Journal of Mathematical Extension
Vol. 10, No. 4, (2016), 87-109
ISSN: 1735-8299
URL: http://www.ijmex.com

Three Step Iteration Process for Two Multivalued Nonexpansive Maps in Hyperbolic Spaces

S. H. Khan*
Qatar University

D. Agbebaku
University of Nigeria

M. Abbas
University of Pretoria

Abstract. In this paper, a three step iteration process has been introduced for two multivalued nonexpansive maps in hyperbolic spaces. Using this process, common fixed points of the two mappings have been approximated through Δ-and strong convergence. A couple of examples have been provided to validate our main results. Our results generalize many reults of the contemporary literature. In particular, the results of [2] are generalized from Banach to hyperbolic spaces, those of [13] from single-valued to multivalued maps in hyperbolic space, and those of [6] to three step iterative scheme in hyperbolic spaces.

AMS Subject Classification: 47H04; 47H09; 47H10
Keywords and Phrases: Iteration process, multivalued, nonexpansive mappings, hyperbolic type spaces

1. Introduction

Many of the mathematical models of real-world problems originating from Biology, Chemistry, Economics, Engineering and Physics, amongst others are

Received: January 2016; Accepted: April 2016
*Corresponding author

usually expressed in the form of functional equations. Such equations can be written in form of fixed point equation

$$Tx = x, \tag{1}$$

where T is an appropriate nonlinear operator and x is an independent variable describing the physical phenomena. The behavior of the physical phenomena that this equation describes can be determined by the nature of the *solutions* to this fixed point equation. In general, the fixed points of the equation (1) are not easily obtained, hence the need for approximate solutions. In this regard, different iterative schemes have been developed and used to approximate fixed points of nonlinear mappings on suitable domains. On the other hand, an iterative process that approximates the fixed points of a nonlinear fixed point equation in a fewer number of iterations is preferable to the iterative schemes with more iteration steps. Different types of iterative schemes have been used in the literature. The very famous Mann iteration scheme is a one-step iteration process while the Ishikawa process is a two-step iteration scheme among others. In [7], Glowinski and Le Tallec used a three-step iteration scheme to approximate the solutions of the elastoviscoplasticity problem in liquid crystal theory and eigenvalue computations. They showed that a three-step iteration process is better in giving numerical results than a one- or two- step iteration process. Haubruge et al. [11] applied the Glowinski and Le Tallec iteration scheme to obtain a new splitting type iterations for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. They showed also that three-step iteration process leads to a highly paralleled iterations under certain conditions. Furthermore, in [2], Abbas and Nazir introduced a new three-step iteration scheme that is faster than the Agarwal et al. two-step iteration scheme. Thus studying a three-step iteration process will yield a better numerical result when applied to real-world problems. In this paper, we study the three-step iteration scheme (to be defined in sequel) introduced by Abbas and Nazir [2].

The role played by ambient spaces involve in a fixed point equation is also very important. In this regard, we note that Banach spaces with convex geometric structures have been studied extensively. Since a Banach space is a vector space, one can introduce a convex structure on it. However, metric spaces do not naturally have this convex structure. The notion of convex metric spaces was introduced by Takahashi [31] who studied the fixed points of nonexpansive mappings in the setting of convex metric spaces. Over time, different convex structures have been introduced on metric spaces. Hyperbolic spaces are specific examples of convex metric spaces. Different definitions of hyperbolic spaces can be found in the literature, see [9, 16, 18, 27], for examples. Although the hyperbolic space defined by Kohlenbach [18] is slightly restrictive than the hyperbolic space introduced in [8], it is, however, more general than the hyperbolic space introduced in [27]. Moreover, this class of hyperbolic spaces contains the Hadamard manifolds, Hilbert balls equipped with the hyperbolic metric [9], $\mathbb{R}$-trees and Cartesian products of Hilbert balls as special cases.

The nonlinear mapping T involved in a fixed point equation (1) is also very important. Nonexpansive mappings constitute one of the important class of nonlinear mappings in fixed point theory.

We recap some necssary ideas as follows. A subset K of a metric space X is proximal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, K) = \inf\{d(x, y) : y \in K\} = d(x, k).$$

Denote by $CB(K)$ the set of closed and bounded subsets of K, by $C(K)$ the compact subset of K and by $P(K)$ the set of proximal bounded subsets of K. Let $H(A, B)$ be the Hausdorff metric induced by the metric d of X, that is

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for all $A, B \in CB(X)$. A multivalued map $T : K \longrightarrow CB(X)$ is nonexpansive if

$$H(Tx, Ty) \leqslant d(x, y),$$

for all $x, y \in K$. A point $x \in K$ is a fixed point of T if $x \in Tx$. We denote by $F(T)$ the set of all fixed ponit of T and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$.

Kohlenbach [18] defined a hyperbolic space as follows:

Definition 1.1. *A metric space (X, d) is a hyperbolic space if there exists a map $W : X^2 \times [0, 1] \longrightarrow X$ such that*

(i) $d(u, W(x, y, \alpha)) \leqslant (1 - \alpha)d(u, x) + \alpha d(u, y)$,

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

(iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,

(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leqslant (1 - \alpha)d(x, y) + \alpha d(z, w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

CAT(0) spaces and Banach spaces are important examples of this type of hyperbolic spaces. A hyperbolic space represents a unified approach for both linear and nonlinear structures simultaneously. There are hyperbolic spaces which are not imbedded in any Banach space, see for example [6]. If a hyperbolic space

(X, d, W) satisfies Definition 1.(i) only, then it coincides with the convex metric space introduced by Takahashi [31].

In the sequel whenever we mention hyperbolic space, we mean the one given above.

We now list some important required concepts.

Definition 1.2. *A subset K of a hyperbolic space X is called convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$* and $\alpha \in [0, 1]$.

Definition 1.3. *A hyperbolic space (X, d, W) is said to be* strictly convex *if for any $x, y \in X$* and $\lambda \in [0, 1]$, *there exists a unique element $z \in X$ such that*

$$d(z, x) = \lambda d(x, y) \quad \text{and} \quad d(z, y) = (1 - \lambda)d(x, y);$$

Definition 1.4. *A hyperbolic space (X, d, W) is* uniformly convex *[32] if for all $u, x, y \in X,\ r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that*

$$\left.\begin{array}{l} d(x, u) \leqslant r \\ d(y, u) \leqslant r \\ d(x, y) \geqslant \epsilon r \end{array}\right\} \Longrightarrow d(W(x, y, \frac{1}{2}), u) \leqslant (1 - \delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \longrightarrow (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called modulus of uniform convexity. The modulus of uniform convexity η is said to be monotone if it decreases with r, for fixed ϵ. Note that a uniformly convex hyperbolic space is strictly convex, see [21].

