \dfrac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} - t^3 \\ g_2(t, a_1, a_2, a_3, b_1, b_2, b_3) = {}_0^cD_t^{\beta(t)}\tilde{v}(t) - t^2\tilde{u}(t) - \dfrac{6t^{3-\beta(t)}}{\Gamma(4-\alpha(t))} + t^4. \end{cases}$$

The values for the free coefficients are obtained by minimizing $\mathbf{e}(a_1, a_2, a_3, b_1, b_2, b_3)$ as following:

$$a_1 = 0 \quad , \quad b_1 = 0 \quad , \quad a_2 = 1 \quad , \quad b_2 = 0 \quad , \quad a_3 = 0 \quad , \quad b_3 = 1.$$

Thus, we get $\tilde{u}(t) = t^2$ and $\tilde{v}(t) = t^3$, which are the exact solutions.

4. Conclusion

In this paper, a class of variable order fractional differential equations (V-FDEs) solved by using an efficient and accurate computational method based on polynomials. The proposed method is very convenient for solving such problems and also requires less computational work to obtain an approximate solution for the problem under study. The main advantage of the proposed method is its fast convergence to the exact solution. Several numerical examples provided to demonstrate the powerfulness of the proposed method. Also, this method has been successfully applied to calculate the approximate solutions for systems of V-FDEs.

References

[1] A. Atangana and D. Baleanu, Numerical solution of a kind of fractional parabolic equations via two difference schemes, *Abstr. Appl. Anal.*, 2013 (2013), 828764.

[2] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, *J. Comout. Phy.*, 293 (2015), 104-114.

[3] T. M. Atanackovic, S. Pilipovic, B. Stankovic, and D. Zorica, Fractional Calculus with Applications in Mechanics, *Wiley, London,* 2014.

[4] Y. Chen, M. Yi, and C. Yu, Error analysis for numerical solution of fractional differential equations by Haar Wavelets Method, *J. Comput. Sci.*, 3 (5) (2012), 367-373.

[5] Y. Chen, L. Liu, B. Li, and Y. Sun, Numerical solution for the variable order linear cable equation with bernstein polynomials, *Appl. Math. Comput.*, 238 (2014), 329-341.

[6] Y. M. Chen, Y. Q. Wei, D. Y. Liu, and H. Yu, Numerical solution for a class of nonlinear variable order fractional differential equations with legendre wavelets, *Appl. Math. Lett.*, 46 (2015), 83-88.

[7] C. F. M. Coimbra and L. E. S. Ramirez, A variable order constitutive relation for viscoelasticity, *Annal. D. Phys.*, 16 (7-8) (2007), 543-552.

[8] C. F. M. Combra, Mechanics with variable-order differential operators, *Annal. D. Phys.*, 12 (11-12) (2003), 692-703.

[9] R. Hilfer, *Applications of Fractional Calculus to Physics,* World Scientific, Singapore, 2000.

[10] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and F. Fereidouni, Two-dimensional legendre wavelets for solving fractional poisson equation with dirichlet boundary conditions, *Engineering Analysis with Boundary Elements*, 37 (11) (2013). 1331-1338.

[11] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi, and C. Cattani, Wavelets method for solving systems of nonlinear singular fractional volterra integro-differential equations, *Commun Nonlinear Sci Numer Simulat*, 19 (2014). 37–48.

[12] M. H. Heydari, M. R. Hooshmandasl, and F. Mohammadi, Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions, *Appl. Math. Comput.*, 234 (2014), 267–276.

[13] M. H. Heydari, M. R. Hooshmandasl, and F. Mohammadi, Two-dimensional legendre wavelets for solving time-fractional telegraph equation, *Advances in Applied Mathematics and Mechanics*, 6 (2) (2014), 247–260.

[14] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and C. Cattani, Wavelets method for the time fractional diffusion-wave equation, *Physics Letters A.*, 379 (2015), 71–76.

[15] D. Ingman and J. Suzdalnitsky, Control of damping oscillations by fractional differential operator with time-dependent order, *Comput. M. Appl. M. E.*, 193 (2004), 5585-5595.

[16] M. M. Khader, Introducing an efficient modification of the homotopy perturbation method by using Chebyshev polynomials, *Arab J. Math. Sci.,* 18 (2012), 61-71.

