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Solution of Stochastic Optimal Control Problems and Financial Applications

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Abstract. In this paper, the stochastic optimal control problems, which frequently occur in economic and finance are investigated. First, using Bellman's dynamic programming method the stochastic optimal control problems are converted to Hamilton-Jacobi-Bellman (HJB) equation. Then, obtained HJB equation is solved through the method of separation of variables by guessing a solution via its terminal condition. Also, the non-linear optimal feedback control law is constructed. Finally, the solution procedure is illustrated for solving some examples that two of them are financial models. In fact, to highlight the applications of stochastic optimal control problems in financial mathematics, some financial models are presented.

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1. Introduction

Optimal control models play a prominent role in a range of application areas, including aerospace, chemical engineering, robotic, economics and

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finance. It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A controlled process is the solution of an ordinary differential equation which some of its parameters can be chosen. Hence, the trajectory of the solution is obtained. Each trajectory has an associated cost, and the goal of optimal control problem is to minimize this cost over all choices of the control parameter. Stochastic optimal control is the stochastic extension of this; In fact, a stochastic differential equation with a control parameter is given. Each choice of the control parameter yields a different stochastic variable as a solution to the stochastic differential equation. Each pathwise trajectory of this stochastic process has an associated cost, and we seek to minimize the expected cost over all choices of the control parameter. In describing a stochastic control model, the kind of information available to the controller at each instant of time, plays an important role. Several situations are possible [2]:

(1) The controller has no information during the system operation. Such controls are often called *open loop*.

(2) The controller knows the state within the system at each instant of time t. We call this, the case of *complete observations*.

Pontryagin's maximum principle method and Bellman's dynamic programming method (HJB equation) represent the most known methods for solving optimal control problems [1-7]. Pontryagin's maximum principle is used to find the necessary conditions for the existence of an optimum solution. This converts the original optimal control problem into a boundary value problem. Another very efficient approach for solving optimal control problems is dynamic programming method. It is a robust approach for solving optimal control problems. The method was originated by R. Bellman in 1950s [3]. Its basic idea is to consider a family of optimal control problems with different initial times and states, to establish relationships among these problems via the HJB equation. If the HJB equation is solvable, then one can obtain an optimal feedback control by taking the maximize or minimize involved in the HJB equation [15]. The HJB equation has not analytical solution in general and finding an approximate solution is at least the most logical way to solve them [8-10]. Also, Kushner presented a survey of the early development of selected areas in non-linear continuous-time stochastic optimal control problem [11]. Here, solution of stochastic optimal control problems are investigated. In fact, obtained HJB equation is solved through the method of separation of variables by guessing a solution via its terminal condition.

This paper is organized into following sections of which this introduction is the first. In Section 2, we introduce deterministic optimal control. Section 3, is about stochastic differential equations. Stochastic optimal control is presented in Section 4, Sections 5 and 6 are about dynamic programming and HJB equation. In Section 7, we present main results with solving some examples, these examples illustrate financial applications of stochastic optimal control. Finally, the paper is concluded with some key points for this work.

2. Deterministic Optimal Control

Optimal control deals with the problem of finding a control variable $u(.)$, which is assumed to be piecewise from class of admissible controls, U. Each choice of control $u(.) \in U \subset \mathbb{R}^m$ yields a state variable $x(t) \in \mathbb{R}^n$ which is the unique solution of:

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

which is called the equation of motion or trajectory, on a fixed interval $[s, T]$ with initial condition:

$$x(s) = y. \tag{2}$$

Along with system of differential equation with initial condition (1)-(2), there is a performance index or cost functional of the form:

$$J(s, y; u) = \psi(T, x(T)) + \int_s^T L(t, x(t), u(t))dt. \tag{3}$$

Here, $L(t, x, u)$ is the running cost, and $\psi(t, x)$ is the terminal cost. The performance index (3) depends on the initial position (s, y) and the choice of control $u(.)$. Therefore the optimization problem is, to minimize J, for each $(s, y; u)$, over all controls $u(t) \in U$. The function u which yields this minimum is called an optimal control. Note that, the optimization problem with performance index as defined in equation (3) is called a Bolza problem. There are two other equivalent optimization problems, which are called Lagrange and Mayer problems [1].