The study of fixed points for multivalued nonexpansive mappings using Hausdorff metric was initiated by Markin [24], see also [1, 4, 14, 15, 25]. Shimizu and Takahashi [32] proved the existence of fixed points for multivalued nonexpansive maps in convex metric spaces.

There is an interesting and rich fixed point theory for multivalued maps. A wide range of theory for such maps has been developed and applied to different areas of mathematics such as control theory, optimization economics and differential equations, just to name a few, for example, see [10]. The existence and convergence of fixed points for multivalued nonexpansive maps in convex metric space has been discussed, for example, in [1] and the references therein.

In [2], Abbas and Nazir considered the following single-valued three-step iterative process in Banach spaces:

$$\left\{\begin{array}{l} z_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1 - \beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} = (1 - \alpha_n)T y_n + \alpha_n T z_n, \end{array}\right.$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$.

Khan [13] translated the above scheme into the language of hyperbolic spaces as follows:

$$\begin{aligned} x_{n+1} &= W\left(Ty_n, Tz_n, \alpha_n\right), \\ y_n &= W\left(Tx_n, Tz_n, \beta_n\right), \\ z_n &= W\left(x_n, Tx_n, \gamma_n\right),\ n \in \mathbb{N}. \end{aligned}$$

Although the results of Khan [13] are interesting, they are single-valued and use only one mapping. In this paper, we not only extend his results to multivalued case but also use two mappings to approximate common fixed points.

Let K be a nonempty convex subset of a hyperbolic space X. Let $S, T : K \longrightarrow C(K)$ be two multivalued mappings and $P_T(x) = \{y \in Tx : d(x,y) = d(x,Tx)\}$. Choose $x_0 \in K$ and define the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), \\ y_n = W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}), \\ x_{n+1} = W(u_n, w_n, \alpha_n), \end{cases} \tag{2}$$

where $v_n \in P_S(x_n)$, $u_n \in P_T(y_n) = P_T(W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}))$, $w_n \in P_T(z_n) = P_T(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}))$ and $\alpha_n, \beta_n \in (0,1)$ such that $\gamma_n \in (0, \frac{1}{2})$.

We verify that the algorithm (2) is well defined. Assume that P_S and P_T are nonexpansive multivalued mappings on K. It follows from the definition of P_T that $d(x,Tx) \leqslant d(x, P_T(x))$ for any x in K. Let a self map $f : K \longrightarrow K$ be defined as

$$f(x) = W(u, w, \alpha_1),$$

for some $u \in P_T(y) = P_T(W(v, w, \frac{\beta_1}{1-\alpha_1}))$, $w \in P_T(z) = P_T(W(x, v, \frac{\gamma_1}{\alpha_1+\beta_1}))$ and $v \in P_S(x_n)$.

For any $x_1, x_2 \in K$, let $v_1 \in P_S(x_1)$, $v_2 \in P_S(x_2)$ such that $d(v_1, v_2) = d(v_1, Sx_2)$,

$w_1 \in P_T(z_1) = P_T(W(x_0, v_1, \frac{\gamma_1}{\alpha_1+\beta_1})), w_2 \in P_T(z_2) = P_T(W(x_0, v_2, \frac{\gamma_1}{\alpha_1+\beta_1}))$ such that $d(w_1, w_2) = d(w_1, Tz_2)$,

$u_1 \in P_T(y_1) = P_T(W(v_1, w_1, \frac{\beta_1}{1-\alpha_1}))$, $u_2 \in P_T(y_2) = P_T(W(v_2, w_2, \frac{\beta_1}{1-\alpha_1}))$ such that $d(u_1, u_2) = d(u_1, Ty_2)$.

Using Definition 1.(iv), we have

$$
\begin{aligned}
& d(f(x_1), f(x_2)) \\
= \;& d(W(u_1, w_1, \alpha_1), W(u_2, w_2, \alpha_1)) \\
\leqslant \;& (1-\alpha_1)d(u_1, u_2) + \alpha_1 d(w_1, w_2) \\
= \;& (1-\alpha_1)d(u_1, Ty_2) + \alpha_1 d(w_1, Tz_2) \\
\leqslant \;& (1-\alpha_1)d(u_1, P_T(W(v_2, w_2, \frac{\beta_1}{1-\alpha_1}))) + \alpha_1 d(w_1, P_T(W(x_0, v_2, \frac{\gamma_1}{\alpha_1+\beta_1}))) \\
\leqslant \;& (1-\alpha_1)H(P_T(W(v_1, w_1, \frac{\beta_1}{1-\alpha_1})), P_T(W(v_2, w_2, \frac{\beta_1}{1-\alpha_1}))) \\
+ \;& \alpha_1 H(P_T(W(x_0, v_1, \frac{\gamma_1}{\alpha_1+\beta_1})), P_T(W(x_0, v_2, \frac{\gamma_1}{\alpha_1+\beta_1}))) \\
\leqslant \;& (1-\alpha_1)\left[1 - \frac{\beta_1}{1-\alpha_1}d(v_1, v_2) + \frac{\beta_1}{1-\alpha_1}d(w_1, w_2)\right] \\
& +\alpha_1\left[\frac{\gamma_1}{\alpha_1+\beta_1}d(v_1, v_2)\right] \\
= \;& (1-\alpha_1-\beta_1)d(v_1, v_2) + \beta_1 d(w_1, w_2) + \alpha_1\left[\frac{\gamma_1}{\alpha_1+\beta_1}d(v_1, v_2)\right] \\
= \;& (1-\alpha_1-\beta_1)d(v_1, Sx_2) + \beta_1 d(w_1, Tz_2) \\
& +\alpha_1\left[\frac{\gamma_1}{\alpha_1+\beta_1}d(v_1, Sx_2)\right]
\end{aligned}
$$

