TWO CLASSES OF MULTICONE GRAPHS DETERMINED BY THEIR SPECTRA

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ABSTRACT. A multicone graph is defined to be the join of a clique and a regular graph. In [1], new classes of multicone graphs are characterized that are determined by their spectra. In this work, we present new classes of multicone graphs that are determined by their adjacency spectrums. Also, we show that these graphs are determined by their Laplacian spectrums. Finally, four problems for further researches is proposed.

Keywords: Adjacency spectrum, Laplacian spectrum, DS graph, Paley graph, Schläfi graph.
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1. INTRODUCTION

In this paper, we are concerned only with finite undirected simple graph (loops and multiple edge are not allowed). All terminology and notations on graphs not defined here can be found in [2, 3, 4, 11]. Let $\Gamma = (V, E)$ be a simple graph, where V be the set of vertices and E be the set of edges of Γ. An edge joining the vertices u and v is denoted by $\{u, v\}$. The complement of a graph G, denoted by $\overline{G}$, is the graph on the vertex set of G such that two vertex of $\overline{G}$, are adjacent if an only if they are not adjacent in G. A graph is called bidegreed if the set of degrees of vertices consists of exactly two distinct elements. Let $A(G)$ denotes the $(0, 1)$-adjacency matrix of graph G. The characteristic polynomial of G is $\det(\lambda I - A(G))$, and is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjaceny eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. The greatest eigenvalues of adjacency spectrum of G is called spectral radius of G and it is denoted by $\varrho(G)$. If G has n vertices, then it has n eigenvalues in descending order as $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$. Let λ_1, λ_2, ..., λ_s be the distinct eigenvalues of G with multiplicity m_1, m_1, ..., m_s, respectively. The multi-set $Spec(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, ..., [\lambda_s]^{m_s}\}$ of eigenvalues of $A(G)$ is called the adjacency spectrum of G. For two graphs G and H, if $Spec(G) = Spec(H)$, we say G and H are cospectral with respect to adjacency matrix. A graph H is said to be determined by its spectrum or DS for short, if for a graph H with $Spec(G) = Spec(H)$, one has G isomorphic to H. Let G be a graph with adjacency matrix A and D be the diagonal matrix of vertex degrees for

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G. The matrix $SL(G) = D + A$ and $L(G) = D - A$ are known as signless Laplacian matrix and Laplacian matrix for G, respectively. In this paper, we present new classes of multicone graphs and we show these graphs are DS with respect to their spectra. This paper is organized as follows. In Section 2, we review some basic information and preliminaries. In Section 3, we characterize graphs that are determined by their adjacency spectrum. In Section 4, we study graphs that are determined by their Laplacian spectrum. In Section 5 and Section 6 similar of Section 3 and Section 4 we characterize another new classes of multicone graphs that are determined by their spectra. In Section 7, we review what is said in previous sections and finally we propose four conjectures for further researches.

The following will be useful in the sequal.

2. Prerequisites

Lemma 2.1. [1, 6] *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*
(*i*) *The number of vertices,*
(*ii*) *The number of edges.*
For the adjacency matrix, the following follows from the spectrum
(*iii*) *The number of closed walks of any length.*
(*iv*) *Being regular or not and the degree of regularity.*
(*v*) *Being bipartite or not.*
For the Laplacian matrix, the following follows from the spectrum
(*vi*) *The number of spanning tree.*
(*vii*) *The number of component.*
(*viii*) *The sum of squares of degrees of vertices.*

Theorem 2.2. [4] *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by:*

$$P_{G_1 \bigtriangledown G_2(x)} = \frac{P_{G_1} P_{G_2}(x)}{(x-r_1)(x-r_2)}((x - r_1)(x - r_2) - n_1 n_2).$$

Proposition 2.3. [9] *Let G be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over G, the graph H ;that is, obtained from G by adding one vertex that is adjacent to all vertices of G, is also determined by its Laplacian spectrum.*