[17] M. M. Khader, Introducing an efficient modification of the variational iteration method by using Chebyshev polynomials, *Appl. Math. Int. J.*, 7 (1) (2012), 283-299.

[18] M. M. Khader and A. S. Hendy, The approximate and exact solutions of the fractional-order dealy differential equations using Legendre pseudospectral method, *Int. J. Pure Appl. Math.*, 74 (3) (2012), 287-297.

[19] Y. Liu, Z. Fang, H. Li, and S. He, A mixed finite element method for a time fractional fourth-order partial differential equation, *Appl. Math. Comput.*, 243 (2014), 703-717.

[20] C. F. Lorenzo and T. T. Hartley, *The Vector Linear Fractional Initialization Problem,* National Aeronautics and Space Administration, Glenn Research Center, 1999.

[21] R. Lin, F. Liu, V. Anh, and I. Turner, Stability and convergence of a new explicit FDM for the variable-order nonlinear fractional diffusion equation, *Appl. Math. Comput.*, 212 (2009), 435-445.

[22] S. Momani and N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.*, 182 (2006), 1083-1092.

[23] M. G. p. Manohar, Mixed method for numerical solution of space-time fractional diffusion-wave equations with three space variables, *Afr. Math.*, 25 (2014), 161-181.

[24] H. T. C. Pedro, M. H. Kobayashi, J. M. C. Pereira, and C. F. M. Coimbra, Variable order modeling of diffusiveconvective effects on the oscillatory flow past a sphere, *J. Vib. Control*, 14 (2008), 1569-1672.

[25] L. E. S. Ramirez and C. F. M. Coimbra, On the selection and meaning of variable order operators for dynamic modelling, *Int. J. Diff. Equ.*, 2010 (2010), 8461-8477.

[26] L. E. S. Ramirez and C. Coimbra, On the variable order dynamics of the nonlinear wake caused by a sedimenting particle, *Physica. D.*, 240 (2011), 1111-1118.

[27] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, *Advances in Fractional Calculus. Theorical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[28] N. H. Sweilam, M. M. Khader, and R. F. Al-Bar, Numerical studies for a multi-order fractional differentional equation, *Phys. Lett. A.*, 371 (2007), 26-33.

[29] N. H. Sweilam and M. M. Khader, A Chebyshev pseudo-spectral method for solving fractional order integro-differential equations, *ANZIAM J.*, 51 (2010), 464-475.

[30] N. H. Sweilam, M. M. Khader, and A. M. Nagy, Numerical solution of two-sided space-fractional wave equation using finite difference method, *J. Comput. Appl. Math.*, 235 (2011), 2832-2841.

[31] M. A. Snyder, *Chebyshev Methods in Numerical Approximation*, Prentice-Hall Inc, Englewood Cliffs, NJ, 1966.

[32] H. Sun, W. Chen, C. Li, and Y. Chen, Finite difference schemes for variable-order time fractional diffusion equation, *Int. J. Bif. Ch.*, 22 (4) (2012), 1250085.

[33] S. Shen, F. Liu, J. Chen, I. Turner, and V. Anh, Numerical techniques for the variable order time fractional diffusion equation, *Appl. Math. Comput.*, 218 (2012), 10861-10870.

[34] N. H. Sweilam, M. M. khader, and H. M. Almarwm, Numerical studies for the variable-order nonlinear fractional wave equation, *Frac. C. Appl. Anal.*, 15 (2012), 669-683.

[35] Y. Tan and S. Abbasbandy, Homotopy analysis method for quadratic Riccati differential equation, *Commun. Nonlinear Sci. Nummer. Simul.*, 13 (3) (2008), 539-546.

[36] Y. Zhang, A finite difference method for fractional partial differential equation, *Appl. Math. Comput.*, 215 (2009), 524-529.

H. Hassani
Department of Mathematics
Ph.D. student of Mathematics
Shahrekord University
Shahrekord, Iran
E-mail: hosseinhassani40@yahoo.com

M. Sh. Dahaghin
Department of Mathematics
Assistant Professor of Mathematics
Shahrekord University
Shahrekord, Iran
E-mail: msh-dahaghin@sci.sku.ac.ir

M. H. Heydari
Department of Mathematics
Assistant Professor of Mathematics
Fasa University
Fasa, Iran
E-mail: heydari@fasau.ac.ir