$$
\begin{aligned}
\leqslant \;& (1-\alpha_1-\beta_1)d(v_1, P_S(x_2)) + \beta_1 d(w_1, P_T(z_2)) \\
& +\alpha_1\left[\frac{\gamma_1}{\alpha_1+\beta_1}d(v_1, P_S(x_2))\right] \\
\leqslant \;& (1-\alpha_1-\beta_1)H(P_S(x_1), P_S(x_2)) \\
+ \;& \beta_1 H(P_T(W(x_0, v_1, \frac{\gamma_1}{\alpha_1+\beta_1})), P_T(W(x_0, v_2, \frac{\gamma_1}{\alpha_1+\beta_1}))) \\
+ \;& \frac{\alpha_1\gamma_1}{\alpha_1+\beta_1}H(P_S(x_1), P_S(x_2)) \\
\leqslant \;& (1-\alpha_1-\beta_1)d(x_1, x_2) + \frac{\beta_1\gamma_1}{\alpha_1+\beta_1}d(v_1, v_2) + \frac{\alpha_1\gamma_1}{\alpha_1+\beta_1}d(x_1, x_2) \\
= \;& (1-\alpha_1-\beta_1)d(x_1, x_2) + \frac{\beta_1\gamma_1}{\alpha_1+\beta_1}d(v_1, Sx_2) + \frac{\alpha_1\gamma_1}{\alpha_1+\beta_1}d(x_1, x_2) \\
\leqslant \;& (1-\alpha_1-\beta_1)d(x_1, x_2) + \frac{\beta_1\gamma_1}{\alpha_1+\beta_1}H(P_S(x_1), P_S(x_2)) + \frac{\alpha_1\gamma_1}{\alpha_1+\beta_1}d(x_1, x_2)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant (1-\alpha_1-\beta_1)d(x_1,x_2)+\frac{\beta_1\gamma_1}{\alpha_1+\beta_1}d(x_1,x_2)+\frac{\alpha_1\gamma_1}{\alpha_1+\beta_1}d(x_1,x_2)\\
&\leqslant (1-\alpha_1-\beta_1+\gamma_1)d(x_1,x_2)\\
&= 2\gamma_1 d(x_1,x_2).
\end{aligned}
$$

Since $2\gamma_1 < 1$, the mapping f is therefore a contraction. Hence by the Banach contraction mapping principle, f has a unique fixed point in K. Thus we have established the existence of x_1. Continuing in this way, the existence of $x_2, x_3, \cdots$ and thus x_n is guaranteed. Hence the above algorithm is well-defined.

The concept of Δ-convergence in a metric space was introduced by Lim [23]. In [18], Kirk and Panyanak applied Lim's concept to CAT(0) spaces and proved a number of results involving weak convergence in Banach spaces. Since then the notion of Δ-convergence has been widely studied and a number of papers have been published in this direction, see[9], [10], [19], [20], [24], [26], [30]. In order to give the definition of Δ-convergence, we first recall the notion of asymptotic radius and asymptotic center. Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define a continuous functional $r(x, \{x_n\})$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,\{x_n\}).$$

$r_K(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$. For any $y \in K$, the set

$$A_K(\{x_n\}) = \{x \in X : r(x,\{x_n\}) \leqslant r(y,\{x_n\})\}$$

is called the asymptotic center of $\{x_n\}$ with respect to $K \subset X$. Asymptotic radius and asymptotic center taken with respect to X are denoted as $r(\{x_n\})$ and $A(\{x_n\})$ respectively. The asymptotic center $A(\{x_n\})$ may, in general, be empty or may contain infinitely many points. It is well known that a complete uniformly convex space with monotone modulus of convexity enjoys the property that bounded sequences have unique asymptotic center with respect to closed convex subsets, see, for example, [22].

Definition 1.5. *A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of all subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, x is called the Δ-limit of $\{x_n\}$.*

Lemma 1.6. [5] **(i)** Every bounded sequence in X has a Δ-convergent subsequence (see [17, p. 3690]).

(ii) If K is a closed convex subset of X and $\{x_n\}$ is a bounded sequence in K, then asymptotic center of $\{x_n\}$ is in K (see [28, Proposition 2.1]).

(iii) If K is a closed convex subset of X and $f : K \longrightarrow X$ is a nonexpansive mapping, then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in K$ and $f(x) = x$ (see [17, Proposition 3.7]).

Lemma 1.7. [5] *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*
The following result is needed in the proof of our main results.

Lemma 1.8. [12] *Let K be a nonempty closed subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in K such that $\lim_{n\to\infty} r(y_m, \{x_n\}) = r(y, \{x_n\})$, then $\lim_{n\to\infty} y_m = y$.*

Lemma 1.9. [12] *Let (X, d, W) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(x_n, x) \leqslant r$, $\limsup_{n\longrightarrow\infty} d(y_n, x) \leqslant r$ and $\lim_{n\longrightarrow\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geqslant 0$, then $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$.*

2. Main Results

In this section we present the main results. The following lemma was proved in [6].

Lemma 2.1. *Let K be a nonempty subset of a metric space X and $T : K \longrightarrow C(K)$ be a multivalued mapping. Then $x \in F(T)$ iff $P_T(x) = \{x\}$ iff $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$.*
We denote $F = F(T) \cap F(S)$ the set of all common fixed points of the multivalued maps T and S.

Lemma 2.2. *Let K be a nonempty closed convex subset of a hyperbolic space X and let $S, T : K \longrightarrow C(K)$ be multivalued mappings such that P_T, P_S are nonexpansive and $F \neq \phi$. Then, for the sequence defined in* (2) , *$d(x_{n+1}, p) \leqslant d(x_n, p)$* for each $p \in F$.

Proof. Let $p \in F$. Then $p \in P_T(p) = \{p\}$ and $p \in P_S(p) = \{p\}$. Using (2) and Definition 1., we have

$$\begin{aligned} d(z_n, p) &= d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n + \beta_n}), p) \\ &\leqslant (1 - \frac{\gamma_n}{\alpha_n + \beta_n}) d(x_n, p) + \frac{\gamma_n}{\alpha_n + \beta_n} d(v_n, p) \end{aligned}$$

$$\begin{aligned}
&\leqslant (1-\frac{\gamma_n}{\alpha_n+\beta_n})d(x_n,p)+\frac{\gamma_n}{\alpha_n+\beta_n}H(P_S(x_n),P_S(p))\\
&\leqslant (1-\frac{\gamma_n}{\alpha_n+\beta_n})d(x_n,p)+\frac{\gamma_n}{\alpha_n+\beta_n}d(x_n,p)\\
&\leqslant d(x_n,p).
\end{aligned}$$

That is,

$$d(z_n,p)\leqslant d(x_n,p). \tag{3}$$

Next,

$$\begin{aligned}
d(y_n,p) &= d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p)\\
&\leqslant (1-\frac{\beta_n}{1-\alpha_n})d(v_n,p)+\frac{\beta_n}{1-\alpha_n}d(w_n,p)\\
&\leqslant (1-\frac{\beta_n}{1-\alpha_n})H(P_S(x_n),P_S(p))+\frac{\beta_n}{1-\alpha_n}H(P_T(z_n),P_T(p))\\
&\leqslant (1-\frac{\beta_n}{1-\alpha_n})d(x_n,p)+\frac{\beta_n}{1-\alpha_n}d(z_n,p)\\
&\leqslant (1-\frac{\beta_n}{1-\alpha_n})d(x_n,p)+\frac{\beta_n}{1-\alpha_n}d(x_n,p)\\
&= d(x_n,p).
\end{aligned}$$

