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# On Characterizing Pairs of Non-Abelian Nilpotent and Filiform Lie Algebras by their Schur Multipliers

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**Abstract.** Let $L$ be an $n$-dimensional non-abelian nilpotent Lie algebra. Niroomand and Russo (2011) proved that $\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $\mathcal{M}(L)$ is the Schur multiplier of $L$ and $s(L)$ is a non-negative integer. They also characterized the structure of $L$, when $s(L) = 0$. Assume that $(N, L)$ is a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian and $N$ is an ideal in $L$ and also $\mathcal{M}(N, L)$ is the Schur multiplier of the pair $(N, L)$. If $N$ admits a complement $K$ say, in $L$ such that $\dim K = m$, then $\dim\mathcal{M}(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$, where $t(K) = \frac{1}{2}m(m-1) - \dim\mathcal{M}(K)$. In the present paper, we characterize the pairs $(N, L)$, for which $0 \leqslant t(K) \leqslant s(L) \leqslant 3$. In particular, we classify the pairs $(N, L)$ such that $L$ is a non-abelian filiform Lie algebra and $0 \leqslant t(K) \leqslant s(L) \leqslant 17$.

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# 1. Introduction

All Lie algebras are considered over a fixed field $\Lambda$ and $[,]$ denotes the Lie bracket. Let $(N, L)$ be a pair of Lie algebras, where $N$ is an ideal in $L$. Then we define the *Schur multiplier* of the pair $(N, L)$ to be the abelian Lie algebra $\mathcal{M}(N, L)$ appearing in the following natural exact sequence of Lie algebras,

$$H_3(L) \to H_3(L/N) \to \mathcal{M}(N, L) \to \mathcal{M}(L) \to \mathcal{M}(L/N)$$
$$\to \frac{L}{[N, L]} \to \frac{L}{L^2} \to \frac{L}{(L^2 + N)} \to 0,$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. Ellis [4] proved that $\mathcal{M}(N, L) \cong \ker(N \wedge L \to L)$, in which $N \wedge L$ denotes the non-abelian exterior product of Lie algebras. Also using the above sequence, one can easily see that if the ideal $N$ possesses a complement in $L$, then $\mathcal{M}(L) = \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$. In this case, for every free presentation $0 \to R \to F \to L \to 0$ of $L$, $\mathcal{M}(N, L)$ is isomorphic to the factor Lie algebra $(R \cap [S, F])/[R, F]$, where $S$ is an ideal in the free Lie algebra $F$ such that $S/R \cong N$. In particular, if $N = L$, then the Schur multiplier of $(N, L)$ will be $\mathcal{M}(L) = (R \cap F^2)/[R, F]$ (see [2] for more detail), which is analogous to the Schur multiplier of a group (see [8]).
In 1994, Moneyhun [9] proved the Lie algebra analogue of Green's result (1956), which states that for an $n$-dimensional Lie algebra L, we have $\dim\mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L)$, where $t(L) \geqslant 0$. Characterization of finite dimensional nilpotent Lie algebras by their Schur multipliers has been already studied by many authors. In [2, 6, 7], all nilpotent Lie algebras are characterized, when $t(L) = 0, 1, 2, \ldots, 8$.

Moreover, Saeedi et al. [12] proved that in a pair $(N, L)$ of finite dimensional nilpotent Lie algebras, if $N$ admits a complement $K$ say, in $L$ with $\dim K = m$, then $\dim\mathcal{M}(N, L) = \frac{1}{2}n(n+2m-1) - t(N, L)$, for a non-negative integer $t(N, L)$. This actually gives us the Moneyhun's result, if $m = 0$. The first author and colleagues [1] characterized the pairs $(N, L)$ of finite dimensional nilpotent Lie algebras, for which $t(N, L) = 0, 1, 2, 3, 4$. Also in a special case, when $L$ is a *filiform* Lie algebra, they determined all pairs $(N, L)$, for $t(N, L) = 0, 1, \ldots, 10$.

Niroomand and Russo [11] presented an upper bound for the dimension of the Schur multiplier of an $n$-dimensional non-abelian nilpotent Lie algebra $L$ and showed that $\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $s(L) \geqslant 0$. They also classified the structure of $L$, when $s(L) = 0$.