3. Stochastic Differential Equations

Stochastic differential equations are often written in the form of:

$$\begin{cases} dx(t) = f(x(t))dt + \sigma(x(t))dw(t), \\ x(0) = x_0, \qquad 0 \leqslant t \leqslant T. \end{cases} \tag{4}$$

Which looks almost like an ordinary differential equation. However, the Ito differentials are not sensible mathematical objects in themselves; rather, we should apply this expression as suggestive notation for the Ito process:

$$x(t) = x_0 + \int_0^t f(x(\tau))d\tau + \int_0^t \sigma(x(\tau))dw(\tau).$$

If there exists a stochastic process $x(t)$ that satisfies this equation, we say that it solves the stochastic differential equation.

Example 3.1. (Scalar linear equation) Consider the scalar linear stochastic differential equation:

$$\begin{cases} dx(t) = ax(t)dt + bx(t)dw(t), \\ x(0) = x_0, \end{cases}$$

driven by a scalar Wiener process $w(t)$, with a and b constants. This stochastic differential equation is said to have multiplicative noise. In fact, we can analytically solve this equation, the solution is [12]:

$$x(t) = x_0 \exp\left(\left(a - \frac{1}{2}b^2\right)t + bw(t)\right).$$

To apply a numerical method to (4) over $[0, T]$, first the interval discretized, then the Euler-Maruyama (EM) method takes the form as follows [13]:

$$x_j = x_{j-1} + f(x_{j-1})dt + \sigma(x_{j-1})\left(w(\tau_j) - w(\tau_{j-1})\right), j = 1, 2, \cdots, L.$$

Where, numerical approximation to $x(\tau_j)$ denoted by x_j, L is a positive integer, $\Delta t = T/L$ and $\tau_j = j\Delta t$.

4. Stochastic Optimal Control

Consider the stochastic differential equation with initial condition as follows:

$$\begin{cases} dx(t) = f(t, x(t), u(t))dt + b(t, x(t), u(t))dw, \\ x(s) = y, \end{cases} \tag{5}$$

where y is a given vector in $\mathbb{R}^n$. Also, $x(t) \in \mathbb{R}^n$ is the state variable or trajectory, $u(t) \in U \subset \mathbb{R}^m$ is the control variable, w is a Wiener process, $f(t, x, u)$ is a drift, and $b(t, x, u)$ is diffusion [1, 2]. The optimal control variable u is presented by:

$$u(t) = \mathbf{u}(t, x(t)),$$

and chosen so as to minimize the following performance index:

$$J(s, y; u) = \mathbb{E}_{sy}\left[\int_s^T L(\tau, x(\tau), u(\tau))d\tau + \psi(x(T))\right]. \tag{6}$$

Now, value function is defined as:

$$V(s, y) = \inf_{u \in U} \mathbb{E}_{sy}\left[\int_s^T L(\tau, x(\tau), u(\tau))d\tau + \psi(x(T))\right] = J(s, y; u^*),$$

i.e. value function V is the minimum cost achievable starting from initial condition $x(s) = y$, and $u^*(.)$ is the optimal control which yields this minimum costs.

5. Dynamic Programming

The use of the principle of optimality or dynamic programming to derive an equation for solving optimal control problem, was first proposed by Bellman [3]. In dynamic programming, a family of fixed initial point control problem is considered. The minimum value of the performance index is considered as a function of this initial point which is called the value function. Whenever the value function is differentiable, it satisfies a non-linear first order hyperbolic partial differential equation called the HJB equation. This equation is used for constructing a non-linear optimal feedback control variable. Note that, much of the notation, theorems, and proofs used throughout are taken from the books [2, 12].

Lemma 5.1. *(Dynamic Programming Principle(See [2, 12]))*

$$V(s,y) = \inf_{u\in U} \mathbb{E}_{sy}\left[\int_s^{s+h} L(\tau, x(\tau), u(\tau))d\tau + V(s+h, x(s+h))\right],$$

where $x(t+h)$ is determined by u from stochastic differential equation with initial condition (5)*.*

Now, the concept of a backwards evolution operator associated with $x(t)$ generated by the stochastic differential equation (5) can be presented as follows:

Lemma 5.2. (See [12]) *The backwards evolution operator associated with $x(t)$ generated by the stochastic differential equation* (5) *with fixed control $u(s) \equiv v$ is as follows:*

$$\mathcal{A}^v\phi = \phi_t + \sum_{i=1}^{n} f_i(t,x,v)\phi_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x,v)\phi_{x_ix_j}, \tag{6}$$

where $a = \sigma\sigma'$.