That is,

$$d(y_n,p)\leqslant d(x_n,p). \tag{4}$$

Therefore,

$$\begin{aligned}
d(x_{n+1},p) &= d(W(u_n,w_n,\alpha_n),p)\\
&\leqslant (1-\alpha_n)d(u_n,p)+\alpha_n d(w_n,p)\\
&\leqslant (1-\alpha_n)H(P_T(y_n),P_T(p))+\alpha_n H(P_T(z_n),P_T(p))\\
&\leqslant (1-\alpha_n)d(y_n,p)+\alpha_n d(z_n,p) \qquad (5)\\
&\leqslant (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p)\\
&= d(x_n,p).
\end{aligned}$$

That is,

$$d(x_{n+1},p)\leqslant d(x_n,p). \quad \square$$

The following results are immediate consequence of Lemma 2.2.

Corollary 2.3. *Let the assumptions of Lemma 2.2 hold. Then for the sequence* $\{x_n\}$ *in* (2) *,we have*

1. $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in F$.
2. $\lim_{n\to\infty} d(x_n, F(T))$ *exists.*
3. $\{x_n\}$ *is bounded.*

Lemma 2.4. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and let $S, T : K \longrightarrow C(K)$ be multivalued maps such that P_T, P_S are nonexpansive maps and $F \neq \emptyset$. Then for the sequence defined in*(2) ,

$$\lim_{n\to\infty} d(x_n, P_S(x_n)) = \lim_{n\to\infty} d(x_n, P_T(y_n)) = \lim_{n\to\infty} d(x_n, P_T(z_n)) = 0.$$

Proof. From Corollary 2.3, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$, call it c for some $c \geqslant 0$. We proceed to proof for the case $c > 0$ as the case $c = 0$ trivially holds. Now $\lim_{n\longrightarrow\infty} d(x_{n+1}, p) = c$ means

$$\lim_{n\longrightarrow\infty} d(W(u_n, w_n, \alpha_n), p) = c.$$

Since P_T is nonexpansive, we have

$$\begin{aligned} d(u_n, p) &\leqslant H(P_T(y_n), P_T(p)) \\ &\leqslant d(y_n, p) \\ &\leqslant d(x_n, p) \text{ by } (4), \end{aligned}$$

so that

$$d(u_n, p) \leqslant d(x_n, p).$$

Taking lim sup of both sides, we have

$$\limsup_{n\to\infty} d(u_n, p) \leqslant \limsup_{n\to\infty} d(x_n, p) = c.$$

Thus

$$\limsup_{n\to\infty} d(u_n, p) \leqslant c. \tag{6}$$

Similarly,

$$\limsup_{n\to\infty} d(v_n, p) \leqslant c \tag{7}$$

and

$$\limsup_{n\to\infty} d(w_n, p) \leqslant c. \tag{8}$$

Furthermore,

$$\begin{aligned}
d(W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}), p) &\leqslant (1 - \frac{\beta_n}{1-\alpha_n})d(v_n, p) + \frac{\beta_n}{1-\alpha_n} d(w_n, p) \\
&\leqslant (1 - \frac{\beta_n}{1-\alpha_n})d(x_n, p) + \frac{\beta_n}{1-\alpha_n} d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

That is

$$d(W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}), p) \leqslant d(x_n, p).$$

Taking lim sup of both sides, we have

$$\limsup_{n\to\infty} d(W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}), p) \leqslant c. \tag{9}$$

Similarly, one can show that

$$\limsup_{n\to\infty} d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p) \leqslant c. \tag{10}$$

Moreover, from (5) , we have

$$\begin{aligned}
& d(x_{n+1}, p) \\
\leqslant\ & (1-\alpha_n)d(y_n, p) + \alpha_n d(z_n, p) \\
\leqslant\ & (1-\alpha_n)\left[(1 - \frac{\beta_n}{1-\alpha_n})d(x_n, p) + \frac{\beta_n}{1-\alpha_n} d(z_n, p)\right] + \alpha_n d(z_n, p) \\
\leqslant\ & (1-\alpha_n)\left[(1 - \frac{\beta_n}{1-\alpha_n})d(x_n, p) + \frac{\beta_n}{1-\alpha_n} d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p)\right] \\
+\ & \alpha_n d(x_n, p) \\
\leqslant\ & (1-\alpha_n-\beta_n)d(x_n, p) + \beta_n d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p) + \alpha_n d(x_n, p) \\
=\ & (1-\beta_n)d(x_n, p) + \beta_n d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p).
\end{aligned}$$

This gives

$$\begin{aligned}
\beta_n d(x_{n+1}, p) &\leqslant (1-\beta_n)d(x_n, p) \\
&+ \beta_n d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p) - (1-\beta_n)d(x_{n+1}, p) \\
&\leqslant \beta_n d(W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}), p) \\
&\quad +(1-\beta_n)\left[d(x_n, p) - d(x_{n+1}, p)\right],
\end{aligned}$$

and so

$$\begin{aligned} d(x_{n+1},p) &\leqslant d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p)+\frac{(1-\beta_n)}{\beta_n}\left[d(x_n,p)-d(x_{n+1},p)\right] \\ &\leqslant d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p)+\frac{(1-a)}{a}\left[d(x_n,p)-d(x_{n+1},p)\right]. \end{aligned}$$

That is,

$$d(x_{n+1},p)\leqslant d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p)+\frac{(1-a)}{a}\left[d(x_n,p)-d(x_{n+1},p)\right].$$

Now taking lim inf of both sides we have

$$c\leqslant \liminf_{n\longrightarrow\infty} d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p). \tag{11}$$

This implies that

$$\begin{aligned} c &\leqslant \liminf_{n\longrightarrow\infty} d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p) \\ &\leqslant \limsup_{n\longrightarrow\infty} d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p)\leqslant c, \end{aligned}$$

and, in turn, we have

$$\lim_{n\longrightarrow\infty} d(W(x_n,v_n,\frac{\gamma_n}{\alpha_n+\beta_n}),p)=c$$