Now, let $(N, L)$ be a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian of dimension $n$. If $N$ possesses a complement $K$ in $L$ such that $\dim K = m$, then by using $\mathcal{M}(L) = \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$, we get $\dim\mathcal{M}(N, L) =$

$\frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$, where $t(K) = \frac{1}{2}m(m - 1) - \dim\mathcal{M}(K)$. In this paper, we intend to classify the pairs $(N, L)$ such that $0 \leqslant t(K) \leqslant s(L) \leqslant 3$. One should notice that if $m = 0$, then the above bound for the dimension of $\mathcal{M}(N, L) = \mathcal{M}(L)$ is actually $\frac{1}{2}(n-1)(n-2) + 1$, which is discussed in [11]. While the results of [1] depend on the Moneyhun's bound $\frac{1}{2}n(n-1)$. This clarifies the differences of our results from the ones in [1].

In particular, we characterize the pairs $(N, L)$ with $0 \leqslant t(K) \leqslant s(L) \leqslant 17$, where $L$ is a finite dimensional non-abelian filiform Lie algebra. Note that our method in the proof of Theorem B of this paper, uses the Ganea's exact sequence [8], which is different from the applied technique in [1].

# 2. Preliminary Results

In this section, we discuss some preliminary results which will be used in the next section.

**Theorem 2.1.** *Let $L$ be a finite dimensional nilpotent Lie algebra. Then*

(*a*) $t(L) = 0$ *if and only if $L$ is abelian;*

(*b*) $t(L) = 1$ *if and only if* $L \cong H(1)$;

(*c*) $t(L) = 2$ *if and only if* $L \cong H(1) \oplus A(1)$;

(*d*) $t(L) = 3$ *if and only if* $L \cong H(1) \oplus A(2)$;

(*e*) $t(L) = 4$ *if and only if* $L \cong H(1) \oplus A(3)$, $L(3, 4, 1, 4)$ *or* $L(4, 5, 2, 4)$;

(*f*) $t(L) = 5$ *if and only if* $L \cong H(1) \oplus A(4)$ *or* $H(2)$;

(*g*) $t(L) = 6$ *if and only if* $L \cong H(1) \oplus A(5)$, $H(2) \oplus A(1)$, $L(4, 5, 1, 6)$, $L(3, 4, 1, 4) \oplus A(1)$ *or* $L(4, 5, 2, 4) \oplus A(1)$;

(*h*) $t(L) = 7$ *if and only if* $L \cong H(1) \oplus A(6)$, $H(2) \oplus A(2)$, $H(3)$, $L(7, 5, 2, 7)$, $L(7, 5, 1, 7)$, $L'(7, 5, 1, 7)$, $L(7, 6, 2, 7)$ *or* $L(7, 6, 2, 7, \beta_1, \beta_2)$;

(*i*) $t(L) = 8$ *if and only if* $L \cong H(1) \oplus A(7)$, $H(2) \oplus A(3)$, $H(3) \oplus A(1)$, $L(4, 5, 1, 6) \oplus A(1)$, $L(3, 4, 1, 4) \oplus A(2)$ *or* $L(4, 5, 2, 4) \oplus A(2)$.

*Here $H(m)$ denotes the Heisenberg Lie algebra of dimension $2m + 1$, $A(n)$ is an $n$-dimensional abelian Lie algebra and $L(a, b, c, d)$ denotes the algebra discovered for the case $t(L) = a$, where $b = dimL$, $c = dimZ(L)$ and $d = t(L)$.*

Table I describes the nilpotent Lie Algebras which are referred in the above theorem (see [2, 6, 7]).

A filiform Lie algebra is an algebra with maximal nilpotent index and defined by Vergne in [13].

**Definition 2.2.** *Let $L$ be a Lie algebra of dimension $n$ over a field $C$ of characteristic zero, and $C^1L = L$, $C^2L = [L, L], \dots,$ and $C^qL = [L, C^{q-1}L], \dots,$ be a* *descending sequence of $L$. Then $L$ is called a filiform Lie algebra if $dimC^qL = n-q$, for $2 \leqslant q \leqslant n$.*

The above definition implies that if $n = q$, then $L$ is nilpotent. Also $\dim C^{q-1}L/C^qL = 1$, for $q = 2, \ldots, n-1$ and $\dim C^1L/C^2L = 2$. In [5], filiform Lie algebras are classified up to dimension 11.