Definition 5.3. (Dynkin's formula(See [1, 2]) *For $s < t$,*

$$\mathbb{E}_{sy}\,\phi(t,x(t)) - \phi(s,y) = \mathbb{E}_{sy}\left[\int_s^t \Big(\phi_t(\tau,x(\tau)) + \mathcal{A}(\tau)\psi(\tau,x(\tau))\Big)d\tau\right].$$

Now, for solving the stochastic optimal control problem, the expected value of a performance index of Bolza type (6) is taken, where the control applied at time t using feedback control u is $u(t, x(t))$. In dynamic programming the optimal expected system performance is considered as a function of the initial data:

$$V(s, y) = \inf_u J(s, y; u).$$

An optimal feedback control law u^* has the property that $V(s, y) = J(s, y; u^*)$ for all (s, y). If u is given then by Dynkin formula with $\psi = V$ and $\mathcal{A} = \mathcal{A}^u$, we have the following equation for the value function V:

$$V(s, y) = -\mathbb{E}_{sy}\left[\int_s^t \big(V_s + A^u(\tau)V\big)d\tau + V(t, x(t))\right]. \tag{7}$$

Now, suppose that the controller uses u for times $s \leqslant \tau \leqslant t$ and uses an optimal control u^* after time s. Its expected performance can be no less than $V(t, x)$. Thus let:

$$u_1(\tau, x) = \begin{cases} u(\tau, x), & \tau \leqslant t, \\ u^*(\tau, x), & \tau > t, \end{cases}$$

by properties of conditional expectation:

$$J(s, y, u_1) = \mathbb{E}_{sy}\left[\int_s^t L(\tau, x(\tau), u(\tau))d\tau + J(t, x(t), u^*)\right].$$

Then:

$$\begin{aligned} &V(s, y) \leqslant J(s, y, u_1), \quad V(t, x(t)) = J(t, x(t), u^*), \\ &V(s, y) \leqslant \mathbb{E}_{sy}\left[\textstyle\int_s^t [L(\tau, x(\tau), u(\tau)]d\tau + V(t, x(t))\right]. \end{aligned} \tag{8}$$

Equality holds, if an optimal control law $u = u^*$ is used during $[s, t]$. Let us subtract (7) from (8) and divide by $t - s$. Since $x(s) = y$, we get $t \to s^+$, with $v = u(s, y)$:

$$0 \leqslant V_s + A^v(s)V + L(s, y, v).$$

Equality holds if $v = u^*(s, y)$, where u^* is an optimal feedback control variable. Thus, for the value function V, the continuous time dynamic programming equation of optimal stochastic optimal control theory have derived:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial s} + \min_{u \in U}\left[L(s,x(s),v) + A^{u(t)}V(s,y)\right] \\ &= \frac{\partial V}{\partial s} + \min_{v(t)\in U}\left[L(s,y,v) + f(s,y,v).D_xV + \tfrac{1}{2}Trace\left(a(s,y,v)D_x^2V\right)\right], \end{aligned}$$

where D_x^2V is Hessian of V and $Trace\left(a\ D_x^2V\right) = \sum_{i,j=1}^{n} a_{ij}V_{x_ix_j}$.