From Lemma 1., (7) , and $\lim_{n\longrightarrow\infty} d(x_n,p)=c$, it follows that

$$\lim_{n\longrightarrow\infty} d(x_n,v_n)=0.$$

Again, from (5) , we have

$$\begin{aligned} d(x_{n+1},p) &\leqslant (1-\alpha_n)d(y_n,p)+\alpha_n d(z_n,p) \\ &\leqslant (1-\alpha_n)d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p)+\alpha_n d(z_n,p) \\ &\leqslant (1-\alpha_n)d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n},p)+\alpha_n d(x_n,p) \\ (1-\alpha_n)d(x_{n+1},p) &\leqslant (1-\alpha_n)d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n},p)) \\ &+ \alpha_n[d(x_n,p)-d(x_{n+1},p)] \\ d(x_{n+1},p) &\leqslant (1-\alpha_n)d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) \\ &+ \frac{\alpha_n}{1-\alpha_n}\left[d(x_n,p)-d(x_{n+1},p)\right] \\ d(x_{n+1},p) &\leqslant d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p)+\frac{b}{1-b}\left[d(x_n,p)-d(x_{n+1},p)\right]. \end{aligned}$$

That is,

$$d(x_{n+1},p) \leqslant d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) + \frac{b}{1-b}\left[d(x_n,p) - d(x_{n+1},p)\right].$$

Now taking lim inf of both sides, we have

$$c \leqslant \liminf_{n\longrightarrow\infty} d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p).$$

Also by (9) ,

$$\limsup_{n\longrightarrow} d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) \leqslant c,$$

so that,

$$\begin{aligned} c &\leqslant \liminf_{n\longrightarrow\infty} d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) \\ &\leqslant \limsup_{n\longrightarrow\infty} d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) \leqslant c, \end{aligned}$$

which implies that

$$\lim_{n\longrightarrow\infty} d(W(v_n,w_n,\frac{\beta_n}{1-\alpha_n}),p) = c. \tag{12}$$

From Lemma 1., (7) , (8) and (12) , it follows that

$$\lim_{n\longrightarrow\infty} d(v_n,w_n) = 0.$$

From basic properties of metric d, we have

$$d(x_n,w_n) \leqslant d(x_n,v_n) + d(v_n,w_n)$$

which implies that

$$\lim_{n\longrightarrow\infty} d(x_n,w_n) \leqslant d(x_n,v_n) + d(v_n,w_n) \longrightarrow 0.$$

Thus

$$\lim_{n\longrightarrow\infty} d(x_n,w_n) = 0.$$

From(2) , we have

$$d(x_{n+1},p) = d(W(u_n,w_n,\alpha_n),p),$$

which implies

$$\lim_{n\longrightarrow\infty} d(W(u_n,w_n,\alpha_n),p) = c.$$

So that from (6) , (8) and Lemma 1., we get

$$\lim_{n\longrightarrow\infty} d(u_n, w_n) = 0.$$

Thus

$$d(x_n, u_n) \leqslant d(x_n, w_n) + d(w_n, u_n),$$

implies that

$$\lim_{n\longrightarrow\infty} d(x_n, u_n) = 0.$$

Finally, since $d(x, P_S(x)) = \inf_{z\in P_S(x)} d(x, z)$, we have

$$\lim_{n\longrightarrow\infty} d(x_n, P_S(x_n)) \leqslant \lim_{n\longrightarrow\infty} d(x_n, v_n) = 0,$$

which implies that

$$\lim_{n\longrightarrow\infty} d(x_n, P_S(x_n)) = 0.$$

Similarly,

$$\lim_{n\longrightarrow\infty} d(x_n, P_S(y_n)) = 0,$$

and

$$\lim_{n\longrightarrow\infty} d(x, P_S(z_n)) = 0.$$

This completes the proof. □

In what follows, we prove that the sequence $\{x_n\}$ given in (2) Δ-converges to a common fixed point of two multivalued nonexpansive mappings.

Theorem 2.5. *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S, T : K \longrightarrow C(K)$ be multivalued nonexpansive maps such that P_T and P_S are nonexpansive. Let $\{x_n\}$ be the sequence in (2). Then $\{x_n\}$ Δ-converges to a common fixed point of S and T (or P_T and P_S).*

Proof. Since $\{x_n\}$ is bounded by Corollary 2.3, it follows that $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x\}$. Let $\{z_n\}$ denote any subsequence of $\{x_n\}$ such that $A(\{z_n\}) = \{z\}$. Then by Lemma 2.4, we have

$$\lim_{n\to\infty} d(z_n, P_T(z_n)) = \lim_{n\to\infty} d(z_n, P_S(z_n)) = 0.$$

Claim: z is a common fixed point of P_S and P_T.

Proof of claim: Take w_m in $P_T(z)$. Then

$$\begin{aligned} r(w_n, \{z_n\}) &= \limsup_{n\to\infty} d(w_n, z_n) \\ &\leqslant \limsup_{n\to\infty} \{d(w_m, P_T(z_n)) + d(P_T(z_n), z_n) \\ &\leqslant \limsup_{n\to\infty} H(P_T(z), P_T(z_n)) \\ &\leqslant \limsup_{n\to\infty} d(z, \{z_n\}) \\ &= r(z, \{z_n\}). \end{aligned}$$

This implies that $|r(w_n, \{z_n\}) - r(z, \{z_n\})| \to 0$ as $m \to \infty$. Therefore, from Lemma 1., we have that $\lim_{n\to\infty} w_m = z$. Since $P_T(K)$ and $P_S(K)$ are closed and bounded subsets of K, it follows that $P_T(z)$ and $P_S(z)$ are closed. Consequently, $\lim_{n\to\infty} w_m = z \in P_T(z)$ and $\lim_{n\to\infty} w_m = z \in P_S(z)$. Hence $z \in F(T)$ and $z \in F(S)$ so that $z \in F$.

By the existence of the limit $\lim_{n\to\infty} d(x_n, z)$ and the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x) \\ &\leqslant \limsup_{n\to\infty} \{d(x_n, x) \\ &\leqslant \limsup_{n\to\infty} d(x_n, z) \\ &= \limsup_{n\to\infty} d(z, \{z_n\}) \\ &= r(z, \{z_n\}). \end{aligned}$$

which leads to a contradiction. Hence $x = z$. Thus $A(\{z_n\}) = \{z\}$ for every subsequence $\{z_n\}$ of $\{x_n\}$ which shows that $\{x_n\}$ Δ-converges to a common fixed point of S and T. $\square$

Next, we give a necessary and sufficient condition for the strong convergence of the iterative scheme (2) .

Theorem 2.6. *Let K be a nonempty, closed and convex subset of a complete hyperbolic space X. Let $S, T : K \longrightarrow C(K)$ be multivalued nonexpansive maps such that P_T and P_S are nonexpansive. Let $\{x_n\}$ be the sequence in* (2) *. Then $\{x_n\}$ converges strongly to a common fixed point of S and T (or P_T and P_S) if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. ($\Longrightarrow$) Suppose that $\{x_n\}$ converges to a common fixed point p of T and S. Then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leqslant d(x_n, F) \leqslant d(x_n, p)$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$ so that $\liminf_{n\to\infty} d(x_n, F) = 0$.