Recall that in [11], Niroomand and Russo proved that if $L$ is an $n$-dimensional non-abelian nilpotent Lie algebra, then $\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2)+1-s(L)$, where $s(L) \geqslant 0$. They also classified the structure of $L$, when $s(L) = 0$. Later in [10], Niroomand characterized all finite dimensional non-abelian nilpotent Lie algebras with $s(L) = 1, 2, 3$. In the following, we have collected all the above results.

**Theorem 2.3.** *Let $L$ be an $n$-dimensional non-abelian nilpotent Lie algebra. Then*

(*a*) $s(L) = 0$ *if and only if* $L \cong H(1) \oplus A(n-3)$;

(*b*) $s(L) = 1$ *if and only if* $L \cong L(4, 5, 2, 4)$;

(*c*) $s(L) = 2$ *if and only if $L$ isomorphic to one of the following Lie algebras*

$$L(3, 4, 1, 4),\ L(4, 5, 2, 4) \oplus A(1),\ H(m) \oplus A(n-2m-1),\ m \geqslant 2;$$

(*d*) $s(L) = 3$ *if and only if $L$ is isomorphic to one of the following Lie algebras*

$$L(4, 5, 1, 6),\ L(5, 6, 2, 7),\ L^{'}(5, 6, 2, 7),\ L(7, 6, 2, 7),\ L^{'}(7, 6, 2, 7),$$
$$L(3, 4, 1, 4) \oplus A(1),\ L(4, 5, 2, 4) \oplus A(2).$$

Now, suppose that $(N, L)$ is a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian of dimension $n$ and $N$ is a non-trivial ideal in $L$. Also, assume that $K$ is the complement of $N$ in $L$ such that $\dim K = m$. Then it follows from $\mathcal{M}(L) = \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$, $\dim\mathcal{M}(L) = \frac{1}{2}(n-1)(n-2)+1-s(L)$ and $\dim\mathcal{M}(K) = \frac{1}{2}m(m-1) - t(K)$ that $\dim\mathcal{M}(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$.

In the following theorem, we characterize the pairs $(N, L)$, for which $0 \leqslant t(K) \leqslant s(L) \leqslant 3$.

**Theorem A.** *If a pair $(N, L)$ satisfies the above assumptions, then*

(*a*) $(s(L), t(K)) = (0, 0)$ *if and only if* $(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n-3))$, $0 \leqslant j \leqslant n-4$.

(*b*) *There is not any pair $(N, L)$, for $(s(L), t(K))$ with $s(L) = 1$ and $t(K) = 0, 1$.*

*(c)* $(s(L), t(K)) = (2, 0)$ *if and only if* $(N, L)$ *is isomorphic to one of the following,*

$$(L(4,5,2,4), L(4,5,2,4) \oplus A(1)), (H(m) \oplus A(j), H(m) \oplus A(n-2m-1))$$
$$\text{such that } 0 \leqslant j \leqslant n-2m-2,\ m \geqslant 2.$$

*(d) There is not any pair* $(N, L)$*, for* $(s(L), t(K))$ *with* $s(L) = 2$ *and* $t(K) = 1, 2$.

*(e)* $(s(L), t(K)) = (3, 0)$ *if and only if* $(N, L)$ *is isomorphic to one of following,*

$$(L(3,4,1,4), L(4,5,1,6)),\ (L(3,4,1,4), L(3,4,1,4) \oplus A(1)),$$
$$(L(4,5,2,4), L(4,5,2,4) \oplus A(2)).$$

*(f)* $(s(L), t(K)) = (3, 1)$ *if and only if* $(N, L)$ *is isomorphic to one of following,*

$$(H(1), L'(5,6,2,7)), \quad (H(1), L(7,6,2,7)).$$

*(g) There is not any pair* $(N, L)$*, for* $(s(L), t(K))$ *with* $s(L) = 3$ *and* $t(K) = 2, 3$.