Theorem 2.4. [1] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

Theorem 2.5. [5] *Let G and H be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geqslant ... \geq \mu_m$, respectively. Then Laplacian spectra of $\overline{G}$ and $G \bigtriangledown H$ are $n-\lambda_1, n-\lambda_2, ..., n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., m+\mu_{m-1}, 0$, respectively.*

Lemma 2.6. [5] *Let G be a graph on n vertices. Then n is Laplacian eigenvalue of G if and only if G is the join of two graphs.*

Proposition 2.7. [2] *For a graph G, the following statements are equivalent:*
(i) G is d-regular.
(ii) $\varrho(G) = d_G$, the average vertex degree.
(iii) G has $v = (1, 1, ..., 1)^t$ as an eigenvector for $\varrho(G)$.

Theorem 2.8. [4, 7] *Let $G - j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j} = P_G(x) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{x-\mu_i}$, where m, α_{ij}^2 and P_G are the number of distinct eigenvalues of graph G, main angle of G and characteristic polynomial of G, respectively.*

3. Main Results

In the following, we always suppose that w is a natural number. Also, P and S denote Paley graph of order 17 and Schläfi graph, respectively.

The main goal of this section is to prove that any graph cospectral with multicone graph $K_w \bigtriangledown P$ must be bidegreed.

3.1. Connected graph cospectral with multicone graph $K_w \bigtriangledown P$.

Proposition 3.1. *Let G be a graph cospectral with multicone graph $K_w \bigtriangledown P$. Then $Spec(G) =$*

$$\left\{[-1]^{w-1}, \left[\frac{-1+\sqrt{17}}{2}\right]^8, \left[\frac{-1-\sqrt{17}}{2}\right]^8, \left[\frac{\Omega+\sqrt{\Omega^2+4\Gamma}}{2}\right]^1, \left[\frac{\Omega+\sqrt{\Omega^2+4\Gamma}}{2}\right]^1\right\},$$

where $\Omega = w+7$ and $\Gamma = 9w+8$.

Proof. It is well-known that $Spec(P) = \left\{[8]^1, \left[\frac{-1+\sqrt{17}}{2}\right]^8, \left[\frac{-1-\sqrt{17}}{2}\right]^8\right\}$. Now, by Theorem 2.2, there is nothing for proof. □

In the following lemma, we show that any graph cospectral with multicone graph $K_w \bigtriangledown P$ must be bidegreed.

Lemma 3.2. *Let G be cospectral with multicone graph $K_w \bigtriangledown P$. Then G is bidegreed in which any vertex of G is of degree $w+8$ or $w+16$.*