6. HJB Equation and its Solution

Section 5. has been shown that value function $V(t,x)$ solves the following partial differential equation with terminal condition which is called HJB equation:

$$(HJB) \quad \begin{cases} \frac{\partial V}{\partial t} + H(t,x,D_xV,D_x^2V) = 0, \\ V(T,x) = \psi(x). \end{cases}$$

Here, H is the Hamiltonian function and has been introduced as follows:

$$\begin{aligned} &H(s,y,D_xV,D_x^2V) = \\ &\min_{v(t)\in U}\left[L(s,y,v) + f(s,y,v).D_xV + \tfrac{1}{2}Trace\left(a(s,y,v)D_x^2V\right)\right]. \end{aligned}$$

Overall, the solution of stochastic optimal control problems will be as follows. First, stochastic optimal control problems are transferred to Hamilton-Jacobi-Bellman (HJB) equation as a non-linear partial differential equation. Then, obtained HJB equation is solved through the method of separation of variables by guessing a solution via its terminal condition to obtain value function $V(t,x)$. Accordingly, the optimal control is derived as $u^*(.)$. In fact, the value function form is determined by guessing via terminal condition of HJB equation with $V(T,x) = \psi(x)$. Inserting these expressions solution into the HJB equation yields an ordinary differential equation. Consequently, by solving the resulting ordinary differential equation, analytic solution of presented stochastic optimal control will be obtained.

7. Case Studies

In this section, some cases of stochastic optimal control problems are investigated. Furthermore, to highlight the applications of stochastic optimal control problems in financial mathematics, some financial models are solved.

Example 7.1. (Stochastic linear regulator problem [6]).
Consider stochastic differential equation:

$$\begin{cases} dx(t) = (3x(t) + u(t))dt + 4dw(t), \\ x(0) = 0.5, \end{cases}$$

with cost functional as:

$$J(s, y; u) = \mathbb{E}_{sy}\left[\int_0^1 \left(\frac{7}{2}x^2(t) + \frac{1}{2}u^2(t)\right)dt + \frac{1}{2}x^2(1)\right].$$

Hamiltonian function in this case can be obtained as follows:

$$H(t, x, D_xV, D_x^2V) = \min_{v\in U}\left\{(3x + v)\frac{\partial V}{\partial x} + 8\frac{\partial^2 V}{\partial x^2} + \frac{7}{2}x^2 + \frac{1}{2}v^2\right\}$$
$$= 3x\frac{\partial V}{\partial x} - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 + 8\frac{\partial^2 V}{\partial x^2} + \frac{7}{2}x^2,$$

where the choice of $v = -\frac{\partial V}{\partial x}$ achieves the minimum of optimal control variable. Consequently, the HJB equation is as follows:

$$(HJB) \quad \begin{cases} \frac{\partial V}{\partial t} + 3x\frac{\partial V}{\partial x} - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 + 8\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}x^2 = 0, \\ V(1, x) = \frac{1}{2}x^2. \end{cases} \tag{9}$$

To solve HJB equation (9) via separation of variables by guessing a solution of the form $V(t, x) = P(t)x^2 + K(t)$ which leads to be the following system of ordinary differential equation:

$$\begin{cases} \dot{P}(t) + 6P(t) - 2P^2(t) + 0.5 = 0, \\ \dot{K}(t) + 16P(t) = 0, \\ P(1) = 0.5, \\ K(1) = 0. \end{cases}$$

Solving the result of this system for $P(t)$ and $K(t)$, it is achieved that the value function $V(t, x)$ is equal to:

$$V(t, x) = \frac{1}{2}\,\frac{\left(7 - 3\,e^{8\,t-8}\right)x^2}{1 + 3\,e^{8\,t-8}} + 8\,\ln\left(\frac{1}{4} + \frac{3}{4}\,e^{8\,t-8}\right) - 56\,t + 56.$$

Taking the partial derivatives of $V(t,x)$ and substituting the formula for the control variable results that:

$$u^*(t,x) = -\frac{\partial V(t,x)}{\partial x} = \frac{\left(3\,\mathrm{e}^{8\,t-8} - 7\right)x}{1 + 3\,\mathrm{e}^{8\,t-8}}.$$

The graphs of value function $V(t,x)$ and optimal control variable $u(t,x)$ are given in Figure 1, and the graph of approximate solutions of trajectory $x(t)$ is given in Figure 2. Also, The exact solution for the performance index is $J^* = V(0, 0.5) = 45.79168679$.