($\Longleftarrow$) Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. Then from Lemma 2.2, we infer that

$$d(x_{n+1}, F) \leqslant d(x_{n+1}, F),$$

so that $\lim_{n\to\infty} d(x_n, F)$ exists. Using the hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$ and the existence of $\lim_{n\to\infty} d(x_n, F)$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

We now show that $\{x_n\}$ is a Cauchy sequence in K. To see this, let $m, n \in \mathbb{N}$ and assume $m > n$. Since from Lemma 2.2, $d(x_{n+1}, p) \leqslant d(x_n, p)$ for all $p \in F$, it follows that

$$d(x_m, p) \leqslant d(x_n, p) \quad \text{for all } p \in F.$$

Thus

$$d(x_m, x_n) \leqslant d(x_m, p) + d(x_n, p) \leqslant 2d(x_n, p).$$

Taking inf on the set F, we have

$$\inf_{p\in F} d(x_m, x_n) \leqslant \inf_{p\in F} 2d(x_n, p) = 2d(x_n, F).$$

This yields

$$d(x_m, x_n) \leqslant 2d(x_n, F).$$

On letting $m \to \infty,\ n \to \infty$, we have

$$0 \leqslant \lim_{n\to\infty} d(x_m, x_n) \leqslant \lim_{n\to\infty} 2d(x_n, F) \longrightarrow 0, \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} d(x_m, x_n) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in K.

Since X is complete, the sequence $\{x_n\}$ converges to a point, say $q \in X$. We now show that q is a common fixed point of T and S. That is, $q \in F$. Indeed,

$$d(x_n, F(P_T)) = \inf_{y\in F(P_T)} d(x_n, y),$$

implies that for each $\epsilon > 0$, there exists $p_n^\epsilon \in F(P_T)$ such that

$$d(x_n, p_n^\epsilon) < d(x_n, F(P_T)) + \frac{\epsilon}{4}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, p_n^\epsilon) < \lim_{n\to\infty} [d(x_n, F(P_T)) + \frac{\epsilon}{4}],$$

which implies

$$\lim_{n\to\infty} d(x_n, p_n^\epsilon) \leqslant \frac{\epsilon}{4}.$$

Observe that

$$d(p_n^\epsilon, q) \leqslant d(x_n, p_n^\epsilon) + d(x_n, q),$$

so that

$$\limsup_{n\to\infty} d(p_n^\epsilon, q) \leqslant \frac{\epsilon}{4}.$$

Finally, we have the following inequality.

$$\begin{aligned} d(P_T(q), q) &\leqslant d(q, p_n^\epsilon) + d(p_n^\epsilon, P_T(q)) \\ &\leqslant d(q, p_n^\epsilon) + H(P_T(p_n^\epsilon), P_T(q)) \\ &\leqslant d(q, p_n^\epsilon) + d(p_n^\epsilon, q) \\ &= 2d(p_n^\epsilon, q). \end{aligned}$$

Taking $\limsup$, we have

$$d(P_T(q), q) \leqslant 2 \limsup_{n\to\infty} d(p_n^\epsilon, q) \leqslant 2\frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon.$$

Since ϵ is arbitrary, therefore $d(P_T(q), q) = 0$. In a way similar way, one can show that $d(P_S(q), q) = 0$. Since F is closed, $q \in F$ as required. □

We now give some examples to illustrate our results.

Example 2.7. Let $K = [0, 1]$ be equipped with the metric $d(x, y) = |x-y|$. Let $S, T : K \longrightarrow CB(K)$ (family of closed and bounded subset of K) be defined by $Sx = [0, \frac{x}{4}]$ and $Tx = [0, \frac{x}{2}]$.Then for any $x, y \in K$

$$H(Tx, Ty) = \max\{\left|\frac{x}{2} - \frac{y}{2}\right|, 0\} = \left|\frac{x}{2} - \frac{y}{2}\right| = \left|\frac{x-y}{2}\right| \leqslant |x-y|.$$

Similarly,

$$H(Sx, Sy) = \max\{\left|\frac{x}{4} - \frac{y}{4}\right|, 0\} = \left|\frac{x}{4} - \frac{y}{4}\right| = \left|\frac{x-y}{4}\right| \leqslant |x-y|.$$

Thus T and S are multivalued nonexpansive maps. Clearly, $F(T) \cap F(S) = \{0\}$. We proceed to show that the sequence $\{x_n\}$ defined in (2) converges to a common fixed point of T and S. In this regard, we define P_T and P_S as follows.

If $x = 0$ then $P_T(x) = P_S(x) = \{0\}$. Let $x \in (0, 1]$, then

$$\begin{aligned} P_T(x) &= \{y \in Tx : d(x, y) = d(x, Tx)\} \\ &= \{y \in Tx : |x - y| = |x - Tx|\} \\ &= \{y \in Tx : |x - y| = |x - [0, \frac{x}{2}]|\} \\ &= \{y \in Tx : |x - y| = \frac{x}{2}\} \\ &= \left\{y = \frac{x}{2}\right\}. \end{aligned}$$

Similarly,

$$P_S(x) = \left\{\frac{3x}{4}\right\}.$$

We define the map $W : \mathbb{R}^2 \times [0, 1] \longrightarrow \mathbb{R}$ in (1.) by

$$W(x, y, \alpha) = (1 - \alpha)x + \alpha y.$$

and choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{3}$ so that the sequence $\{x_n\}$ is define as follows

$$\begin{cases} z_n = W(x_n, v_n, \frac{\gamma_n}{\alpha_n + \beta_n}) = \frac{1}{2}(x_n + v_n), \\ y_n = W(v_n, w_n, \frac{\beta_n}{1 - \alpha_n}) = \frac{1}{2}(v_n + w_n), \\ x_{n+1} = W(u_n, w_n, \alpha_n) = \frac{2}{3}u_n + \frac{1}{3}w_n, \end{cases}$$

with $v_n \in P_S(x_n) = \left\{\frac{3x_n}{4}\right\}$, that is,

$$v_n = \frac{3x_n}{4},$$

$u_n \in P_T(y_n) = \left\{\frac{y_n}{2}\right\}$, that is,

$$u_n = \frac{y_n}{2},$$

and $w_n \in P_T(z_n) = \left\{\frac{z_n}{2}\right\}$, that is,

$$w_n = \frac{z_n}{2}.$$

The table below show the result of our computation. Note that only a few iterations are displayed here, the other iterations follow a similar trend and are thus ignored.