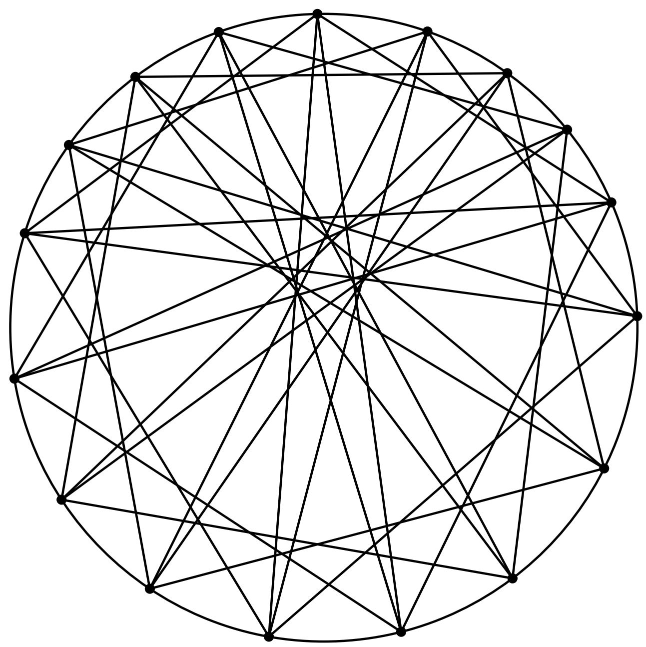

FIGURE 1. Paley graph of order 17

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph G consists of at least three number. Hence the equality in Theorem 2.4 cannot happen for any δ. But, if we put $\delta = w + 8$, then the equality in Theorem 2.4 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 16$. By contrary, we suppose that $\Delta < w + 16$. Therefore, the equality in Theorem 2.4 cannot hold for any δ. But, if we put $\delta = w + 8$, then this equality holds . This is a contradiction and so $\Delta = w + 49$. Now, $\delta = w + 8$, since G is bidegreed and G has $w + 17$, $\Delta = w + 16$ and $w(w + 16) + 17(w + 8) = w\Delta + 17(w + 7) = \sum_{i=1}^{w+17} \deg v_i$. This completes the proof. □

In the following lemma, we prove that multicone graphs $K_1 \bigtriangledown P$ are DS.

Lemma 3.3. *Any graph cospectral with multicone graph $K_1 \bigtriangledown P$ is DS.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown P$. By Lemma 3.2, it is easy to see that G has one vertex of degree 17, say j. Now, Theorem 2.8

implies that $P_{G-j} = (x-\mu_3)^{54}(x-\mu_4)^{20}[\alpha_{1j}^2 A_1 + \alpha_{2j}^2 A_2 + \alpha_{3j}^2 A_3 + \alpha_{4j}^2 A_4]$, where

$\mu_1 = \frac{8+\sqrt{132}}{2}$, $\mu_2 = \frac{8-\sqrt{132}}{2}$, $\mu_3 = \frac{-1+\sqrt{17}}{2}$ and $\mu_4 = \frac{-1-\sqrt{17}}{2}$.
$A_1 = (x-\mu_2)(x-\mu_3)(x-\mu_4)$,
$A_2 = (x-\mu_1)(x-\mu_3)(x-\mu_4)$,
$A_3 = (x-\mu_1)(x-\mu_2)(x-\mu_4)$,
$A_4 = (x-\mu_1)(x-\mu_2)(x-\mu_3)$.

Now, we have:

$x + y + 8 = -(54\mu_3 + 20\mu_4)$,
$x^2 + y^2 + 16 = 136 - (54\mu_3^2 + 20\mu_4^2)$,

where x and y are the eigenvalues of graph $G-j$. If we solve above equations, then $x = \frac{-1+\sqrt{17}}{2}$ and $y = \frac{-1-\sqrt{17}}{2}$. Hence $Spec(G-j) = Spec(P)$ and so $G - j \cong P$. This completes the proof. □

Up to now, we show that multicone graph $K_1 \bigtriangledown P$ are DS with respect to their adjacency spectrums. The natural question is what happen for multicone graph $K_w \bigtriangledown P$? we answer this question in the next theorem.

Theorem 3.4. *multicone graph $K_w \bigtriangledown P$ are DS with respect to their adjacency spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown P)$, then $G_1 \cong K_w \bigtriangledown P$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown P)$, then $G \cong K_{w+1} \bigtriangledown P$, where G is a graph. By Lemma 3.2, Theorem 2.4 , Lemma 2.1 (iii) and in a similar manner of Lemma 3.3 for $G-j$, where j is a vertex of degree $w+17$ belong to G, we obtain $Spec(G-j) = Spec(K_w \bigtriangledown P)$. This completes the proof. □

In the following, we show that multicone graphs $K_w \bigtriangledown P$ are DS with respect to Laplacian spectrums.