Also in the following result, we classify the pairs $(N, L)$ with $0 \leqslant t(K) \leqslant s(L) \leqslant 17$, where $L$ is a finite dimensional non-abelian filiform Lie algebra.

**Theorem B.** *Let* $(N, L)$ *be a pair of finite dimensional filiform Lie algebras, in which* $L$ *is non-abelian and* $N$ *is a non-trivial ideal in* $L$*. Then*

*(a)* $(s(L), t(K)) = (7, 0)$ *if and only if* $(N, L) \cong (N, L'(11, 6, 1, 11))$ *such that* $\{x, y, z, c, r, t\}$ *is a basis for* $L'(11, 6, 1, 11)$ *and*

$$N = \langle y, z, c, r, t |\ [z, c] = -t,\ [y, r] = t,\ [y, z] = r + \alpha t\rangle$$

*is an ideal of* $L'$ $(\alpha \in \Lambda)$.

*(b) There is not any pair* $(N, L)$ *for* $(s(L), t(K))$ *such that* $s(L) \in \{0, 1, 2, \cdots, 17\} - \{7\}$ *and* $t(K) \in \{1, 2, \cdots, 17\}$.

## 3. Proof of the Main Theorems

The following results are needed for proving Theorem A and B (see [2, 3, 9]).

**Lemma 3.1.** *Let* $A(n)$ *be an* $n$*-dimensional abelian Lie algebra and* $H(m)$ *the Heisenberg Lie algebra of dimension* $2m + 1$*. Then*

*(a)* $dim\mathcal{M}(A(n)) = \frac{1}{2}n(n-1)$;

(*b*) *dim*$\mathcal{M}(H(1)) = 2$;
(*c*) *dim*$\mathcal{M}(H(m)) = 2m^2 - m - 1$, *for* $m \geqslant 2$.

**Lemma 3.2.** Let $L = A \oplus B$. Then

$$dim\mathcal{M}(L) = dim\mathcal{M}(A) + dim\mathcal{M}(B) + dim(A/A^2 \otimes B/B^2).$$

**Theorem 3.3.** *If $L$ is a finite dimensional nilpotent Lie algebra of dimension greater than 1 and class $c$, then $\mathcal{M}(L) \neq 0$.*

By using [11], we may easily prove the following results. Observe that the results discussed in [11] are based on the equality $t(K) = \frac{1}{2}n(n-1) - \dim\mathcal{M}(L)$ in [9]. But our similar results depend on $s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim\mathcal{M}(L)$.

**Proposition 3.4.** *Let $L$ be a non-abelian filiform Lie algebra of dimension $n$, $K = Z(L)$ and $H = L/K$. Then $s(H) + \dim L^2 \leqslant s(L) + 1$.*

**Proposition 3.5.***Let $L$ be a non-abelian filiform Lie algebra of dimension $n$. Then* $\dim L^2 \leqslant s(L) + 1$.

**Theorem 3.6.** *Let $L$ be an $n$-dimensional non-abelian filiform Lie algebra and* $\dim L^2 = c$. *Then $c^2 - c \leqslant 2s(L)$.*

**Theorem 3.7.** *Let $L$ be an $n$-dimensional non-abelian filiform Lie algebra. Then*
(*a*) *There is no filiform Lie algebra, with* $s(L) = 1, 3, 4, 5, 6,$ 9, 10, 11, 14, 15, 16, 17;
(*b*) $s(L) = 0$ *if and only if* $L \cong H(1)$;
(*c*) $s(L) = 2$ *if and only if* $L \cong L(3, 4, 1, 4)$;
(*d*) $s(L) = 4$ *if and only if* $L \cong L(11, 6, 1, 11)$ *or* $L'(11, 6, 1, 11)$;
(*e*) $s(L) = 8$ *if and only if* $L \cong L(11, 6, 1, 12)$ *or* $L'(11, 6, 1, 12)$;
(*f*) $s(L) = 12$ *if and only if* $L \cong L(17, 7, 1, 17)$ *or* $L'(17, 7, 1, 17)$;
(*g*) $s(L) = 13$ *if and only if* $L \cong L(17, 7, 1, 18)$.