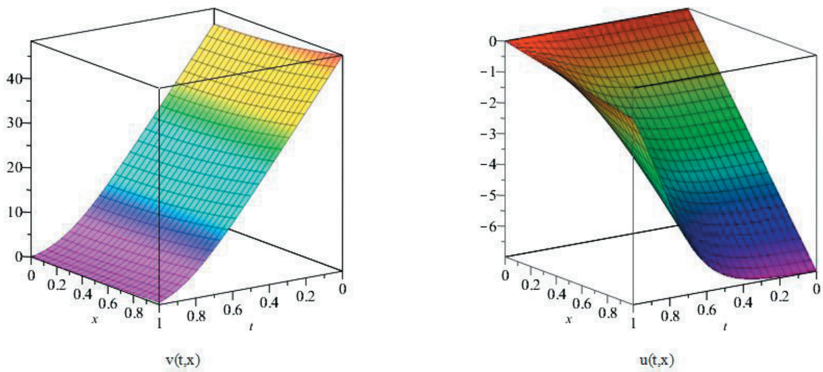

Figure 1: The graphs of value function $V(t,x)$ and control variable $u(t,x)$ for Example 7.1.

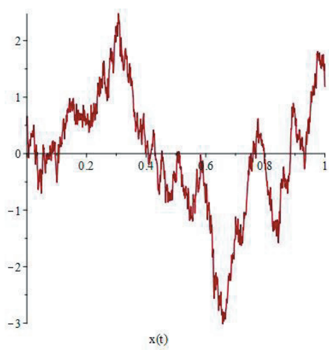

Figure 2: The graph of EM approximate of $x(t)$ with $\Delta t = 0.001$ for Example 7.1

Example 7.2. (Dynamic voluntary provision of public goods [14]) Consider stochastic differential equation:

$$\begin{cases} dx(t) = (-x(t) + u(t))dt + \sqrt{2}x(t)dw(t), \\ x(0) = \sqrt{2}, \end{cases}$$

with cost functional as:

$$J(s, y; u) = \mathbb{E}_{sy}\left[\int_0^1 \left(x^2(t) - \frac{1}{2}u^2(t)\right) dt\right].$$

Hamiltonian function in this case can be as follows:

$$H(t, x, D_x V, D_x^2 V) = \min_{v \in U} \left\{(-x + v)\frac{\partial V}{\partial x} + x^2 \frac{\partial^2 V}{\partial x^2} + x^2 - \frac{1}{2}v^2\right\}$$
$$= -x\frac{\partial V}{\partial x} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 + x^2 \frac{\partial^2 V}{\partial x^2} + x^2,$$

where the choice of $v = \frac{\partial V}{\partial x}$ achieves the minimum of optimal control variable. Consequently, the HJB equation is as follows:

$$(HJB) \qquad \begin{cases} \frac{\partial V}{\partial t} - x\frac{\partial V}{\partial x} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 + x^2 \frac{\partial^2 V}{\partial x^2} + x^2, \\ V(1, x) = 0. \end{cases} \tag{10}$$

To solve HJB equation (10) via separation of variables by guessing a solution of the form $V(t, x) = P(t)x^2$ which leads to be the following ordinary differential equation:

$$\begin{cases} \dot{P}(t) + 2P^2(t) + 1 = 0, \\ P(1) = 0. \end{cases}$$

Solving the result of this ordinary differential equation for $P(t)$, it is achieved that the value function $V(t, x)$ is equal to:

$$V(t, x) = -\frac{\sqrt{2}}{2}\tan\left(\sqrt{2}\,(t-1)\right)x^2.$$

Taking the partial derivatives of $V(t, x)$ and substituting the formula for the control variable results that:

$$u^*(t, x) = \frac{\partial V(t, x)}{\partial x} = -\sqrt{2}\tan\left(\sqrt{2}\,(t-1)\right)x.$$

The graphs of value function $V(t, x)$ and optimal control variable $u(t, x)$

are given in Figure 3, and the graph of approximate solutions of trajectory $x(t)$ is given in Figure 4. Also, The exact solution for the performance index is $J^* = V(0, \sqrt{2}) = 8.957797208$.