Table 1: Computations Table

n	x_n	v_n	z_n	w_n	y_n	u_n	x_{n+1}
n	x_n	$\frac{3x_n}{4}$	$\frac{1}{2}(x_n + v_n)$	$\frac{z_n}{2}$	$\frac{1}{2}(v_n + w_n)$	$\frac{y_n}{2}$	$\frac{2}{3}u_n + \frac{1}{3}w_n$
1	0.5	0.375	0.4375	0.21875	0.296875	0.1484375	0.171875
2	0.171875	0.128906	0.150391	0.075195	0.102051	0.051025	0.059082
3	0.059082	0.044312	0.051697	0.025848	0.035080	0.017540	0.020309
4	0.020309	0.015232	0.017771	0.008885	0.012059	0.006029	0.006981
5	0.006981	0.005236	0.006109	0.003054	0.004145	0.002073	0.002400
6	0.002400	0.001800	0.002100	0.001050	0.001425	0.000712	0.000825
7	0.000825	0.000619	0.000722	0.000361	0.000490	0.000245	0.000284
8	0.000284	0.000213	0.000248	0.000124	0.000168	0.000084	0.000097
9	0.000097	0.000073	0.000085	0.000043	0.000058	0.000029	0.000034
10	0.000034	0.000025	0.000029	0.000015	0.000020	0.000010	0.000012
11	0.000012	0.000009	0.000010	0.000005	0.000007	0.000003	0.000004
12	0.000004	0.000003	0.000003	0.000002	0.000002	0.000001	0.000001
13	0.000001	0.000001	0.000001	0.000001	0.000001	0.000000	0.000000
14	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

From the table it is clear that as $n \longrightarrow \infty$, the sequence $x_n \longrightarrow 0 \in F$. Hence result follows.

Example 2.8. Let $K = [1, \infty)$ be equipped with the metric $d(x, y) = |x - y|$. Let $S, T : K \longrightarrow CB(K)$ (family of closed and bounded subset of K) be defined by $Sx = [1, 1 + \frac{x}{4}]$ and $Tx = [1, 1 + \frac{x}{2}]$. The sets of fixed points of the maps T and S are $F(T) = [1, 2]$ and $F(S) = [1, \frac{4}{3}]$, respectively. The set of common fixed points $F = [1, \frac{4}{3}]$.

$$H(Tx, Ty) = \max\left\{\left|\frac{x}{2} - \frac{y}{2}\right|, 0\right\} = \left|\frac{x}{2} - \frac{y}{2}\right| \leqslant |x - y|,$$

$$H(Sx, Sy) = \max\left\{\left|\frac{x}{4} - \frac{y}{4}\right|, 0\right\} = \left|\frac{x}{4} - \frac{y}{4}\right| \leqslant |x - y|.$$

We proceed to show that the sequence $\{x_n\}$ defined in (2) converges to a common fixed point of T and S. In this regard, we define P_T and P_S as follows.
If $x \in [1, \frac{4}{3}]$, then $P_T(x) = \{x\} = P_S(x)$.
If $x \in (2, \infty)$,then

$$P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\} = \left\{y = \frac{x+2}{2}\right\}.$$

Similarly,

$$P_S(x) = \left\{\frac{x+4}{4}\right\}.$$

However, if $x \in (\frac{4}{3}, 2]$ then $P_T(x) = \{x\}$ and $P_S(x) = \left\{\frac{x+4}{4}\right\}$. We define the map $W : \mathbb{R}^2 \times [0, 1] \longrightarrow \mathbb{R}$ in (1.) by

$$W(x, y, \alpha) = (1 - \alpha)x + \alpha y.$$

and choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{3}$ so that the sequence $\{x_n\}$ is define as follows

$$\begin{cases} z_n = W(x_n, v_n, \frac{\gamma_n}{\alpha_n+\beta_n}) = \frac{1}{2}(x_n + v_n), \\ y_n = W(v_n, w_n, \frac{\beta_n}{1-\alpha_n}) = \frac{1}{2}(v_n + w_n), \\ x_{n+1} = W(u_n, w_n, \alpha_n) = \frac{2}{3}u_n + \frac{1}{3}w_n, \end{cases}$$

with $v_n \in P_S(x_n) = \left\{\frac{x_n+4}{4}\right\}$, that is, $v_n = \frac{x_n+4}{4}$, $u_n \in P_T(y_n) = \left\{\frac{y_n+2}{2}\right\}$, that is, $u_n = \frac{y_n+2}{2}$, and $w_n \in P_T(z_n+2) = \left\{\frac{z_n+2}{2}\right\}$ that is, $w_n = \frac{z_n+2}{2}$.
The table below show the result of our computation. Note that only a few iterations are displayed here, the other iterations follow a similar trend and are thus ignored.

Table 2: Computations Table

n	x_n	v_n	z_n	w_n	y_n	u_n	x_{n+1}
n	x_n	$\frac{x_n+4}{4}$	$\frac{1}{2}(x_n+v_n)$	$\frac{z_n+2}{2}$	$\frac{1}{2}(v_n+w_n)$	$\frac{y_n+2}{2}$	$\frac{2}{3}u_n+\frac{1}{3}w_n$
1	3	1.75	2.375	2.1875	1.96875	1.984375	2.052083333
2	2.052083333	1.513020833	1.782552083	0.945638021	1.229329427	1.614664714	1.391655816
3	1.391655816	1.347913954	1.369784885	0.842446221	1.095180088	1.547590044	1.312542103
4	1.312542103	1.328135526	1.320338814	0.830084704	1.079110115	1.539555057	1.303064939
5	1.303064939	1.325766235	1.314415587	0.828603897	1.077185066	1.538592533	1.301929654
6	1.301929654	1.325482414	1.313706034	0.828426508	1.076954461	1.538477231	1.301793656
7	1.301793656	1.325448414	1.313621035	0.828405259	1.076926836	1.538463418	1.301777365
8	1.301777365	1.325444341	1.313610853	0.828402713	1.076923527	1.538461764	1.301775414
9	1.301775414	1.325443853	1.313609633	0.828402408	1.076923131	1.538461565	1.30177518
10	1.30177518	1.325443795	1.313609487	0.828402372	1.076923083	1.538461542	1.301775152
11	1.301775152	1.325443788	1.31360947	0.828402367	1.076923078	1.538461539	1.301775148
12	1.301775148	1.325443787	1.313609468	0.828402367	1.076923077	1.538461539	1.301775148
13	1.301775148	1.325443787	1.313609467	0.828402367	1.076923077	1.538461538	1.301775148

From the table it is clear that as $n \longrightarrow \infty$, the sequence $x_n \longrightarrow 1.301775148 \in F$. Hence result follows.