In Table II, we describe the filiform Lie Algebras which are referred in the above theorem, using [2, 6, 7].

Now, we are ready to prove our main theorems.

**Proof of Theorem A.** Case $s(L) = 0$. In this case $t(K) = 0$ and Theorem 2.1 and 2.3 imply that $L \cong H(1) \oplus A(n-3)$ and $K$ is an abelian subalgebra of $L$. According to the structure of $L$ by choosing a suitable subalgebra $K$ and an ideal $N$ in $L$, similar to the proof of Theorems A and B in [1], one can easily verify that

$$(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n-3)),$$

where $0 \leqslant j \leqslant n-4$.

Case $s(L) = 1$. In this case $t(K) = 0, 1$ and by Theorem 2.3, $L \cong L(4, 5, 2, 4)$. Let $t(K) = 0$. Thus $K$ is abelian. Now if $\dim N = 4$ and $\dim K = 1$, then

$$N = \langle x_1, x_3, x_4, x_5 | [x_1, x_4] = x_5 \rangle \cong H(1) \oplus A(1),$$

such that $\{x_1, x_2, x_3, x_4, x_5\}$ is a basis for $L$. Hence Theorem 2.1 implies that $t(N) = 2$ and $\dim \mathcal{M}(N) = 4$. But by Lemma 3.2, we have $\dim \mathcal{M}(N) = 3$, which is a contradiction. Also if $\dim N = 3$ and $\dim K = 2$ or $\dim N = 2$ and $\dim K = 3$, then there do not exist any ideal and subalgebra satisfying in the assumptions of the theorem. If $t(K) = 1$, then $K \cong H(1)$ and there do not exist any ideal and subalgebra satisfying in the theorem.

Case $s(L) = 2$. In this case $t(K) = 0, 1, 2$. By using Theorem 2.3, we get $L \cong L(3, 4, 1, 4)$, $L(4, 5, 2, 4) \oplus A(1)$ or $H(m) \oplus A(n-2m-1)$ such that $m \geqslant 2$. First suppose that $L \cong L(3, 4, 1, 4)$ and $t(K) = 0$, then $K$ is abelian. If $\dim N = 3$ and $\dim K = 1$ or $\dim K = \dim N = 2$, then choosing a suitable subalgebra $K$ and an ideal $N$ in $L$ and Lemma 3.2 imply that there is no such a pair. If $t(K) = 1$, then $K \cong H(1)$. By choosing a suitable ideal $N$ in $L$ and Lemma 3.2, there is not any desired pair. And finally if $t(K) = 2$, then ideal $N$ is trivial.

Now, let $L \cong L(4, 5, 2, 4) \oplus A(1)$. If $t(K) = 0$, then $K$ is abelian. Therefore $(N, L) \cong (L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(1))$. If $t(K) = 1, 2$, then Theorem 2.1, Lemma 3.2 and choosing a suitable subalgebra $K$ and an ideal $N$ of $L$ imply that there is not any pair.

Finally, let $L \cong H(m) \oplus A(n-2m-1)$ such that $m \geqslant 2$. If $t(K) = 0$, then similar to the first case, we can easily see that

$$(N, L) \cong (H(m) \oplus A(j), H(m) \oplus A(n-2m-1)),$$

where $0 \leqslant j \leqslant n-2m-2$ and $m \geqslant 2$. If $t(K) = 1$, then there is not any suitable ideal $N$ in $L$, and if $t(K) = 2$, then by Theorem 2.1 and Lemma 3.2, there is no such a pair.