4. Connected graph cospectral with multicone graph $K_w \bigtriangledown P$ with respect to Laplacian spectrum

Theorem 4.1. *multicone graph $K_w \bigtriangledown P$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. let the claim be true for values less than w; that is, $Spec(L(G_1)) = Spec(L(K_i \bigtriangledown P)) = \left\{ [0]^1,\ [17+i]^i,\ \left[\frac{\sqrt{17}+17}{2}+i\right]^8,\ \left[\frac{-\sqrt{17}+17}{2}+i\right]^8 \right\}$ follows that $G_1 \cong K_i \bigtriangledown P$, where $i = 1, 2, ..., w-1$. We show that the problem is true for w; that is, we show that $Spec(L(G)) = Spec(L(K_w \bigtriangledown P)) =$

$\left\{[0]^1, [17+w]^w, \left[\frac{\sqrt{17}+17}{2}+w\right]^8, \left[\frac{-\sqrt{17}+17}{2}+w\right]^8\right\}$ follows that $G \cong K_w \bigtriangledown P$, where G is a graph. Theorem 2.6 implies that G_1 and G are the join of two graphs. On the other hand, $Spec(L(K_j \bigtriangledown G_1)) = Spec(L(G)) = spec(L(K_w \bigtriangledown P))$, where $j+i=w$ and also G has one vertex and $i+17$ edges more than G_1. Therefore, we must have $G \cong K_j \bigtriangledown G_1$. Because, G is the join of two graph and also according to spectrum of G, must K_j be joined to G_1 and this is only available state. □

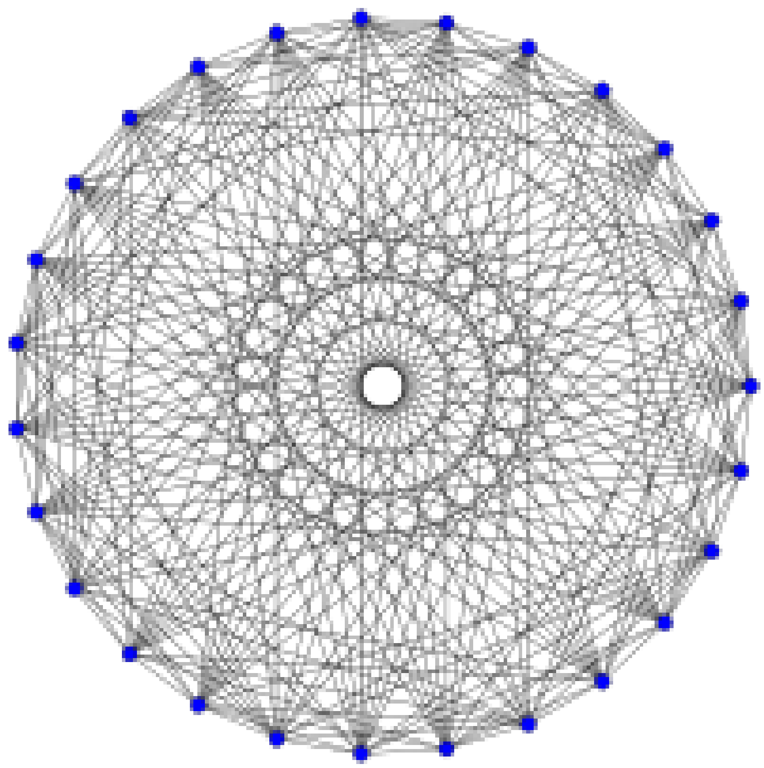

FIGURE 2. Schläfi graph

In the following section, we show that any graph cospectral with multicone graph $K_w \bigtriangledown S$ must be bidegreed.

Hereafter, with similar aguments of above results, we characterize another new classes of multicone graphs that are DS with respect to their spectra.

5. Connected graph cospectral with multicone graph $K_w \bigtriangledown S$

Proposition 5.1. *Let G be a graph cospectral with multicone graph $K_w \bigtriangledown S$. Then $Spec(G) =$*

$\left\{[-1]^{w-1}, [1]^{20}, [-5]^{6}, \left[\frac{\Omega+\sqrt{\Omega^2+4\Gamma}}{2}\right]^{1}, \left[\frac{\Omega+\sqrt{\Omega^2+4\Gamma}}{2}\right]^{1}\right\}$, *where* $\Omega = 9 + w$ *and* $\Gamma = 17w + 10$.

Proof. It is well-known that $Spec(S) = \left\{[10]^{1}, [1]^{20}, [-5]^{6}\right\}$. Now, by Theorem 2.2, there is nothing to prove. □

In the following lemma, we show that any graph cospectral with multicone graph $K_w \bigtriangledown S$ must be bidegreed.