3. Concluding Remarks

The result presented in this paper concern the convergence of the three-step iteration scheme (2) to common fixed point of two multivalued nonexpansive maps in a hyperbolic space. Our results extend the work of Khan [13] to multivalued maps in hyperbolic spaces. Furthermore, our results can also be considered as an extension of the work of Khan et al. [6] to three steps iterative schemes in hyperbolic spaces.

References

[1] M. Abbas, S. H. Khan, A. R. Khan, and R. P. Agarwal. Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme. *Applied Mathematics Letters*, 24 (2) (2011), 97-102.

[2] M. Abbas and T. Nazir. A new faster iteration process applied to constrained minimization and feasibility problems. *Matematicki Vesnik, Article*, 66 (2) (2014), 223-234.

[3] R. P. Agarwal, D. O. Regan, and D. R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *J. Nonlinear Convex Anal.*, 8 (1), (2007), 61-79.

[4] H. Covitz and S. B. Nadler. Multi-valued contraction mappings in generalized metric spaces. *Israel Journal of Mathematics*, 8 (1) (1970), 5-11.

[5] S. Dhompongsa and B. Panyanak. On $\triangle$-convergence theorems in cat(0) spaces. *Computers & Mathematics with Applications*, 56 (10), 2572-2579, 2008.

[6] H. Fukhar-ud-din, S. H. Khan, and A. Kalsoom. Common fixed points of two multivalued nonexpansive maps by a one-step implicit algorithm in hyperbolic spaces, *Matematicki Vesnik*, 4 (2014), 1-13.

[7] R. Glowinski and P. Le Tallec. *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, Volume 9. SIAM, 1989.

[8] K. Goebel and W. A. Kirk. Iteration processes for nonexpansive mappings. *Topological Methods in Nonlinear Functional Analysis. Contemporary Mathematics*, 21 (1983), 115-123.

[9] K. Goebel and S. Reich. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings.* M. Dekker, 1984.

[10] L. Górniewicz. *Topological Fixed Point Theory of Multivalued Mappings*, volume 4. Springer, 2006.

[11] S. Haubruge, V. H. Nguyen, and J. Strodiot. Convergence analysis and applications of the glowinski–le tallec splitting method for finding a zero of the sum of two maximal monotone operators. *Journal of Optimization Theory and Applications*, 97 (3) (1998), 645-673.

[12] A. R. Khan, M. A. A. Khan, and et al. An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. *Fixed Point Theory and Applications*, 1 (2012), 1-12.

[13] S. H. Khan, Fixed point approximation of nonexpansive mappings on a nonlinear domain. *Abstract and Applied Analysis,* Volume 2014, Article ID 401650, 5 pages http://dx.doi.org/10.1155/2014/401650.

[14] S. H. Khan, M. Abbas, and B. E. Rhoades, A new one-step iterative scheme for approximating common fixed points of two multivalued nonexpansive mappings, *Rendiconti del Circolo Matematico di Palermo*, 59 (1) (2010), 151-159.

[15] S. H. Khan and I. Yildirim, Fixed points of multivalued nonexpansive mappings in Banach spaces. *Fixed Point Theory and Appl.,* 2012, Article ID, 73 (2012).

[16] W. A. Kirk. Krasnoselskii's iteration process in hyperbolic space, *Numerical functional Analysis and optimization*, 4 (4) (1982), 371-381.

[17] W. A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Analysis: Theory, Methods & Applications*, 68 (12) (2008), 3689-3696.

[18] U. Kohlenbach, Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357 (1) (2005), 89-128.

[19] T. Laokul and B. Panyanak, Approximating fixed points of nonexpansive mappings in CAT (0) spaces, *International Journal of Mathematical Analysis*, 3 (27) (2009), 1305-1315.

[20] W. Laowang and B. Panyanak, Strong and convergence theorems for multivalued mappings in CAT (0) spaces, *J. Inequal.Appl.,* volume 2009, Article ID 730132, 16 pages doi: 10.1155/2009/730132.

[21] L. Leustean, A quadratic rate of asymptotic regularity for CAT (0)-spaces, *Journal of Mathematical Analysis and Applications*, 325 (1) (2007), 386-399.

[22] L. Leustean, Nonexpansive iterations in uniformly convex w-hyperbolic spaces, Nonlinear analysis and optimization I. *Nonlinear analysis,* (2010), 193-210.

[23] T. C. Lim, Remarks on some fixed point theorems, *Proceedings of the American Mathematical Society*, (1976), 179-182.

[24] J. T. Markin, Continuous dependence of fixed point sets, *Proceedings of the American Mathematical Society*, 38 (3) (1973), 545-547.

[25] S. B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.*, 30 (2) (1969), 475-488.

[26] B. Nanjaras, B. Panyanak, and W. Phuengrattana, Fixed point theorems and convergence theorems for suzuki-generalized nonexpansive mappings in cat (0) spaces, *Nonlinear Analysis: Hybrid Systems*, 4 (1) (2010), 25-31.

[27] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Analysis: Theory, Methods and Applications*, 15 (6) (1990), 537-558.

[28] B. Panyanak, S. Dhompongsa, W.A. Kirk. Nonexpansive set-valued mappings in metric and banach spaces, *J. Nonlinear Convex Analysis*, 8 (1) (2007), 35-45.

[29] D. R. Sahu, Applications of the s-iteration process to constrained minimization problems and split feasibility problems, *Fixed Point Theory*, 12 (1) (2011), 187-204.

[30] K. P. R. Sastry and G. V. R. Babu, Convergence of ishikawa iterates for a multi-valued mapping with a fixed point, *Czechoslovak Mathematical Journal*, 55 (4) (2005), 817-826.

[31] W. Takahashi et al., A convexity in metric space and nonexpansive mappings, *Kodai Mathematical Seminar Reports*, 22 (2) (1970), 142-149.

[32] W. Takahashi and T. Shimizu, Fixed points of multivalued mappings in certain convex metric spaces, *Topol. Methods Nonlinear Anal.,* 8 (1) (1996), 197-203.

Safeer Hussain Khan
Department of Statistics
Statistics and Physics
Professor of Mathematics
Qatar University
Doha 2713, Qatar
E-mail: safeerhussain5@yahoo.com

Dennis Agbebaku
Department of Mathematics
Lecturer of Mathematics
University of Nigeria
410001 Nsukka, Nigeria
E-mail: dennis.agbebaku@unn.edu.ng

Mujahid Abbas
Department of Mathematics
Applied Mathematics
Professor of Mathematics
University of Jeddah
P. O. Box 80327, Jeddah 21589
Saudi Arabia
E-mail: abbas.mujahid@gmail.com