Case $s(L) = 3$. By using Theorem 2.3, we have $L \cong L(4, 5, 1, 6)$, $L(3, 4, 1, 4) \oplus A(1)$, $L(4, 5, 2, 4) \oplus A(2)$, $L'(5, 6, 2, 7)$ or $L(7, 6, 2, 7)$. By an analogous manner in [1] and the previous cases, we can show that $(N, L)$ is isomorphic to one of the following pairs

$$\begin{aligned} &(L(3,4,1,4), L(4,5,1,6)), \quad (L(3,4,1,4), L(3,4,1,4) \oplus A(1)), \\ &(L(4,5,2,4), L(4,5,2,4) \oplus A(2)), \quad (H(1), L'(5,6,2,7)), \\ &(H(1), L(7,6,2,7)). \quad \square \end{aligned}$$

**Proof of Theorem B.** By Theorem 3.7 there is not any filiform Lie algebra, with $s(L) = 1, 3, 4, 5, 6, 9, 10, 11, 14, 15, 16, 17$. Similar to the proof of Theorem A, one can easily check that in cases $s(L) = 0, 2$, there is not any suitable pair. Thus consider only the cases $s(L) = 7, 8, 12, 13$.

Case $s(L) = 7$. In this case $t(K) = 0, 1, 2, \ldots, 7$. Theorem 3.7 implies that $L \cong L(11, 6, 1, 11)$ or $L'(11, 6, 1, 11)$.
Let $L \cong L(11, 6, 1, 11)$ and $t(K) = 0$. Now if $\dim N = 5$, $\dim K = 1$ and $\{x, y, z, c, r, t\}$ is a basis for $L(11, 6, 1, 11)$, then $N = \langle x, z, c, r, t\rangle$ such that $[x, z] = c$, $[x, c] = r$, $[x, r] = \alpha t$ and $[z, c] = -t$ or $N = \langle y, z, c, r, t\rangle$ such that $[y, z] = \beta t$, $[y, r] = t$ and $[z, c] = -t$, where $\alpha, \beta \in \Lambda$. If $N = \langle x, z, c, r, t\rangle$, then $\dim \mathcal{M}(N) = 10-l$, for a non-negative integer $0 \leqslant l \leqslant 10$. If $l = 0, 1, 2, \ldots, 8$ or 10, then by Theorem 2.1 and Theorem 3.3, we get a contradiction. Therefore, $l = 9$ and $\dim \mathcal{M}(N) = 1$. But Lemma 3.2 implies that $\dim \mathcal{M}(N) = 2$, which is impossible. Now, assume that $N = \langle y, z, c, r, t\rangle$, then by Lemma 3.2, $\dim \mathcal{M}(N) = 0$, which is a contradiction by Theorem 3.3. In the other possibilities for the dimensions of $N$ and $K$, one may easily see that there do not exist any ideal and subalgebra satisfying in the hypothesis of the theorem. Similar to previous cases, we may check that if $t(K) = 1, 2, 3, 4, 5, 6, 7$, then there is not any pair.
Let $L \cong L'(11, 6, 1, 11)$ and $t(K) = 0$. Hence K is abelian. Suppose that $\dim N = 5$, $\dim K = 1$ and $\{x, y, z, c, r, t\}$ is a basis for $L'(11, 6, 1, 11)$. If $N = \langle x, z, c, r, t | [x, z] = c, [x, c] = r, [x, r] = \alpha t, [z, c] = -t\rangle$, where $\alpha \in \Lambda$, then by Theorem 2.1 and Lemma 3.2, we get a contradiction. So assume that $N = \langle y, z, c, r, t | [y, z] = r + \beta t, [y, r] = t, [z, c] = -t\rangle$, where $\beta \in \Lambda$. Thus $\dim \mathcal{M}(N) = 1$ and we have $(N, L) \cong (N, L'(11, 6, 1, 11))$. In the other cases for the dimensions of $N$ and $K$, one may check that there do not exist any ideal and subalgebra satisfying in the theorem. Also, if $t(K) = 1, 2, 3, 4, 5, 6, 7$, then there is not any pair.

Case $s(L) = 8$. In this case, similar to the previous cases, there is not any pair.