Lemma 5.2. *Let G be cospectral with multicone graph $K_w \bigtriangledown S$. Then G is bidegreed in which any vertex of G is of degree $w + 10$ or $w + 26$.*

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph G consists of at least three number. Hence the equality in Theorem 2.4 cannot happen for any δ. But, if we put $\delta = w+10$, then the equality in Theorem 2.4 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 26$. By contrary, we suppose that $\Delta < w + 26$. Therefore, the equality in Theorem 2.4 cannot hold for any δ. But, if we put $\delta = w + 10$, then this equality holds . This is a contradiction and so $\Delta = w+26$. Now, $\delta = w+10$, since G is bidegreed and G has $w+27$ vertices, $\Delta = w + 26$ and $w(w + 26) + 27(w + 10) = w\Delta + 27(w + 10) = \sum_{i=1}^{w+27} \deg v_i$. This completes the proof. □

In the following, we show that any graph cospectral with multicone graph $K_1 \bigtriangledown S$ is isomorphic to $K_1 \bigtriangledown S$.

5.1. **Connected graph cospectral with multicone graph $K_1 \bigtriangledown S$.**

Lemma 5.3. *Any graph cospectral with multicone graph $K_1 \bigtriangledown S$ is isomorphic to $K_1 \bigtriangledown S$.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown S$. By Lemma 5.2 , it is easy to see that G has one vertex of degree 27, say j. Now, Theorem 2.8 implies that $P_{G-j} = (x - \mu_3)^{19}(x - \mu_4)^5[\alpha_{1j}^2 D_1 + \alpha_{2j}^2 D_2 + \alpha_{3j}^2 D_3 + \alpha_{4j}^2 D_4]$, where

$\mu_1 = \frac{10+\sqrt{208}}{2}$, $\mu_2 = \frac{10-\sqrt{208}}{2}$, $\mu_3 = 1$ and $\mu_4 = -5$.
$D_1 = (x - \mu_2)(x - \mu_3)(x - \mu_4)$,
$D_2 = (x - \mu_1)(x - \mu_3)(x - \mu_4)$,
$D_3 = (x - \mu_1)(x - \mu_2)(x - \mu_4)$,
$D_4 = (x - \mu_1)(x - \mu_2)(x - \mu_3)$.

Now, we have:

$x + y + 10 = -(19\mu_3 + 5\mu_4)$,
$x^2 + y^2 + 100 = 270 - (19\mu_3^2 + 5\mu_4^2)$,

where x and y are the eigenvalues of graph $G-j$. If we solve above equation, then $x = 1$ and $y = -5$. Hence $Spec(G - j) = Spec(S)$ and so $G - j \cong S$. This completes the proof. □

Theorem 5.4. *Multicone graph $K_w \bigtriangledown S$ are DS with respect to their adjacency spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown S)$, then $G_1 \cong K_w \bigtriangledown S$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown S)$, then $G \cong K_{w+1} \bigtriangledown S$, where G is a graph. By Lemma 5.2 , Theorem 2.4 , Lemma 2.1 (iii) and in a similar manner of Lemma 5.3 for $G-j$, where j is a vertex of degree $w+56$ belong to G, we obtain $Spec(G-j) = Spec(K_w \bigtriangledown S)$. This completes the proof. □

In the following, we show that multicone graphs $K_w \bigtriangledown S$ are DS with respect to their Laplacian spectrum.