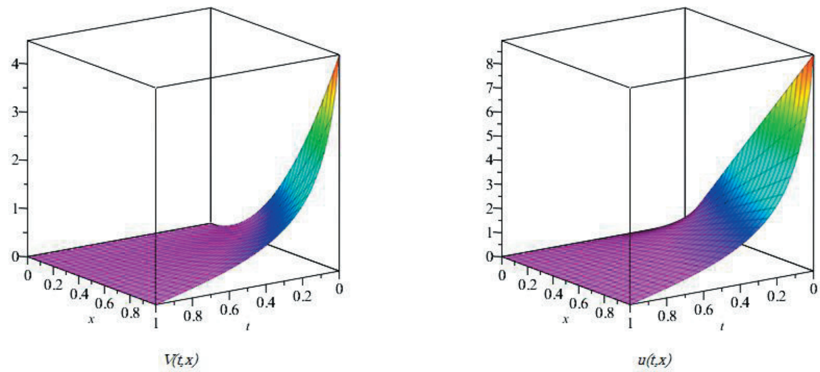

Figure 3: The graphs of value function $V(t,x)$ and control variable $u(t,x)$ for Example 7.2.

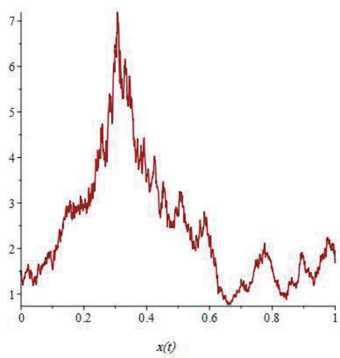

Figure 4: The graph of EM approximate of $x(t)$ with $\Delta t = 0.001$ for Example 7.2.

Example 7.3. (A non-linear stochastic optimal control problem [15]) Consider the following stochastic differential equation:

$$\begin{cases} dx(t) = \left(\sqrt{x(t)} - \dfrac{1}{16}x(t) - u(t)\right)dt + \dfrac{1}{2}x(t)dw(t), \\ x(0) = 1, \end{cases}$$

with cost functional as:

$$J(s,y;u)=\mathbb{E}_{sy}\Big[\int_0^{\frac{16}{3}}\Big(\sqrt{u(t)}-\frac{u(t)}{\sqrt{x(t)}}\Big)dt+\frac{5}{2}\sqrt{x(\frac{16}{3})}\Big],$$

Hamiltonian function in this case can be as follows:

$$H(t,x,D_xV,D_x^2V)=\min_{v\in U}\Big\{\sqrt{v}-\frac{v}{\sqrt{x(t)}}+(\sqrt{x(t)}-\frac{1}{16}x(t)-v)\frac{\partial V}{\partial x}+\frac{1}{8}x^2\frac{\partial^2 V}{\partial x^2}\Big\},$$

where the choice of $v=\frac{x}{4(1+\sqrt{x(t)}\frac{\partial V}{\partial x})^2}$ achieves the minimum of optimal control variable. Consequently, the HJB equation is as follows:

$$(HJB)\quad\begin{cases}\frac{\partial V}{\partial t}+\frac{\sqrt{x(t)}}{2(1+\sqrt{x(t)}\frac{\partial V}{\partial x})}-\frac{\sqrt{x(t)}}{4(1+\sqrt{x(t)}\frac{\partial V}{\partial x})^2}-\frac{x(t)\frac{\partial V}{\partial x}}{4(1+\sqrt{x(t)}\frac{\partial V}{\partial x})^2}\\ \qquad+\sqrt{x(t)}\frac{\partial V}{\partial x}-\frac{1}{16}x(t)\frac{\partial V}{\partial x}+\frac{1}{8}x(t)^2\frac{\partial^2 V}{\partial x^2}=0,\\ V(\frac{16}{3},x)=\frac{5}{2}\sqrt{x}.\end{cases}\tag{11}$$

To solve HJB equation (11) via separation of variables by guessing a solution of the form $V(t,x)=P(t)\sqrt{x}+K(t)$ which leads to be the following system of ordinary differential equation:

$$\begin{cases}\dot{P}(t)-\frac{1}{16}P(t)+\frac{1}{2(P(t)+2)}=0,\\ \dot{K}(t)+\frac{1}{2}P(t)=0,\\ P(\frac{16}{3})=\frac{5}{2},\\ K(\frac{16}{3})=0.\end{cases}$$