Case $s(L) = 12$. By using Theorem 3.7, we have $L \cong (17, 7, 1, 17)$ or $L'(17, 7, 1, 17)$. First, let $L \cong L(17, 7, 1, 17)$ and $t(K) = 0$. Then $K$ is abelian. Suppose that $\dim N = 6$ and $\dim K = 1$ and $\{x, y, z, c, r, t, u\}$ is a basis for $L(17, 7, 1, 17)$. Now if

$$N = \langle y, z, c, r, t, u | [y, z] = \alpha t + \beta u, [y, c] = \gamma u\rangle,$$

where $\gamma = \alpha$, then by Lemma 3.2, we have $\dim \mathcal{M}(N) = 0$, which is a contradiction. If $N = \langle x, z, c, r, u, t | [x, c] = r, [x, z] = c, [x, r] = t\rangle$, then Lemma 3.2 implies that $\dim \mathcal{M}(N) = 1$. Now, consider the Lie algebra analogue of Ganea's exact sequence (see [8], Theorem 2.6.5) as follows

$$Z \otimes N/N^2 \to \mathcal{M}(N) \to \mathcal{M}(N/Z) \to N^2 \cap Z \to 0,$$

where $Z$ is a central ideal of $N$. Thus $Z(N) = \langle u, t\rangle$ and $\dim \mathcal{M}(N/Z) \leqslant \dim \mathcal{M}(N) + \dim(N^2 \cap Z)$. If $Z = \langle u\rangle$, then $\dim \mathcal{M}(N/\langle u\rangle)) = 2 - l$, for some non-negative integer $l$. Now if $l = 0$, then Theorem 2.1 implies that $N/\langle u\rangle$ is abelian, which is impossible. If $l = 1, 2$, then $N/\langle u\rangle \cong H(1)$ or $H(1) \oplus A(1)$, which is a contradiction and also if $Z = \langle u, t\rangle$, then by an analogous manner, we can get a contradiction. For the other cases of dimensions of $N$ and $K$, there are no ideal and subalgebra satisfying in the conditions of the theorem. Also, if $t(K) = 1, 2, \ldots, 8$, then there is not any pair.

Now, assume that $t(K) = 9$. If $\dim K = 1, 2, 3$, then $K$ is abelian or $H(1)$. Therefore $t(K) = 0$ or $1$ , which is a contradiction. If $\dim K = 4$, then $\dim \mathcal{M}(K) < 0$, which is impossible. If $\dim K = 5$ and $\dim N = 2$, then there are no ideal and subalgebra satisfying in the theorem. And finally, if $\dim K = 6$ and $\dim N = 1$, then $\dim \mathcal{M}(N, L) < 0$, which is impossible. In the case $t(K) = 10, 11, 12$ by a similar method, we may get a contradiction.
Let $L \cong L'(17, 7, 1, 7)$ and $t(K) = 0$. Similarly (the case $(s(L), t(K)) = (12, 0)$ and by using Ganea's exact sequence), there is not any pair $(N, L)$. Also, there are no ideal and subalgebra satisfying in the theorem, for $t(K) = 1, 2, \ldots, 12$.

Case $s(L) = 13$. In this case $L \cong L(17, 7, 1, 18)$ and we can check that there is not any pair. The proof is complete. $\square$