6. Connected graph cospectral with multicone graph $K_w \bigtriangledown S$ with respect to Laplacian spectrum

Theorem 6.1. *Multicone graph $K_w \bigtriangledown S$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. let the claim be true for values less than w; that is, $Spec(L(G_1)) = Spec(L(K_i \bigtriangledown S)) = \left\{[i+27]^i,\ [i+9]^{20},\ [i+15]^6,\ [0]^1\right\}$ follows that $G_1 \cong K_i \bigtriangledown S$, where $i = 1, 2, ..., w-1$. We show that the problem is true for w; that is, we show that $Spec(L(G)) = Spec(L(K_w \bigtriangledown S)) = \left\{[w+27]^w,\ [w+9]^{20},\ [w+15]^6,\ [0]^1\right\}$ follows that $G \cong K_w \bigtriangledown S$, where G is a graph. Theorem 2.6 implies that G_1 and G are the join of two graphs. On the other hand, $Spec(L(K_j \bigtriangledown G_1)) = Spec(L(G)) = spec(L(K_w \bigtriangledown S))$, where $j+i = w$ and also G has one vertex and $i+27$ edges more than G_1. Therefore, we must have $G \cong K_j \bigtriangledown G_1$. Because, G is the join of two graphs and also according to spectrum of G, must K_j be joined to G_1 and this is only available state. □

Now, we review what is stated in previous sections and finally we propose four problems.

7. Corollaries and Conjectures

Corollary 7.1. *Let G be a graph. The following statemants are equivalent:*
(i) $G \cong K_w \bigtriangledown P$.
(ii) G is cospectral with multicone graph $K_w \bigtriangledown P$.
(iii) $Spec(L(G)) = Spec((L(K_w \bigtriangledown P)))$

Corollary 7.2. *Let G be a graph. The following statemants are equivalent:*

(i) $G \cong K_w \bigtriangledown S$.
(ii) G is cospectral with multicone graph $K_w \bigtriangledown S$.
(iii) $Spec(L(G)) = Spec((L(K_w \bigtriangledown S)))$

Now, we pose the following conjectures.

Conjecture 7.3. *Graphs* $\overline{K_w \bigtriangledown P}$ *are DS with respect to their adjacency spectrums.*

Conjecture 7.4. *Multicone graphs* $K_w \bigtriangledown P$ *are DS with respect to signless Laplacian spectrums.*

Conjecture 7.5. *Graphs* $\overline{K_w \bigtriangledown S}$ *are DS with respect to their adjacency spectrums.*

Conjecture 7.6. *Multicone graphs* $K_w \bigtriangledown S$ *are DS with respect to signless Laplacian spectrums.*

References

[1] A. Abdollahi, S. Janbaz and M. Oubodi, *Graphs Cospectral with A Friendship Graph Or its Complement*, Trans. Combin., **2**, no. 4 (2013), 37–52.

[2] A. Z. Abdian, and S. M. Mirafzal, *On new classes of multicone graphsdetermined by their spectrums*, Algebr. Struct. Appl., **2**, no. 1 (2015), 21-32.

[3] N. L. Biggs, *Algebraic Graph Theory*, (second edition), Cambridge University press, Cambridge, (1933).

[4] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of graph Spectra*, London Mathematical Society Student Texts, **75**, Cambridge University Press, Cambridge, (2010).

[5] A. Mohammadian, B. Tayfeh-Rezaie, *Graphs with four distinct Laplacian eigenvalues*, J. Algebraic Combin., **34** (2011), 671–682.

[6] G. R. Omidi, *On graphs with largest Laplacian eignnvalues at most* 4, Australas. J. Combin., **44** (2009) 163–170.

[7] P. Rowlinson, *THE MAIN EIGENVALUES OF A GRAPH: A SURVEY*, Appl. Anal. Discrete Math., **1** (2007): 445-471.

[8] E. R. van Dam and W. H. Haemers, *Which graphs are determined by their spectrum?*, Linear Algebra. Appl. **373** (2003) 241–272.

[9] E. R. van Dam, W. H. Haemers, *Developments on spectral characterizations of graphs*, Discrete Math., **309** (2009) 576–586.

[10] J. F. Wang, H. Zhao, and Q. Haung, *Spectral charactrization of multicone graphs*, Czec. Math. J., **62**, no. 137 (2012), 117–126.

[11] D. B. West, *Introduction to Graph Theory* (Second Edition), University of Illinios–Urbana (2001).

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