Solving the result of this system of ordinary differential equation for $P(t)$ and $K(t)$, leads to:

$$P(t)=\frac{1}{4}\,\frac{\left(-512+52\,\beta\,(t)+4\,\sqrt{-3328\,\beta\,(t)+169\,\beta^2\,(t)}\right)^{2/3}+64}{\sqrt[3]{-512+52\,\beta\,(t)+4\,\sqrt{-3328\,\beta\,(t)+169\,\beta^2\,(t)}}},$$

$$K(t)=\frac{1}{2}\int_t^{\frac{16}{3}}P(s)ds\approx\frac{20}{3}-\frac{5}{4}t-\frac{13}{1152}\left(t-\frac{16}{3}\right)^2-\frac{1469}{4478976}\left(t-\frac{16}{3}\right)^3$$
$$-\frac{51727}{7739670528}\left(t-\frac{16}{3}\right)^4,$$

where $\beta(t) = \exp(\frac{3}{16}t - 1)$, and the feedback control variable is as follows:

$$u^*(t,x) = \frac{x}{4(1+\sqrt{x(t)}\frac{\partial V}{\partial x})^2} = \frac{2x}{(2+P(t))^2}.$$

The graphs of value function $V(t,x)$ and optimal control variable $u(t,x)$ are given in Figure 5, and the graph of approximate solutions of trajectory $x(t)$ is given in Figure 6. Also, The exact solution for the performance index is $J^* = V(0,1) = 8.697876607$.

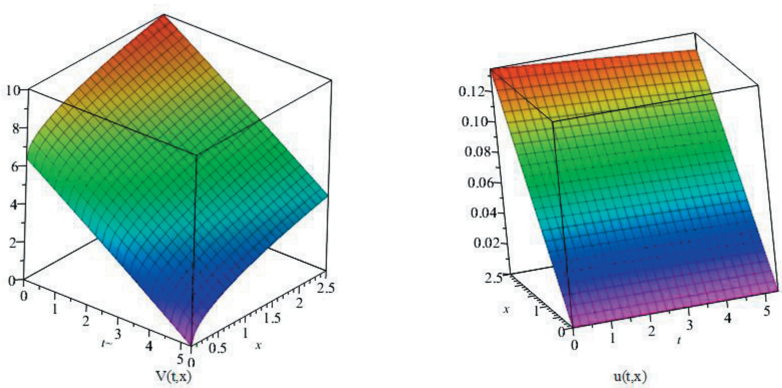

Figure 5: The graphs of value function $V(t,x)$ and control variable $u(t,x)$ for Example 7.3.

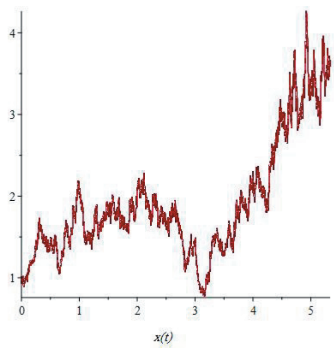

Figure 6: The graph of EM approximate of $x(t)$ with $\Delta t = 0.001$ for Example 7.3.

Example 7.4. (Merton's portfolio selection model [1])
Consider stochastic differential equation:

$$\begin{cases} dx = 0.05(1-u_1)xdt + u_1x(0.11dt + 0.4dw) - u_2dt, \\ x(0) = 10^5. \end{cases}$$

Where, $u_1(t)$ be the proportion of the money invested in the risky asset at time t and $u_2(t)$ the consumption rate. Now, the control variable is the 2-dimensional vector as $u(t) = (u_1(t), u_2(t))$ where $0 \leqslant u_1(t) \leqslant 1, u_2(t) \geqslant 0$. The standard problem is to maximize the cost functional:

$$J = \mathbb{E}_{sy}\Big[\int_0^{10} (e^{-0.11t}\sqrt{u_2(t)})dt\Big],$$

Here, the Hamiltonian function can be obtained as follows:

$$H(t,x,V_x,V_{xx}) = \max_{v_1,v_2}\big\{\big(0.05(1-v_1)x + 0.11v_1x - v_2\big)V_x + \frac{0.16v_1^2x^2}{2}V_{xx} \\ + e^{-0.11t}\sqrt{v_2}\big\},$$

where the optimal control $u^* = (u_1^*, u_2^*)$ is found to be:

$$u_1^* = -\frac{3V_x}{8xV_{xx}}, u_2^* = \frac{1}{4e^{0.22t}V_x^2}.$$

Furthermore, the HJB equation can be as follows:

$$(HJB) \qquad \begin{cases} V_t - \frac{9}{800}\frac{V_x^2}{V_{xx}} + \frac{1}{20}xV_x + \frac{1}{4}\frac{e^{-0.22t}}{V_x} = 0, \\ V(10,x) = 0. \end{cases} \tag{12}$$