**Table I** :

| t(L) | dim L | Non Zero Multiplication | Nilpotent Lie algebra |
|---|---|---|---|
| 0 | | | Abelian |
| 1 | 3 | $[x_1, x_2] = x_3$ | $H(1)$ |
| 2 | 4 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(1)$ |
| 3 | 5 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(2)$ |
| 4 | 4 | $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$ | $L(3, 4, 1, 4)$ |
| 4 | 5 | $[x_1, x_2] = x_3,\ [x_1, x_4] = x_5$ | $L(4, 5, 2, 4)$ |
| 4 | 6 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(3)$ |
| 5 | 5 | $[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$ | $H(2)$ |
| 5 | 7 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(4)$ |
| 6 | 5 | $[x_1, x_2] = x_3,\ [x_1, x_3] = x_5$ | $L(3, 4, 1, 4) \oplus A(1)$ |
| 6 | 5 | $[x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5$ | $L(4, 5, 1, 6)$ |
| 6 | 6 | $[x_1, x_2] = x_5, [x_1, x_3] = x_5, [x_3, x_4] = x_5$ | $H(2) \oplus A(1)$ |
| 6 | 6 | $[x_1, x_2] = x_3,\ [x_1, x_4] = x_6$ | $L(4, 5, 2, 4) \oplus A(1)$ |
| 6 | 8 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(5)$ |
| 7 | 5 | $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$ | $L(7, 5, 2, 7)$ |
| 7 | 5 | $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$ | $L(7, 5, 1, 7)$ |
| 7 | 5 | $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$ <br> $[x_1, x_4] = x_5$ | $L'(7, 5, 1, 7)$ |
| 7 | 6 | $[x_1, x_2] = x_3, [x_1, x_4] = x_6, [x_2, x_5] = x_6$ | $L(5, 6, 2, 7)$ |
| 7 | 6 | $[x_1, x_2] = x_3,\ [x_4, x_5] = x_6$ | $L'(5, 6, 2, 7)$ |
| 7 | 6 | $[x_1, x_2] = x_5,\ [x_3, x_4] = x_6$ | $L(7, 6, 2, 7)$ |
| 7 | 6 | $[x_1, x_2] = x_5 + \beta_1 x_6,\ [x_3, x_4] = x_5$ <br> $[x_1, x_4] = x_6,\ [x_3, x_2] = \beta_2 x_6$ | $L(7, 6, 2, 7, \beta_1, \beta_2)$ |
| 7 | 7 | $[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$ | $H(2) \oplus A(2)$ |
| 7 | 7 | $[x_1, x_2] = x_7, [x_3, x_4] = x_7, [x_5, x_6] = x_7$ | $H(3)$ |
| 8 | 6 | $[x_1, x_2] = x_3,\ [x_1, x_3] = x_6$ | $L(3, 4, 1, 4) \oplus A(2)$ |
| 8 | 6 | $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_6$ | $L(4, 5, 1, 6) \oplus A(1)$ |
| 8 | 7 | $[x_1, x_2] = x_3,\ [x_1, x_4] = x_7$ | $L(4, 5, 2, 4) \oplus A(2)$ |
| 8 | 8 | $[x_1, x_2] = x_5,\ [x_3, x_4] = x_5$ | $H(2) \oplus A(3)$ |
| 8 | 8 | $[x_1, x_2] = x_7, [x_3, x_4] = x_7, [x_5, x_6] = x_7$ | $H(3) \oplus A(1)$ |
| 8 | 10 | $[x_1, x_2] = x_3$ | $H(1) \oplus A(7)$ |

**Table II** :

| s(L) | dim L | Non Zero Multiplication | Filiform Lie algebra |
|---|---|---|---|
| 4 | 4 | $[x,y]=z,\ [x,z]=r$ | $L(3,4,1,4)$ |
| 7 | 5 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$ | $L(7,5,1,7)$ |
| 11 | 6 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=\alpha_4 t,\ [y,z]=\alpha_5 t$<br>[y,r]=t, [z,c]=-t | $L'(7,5,1,7)$ |
| 11 | 6 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=\alpha_4 t,\ [y,z]=r+\alpha_5 t$<br>$[y,c]=\alpha_4 t,\ [y,r]=t,\ [z,c]=-t$ | $L'(7,5,1,7)$ |
| 12 | 6 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=t,\ [y,z]=\alpha_5 t$ | $L(11,6,1,12)$ |
| 12 | 6 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=t,\ [y,c]=t$<br>$[y,z]=r+\alpha_5 t$ | $L'(11,6,1,12)$ |
| 17 | 7 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=t,\ [y,z]=\alpha_5 t+\beta_5 u$<br>$[y,c]=\beta_6 u$ | $L(17,7,1,17)$<br>$\beta_6=\alpha_5$ |
| 17 | 7 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[y,z]=r+\alpha_5 t+\beta_5 u$<br>$[x,r]=t+\beta_4 u,\ [y,c]=t+\beta_6 u$<br>$[y,r]=\beta_7 u,\ [z,c]=\beta_8 u$ | $L(17,7,1,17)$<br>$\beta_6=\alpha_5$<br><br>$\beta_8=1-\beta_7$ |
| 18 | 7 | $[x,y]=z,\ [x,z]=c,\ [x,c]=r$<br>$[x,r]=t,\ [y,z]=\alpha_5 t+\beta_5 u$<br>$[y,c]=t+\beta_6 u,\ [y,r]=\beta_7 u$<br>$[z,c]=\beta_8 u$ | $L(17,7,1,18)$<br>$\beta_7=-\beta_8$ |

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