To solve HJB equation (12) via separation of variables by guessing a solution of the form $V(t,x) = P(t)\sqrt{x}$ which leads to be the following ordinary differential equation:

$$\begin{cases} \dot{P}(t) + \frac{29}{800}P(t) + \frac{e^{-0.22t}}{2P(t)} = 0, \\ P(10) = 0. \end{cases}$$

Solving the result of this ordinary differential equation for $P(t)$, it is achieved that the value function $V(t,x)$ is equal to:

$$V(t,x) = \frac{20\sqrt{59x}}{59}\sqrt{\mathrm{e}^{-\frac{11t}{50}} - \mathrm{e}^{-\frac{29t}{400} - \frac{59}{40}}},$$

Taking the partial derivatives of $V(t,x)$ and substituting the formula for the control variables yields to:

$$u_1^* = -\frac{3V_x}{8xV_{xx}} = \frac{3}{4},$$
$$u_2^* = \frac{1}{4e^{0.22t}V_x^2} = \frac{x}{e^{0.22t}P^2(t)} = \frac{59x}{400(1-\mathrm{e}^{\frac{59\,t}{400}-\frac{59}{40}})}.$$

The graphs of value function $V(t,x)$ and optimal control variable $u_2(t,x)$ are given in Figure 7, and the graph of approximate solutions of trajectory $x(t)$ is given in Figure 8. Also, The exact solution for the performance index is $J^* = V(0,10^5) = 723.0918887$.

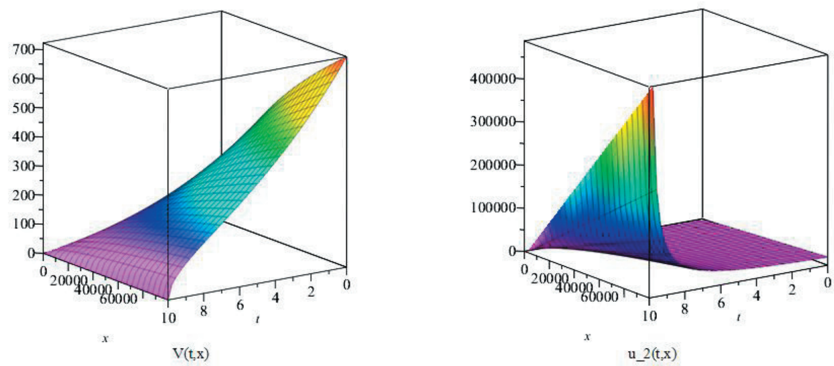

Figure 7: The graphs of value function $V(t,x)$ and control variable $u(t,x)$ for Example 7.4.

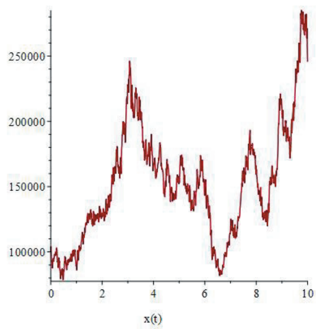

Figure 8: The graph of EM approximate of $x(t)$ with $\Delta t = 0.001$ for Example 7.4.

8. Conclusion

Only a small class of stochastic optimal control problems admits analytic solutions for the value function and the corresponding optimal strategies. Dynamic programming method represents the most known method for solving optimal control problems. In this paper, stochastic optimal control problems and Bellman's dynamic programming method (Hamilton-Jacobi-Bellman equation) were investigate. In fact, a stochastic optimal control problem was converted to HJB equation and obtained HJB equation was solved through the method of separation of variables by guessing a solution via its terminal condition. Also, the non-linear optimal feedback control law was constructed. Finally, some financial examples were simulated to highlight the applications of stochastic optimal control problems.

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