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# Families of Primitive Pythagorean Triples Involving Terms of Generalized Fibonacci and Lucas Sequences

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**Abstract.** In this paper, we consider primitive Pythagorean triples with terms of the generalized Fibonacci and Lucas sequences. We give families of primitive Pythagorean triples whose coefficient may all be simply expressed in terms of the generalized Fibonacci and Lucas sequences.

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## 1. Introduction

The second order sequence $\{W_n(a,b;r,s)\}$ is defined for $n>1$ by

$$W_n(a,b;r,s) = rW_{n-1}(a,b;r,s) - sW_{n-2}(a,b;r,s),$$

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in which $W_0(a,b;r,s)=a$, $W_1(a,b;r,s)=b$, where $a,b,r,s$ are arbitrary integers. As some special cases of $\{W_n(a,b;r,s)\}$, denote $W_n(0,1;r,-1)$, $W_n(2,r;r,-1)$, $W_n(0,1;R,1)$ and $W_n(2,R;R,1)$ by $U_n, V_n, u_n$ and $v_n$, respectively, where $|R|>2$ is an integer. Clearly, when $r=1$, $U_n=F_n$ ($n$th Fibonacci number) and $V_n=L_n$ ($n$th Lucas number). The Binet forms of the sequences $\{U_n\}$, $\{V_n\}$, $\{u_n\}$ and $\{v_n\}$ are given by

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ V_n=\alpha^n+\beta^n \text{ and } u_n=\frac{\gamma^n-\delta^n}{\gamma-\delta},\ v_n=\gamma^n+\delta^n,$$

where $\alpha,\beta$ and $\gamma,\delta$ are the roots of equations $x^2-rx-1=0$ and $x^2-Rx+1=0$, respectively [5].
Pythagoras have a big importance and interest in mathematics, science and philosophy. He is known by many people because of the Pythagorean Theorem that is about a property of all triangles with a right-angle(an angle of 90°). Due to Pythagorean Theorem, Pythagorean triples correspond to right triangles with integer sides.

1. A Pythagorean triple is a triple $(a,b,c)$ of positive integers $a,b,c$ such that $a^2+b^2=c^2$.

2. A primitive Pythagorean triple is a Pythagorean triple in which $gcd(a,b,c)=1$.

Throughout this paper, we will take primitive Pythagorean triple $(a,b,c)$ such that $a\equiv 1(mod2)$ and $b\equiv 0(mod2)$.

**Theorem 1.1.** [3] *Let $e$ and $f$ be positive coprime integers of opposite parity with $e>f$. If $a,b,c$ are integers such that*

$$a=e^2-f^2,\ b=2ef,\ \text{and } c=e^2+f^2, \tag{1}$$

*then $(a,b,c)$ is a primitive Pythagorean triple.*

**Corollary 1.2.** [3] *Let $e'$ and $f'$ be odd positive coprime integers with $e'>f'$. If $a,b,c$ are integers such that*

$$a=e'f',\ b=\frac{1}{2}(e'^2-f'^2),\ \text{and } c=\frac{1}{2}(e'^2+f'^2), \tag{2}$$

*then $(a,b,c)$ is a primitive Pythagorean triple.*

Let $\Pi$ be a set as follows: for Pythagorean triple $(a, b, c)$,

$$\Pi = \{(a, b, c)pa + qb + rc = t, a \equiv 1(mod2), b \equiv 0(mod2), \gcd(a, b, c) = 1\}. \quad (3)$$

In [4], let $m$ and $n$ be fixed coprime integers and let $t$ be an arbitrary fixed integer. Terr gave some definitions regarding families of primitive Pythagorean triple as follows: for $m < n$, the families of primitive Pythagorean triples

$$\Pi(*, m, n \mid t) = \{(a, b, c) \in \Pi : mc - nb = t\},$$

$$\Pi(m, *, n \mid t) = \{(a, b, c) \in \Pi : mc - na = t\},$$

and for any integers $n, m$, familiy of primitive Pythagorean triple

$$\Pi(m, n, * \mid t) = \{(a, b, c) \in \Pi : mb - na = t\},$$

and the primitive Pythagorean triple family parametrizations

$$P(*, m, n \mid t) = \left\{(e, f) : m\left(e^2 + f^2\right) - 2nef = t\right\},$$

$$P^{'}(*, m, n \mid t) = \left\{(e', f') : m\left(e'^2 + f'^2\right) - n(e'^2 - f'^2) = 2t\right\}. \quad (4)$$

Similarly, primitive Pythagorean triple family parametrizations

$$P(m, *, n \mid t),\ P(m, n, * \mid t) \text{ and } P'(m, *, n \mid t),\ P'(m, n, * \mid t),$$

are defined.

The author investigate families of primitive Pyhagorean triples of the form $(a, b, c)$, where $mc - nb = t$, $mc - na = t$ or $mb - na = t$ for some fixed positive coprime integers $m$ and $n$, and a fixed nonzero integer $t$. A few of these cases are especially interesting since the solutions may be simply written in terms of Fibonacci and Lucas numbers.

In this paper, inspiring by the work in [4], we consider some interesting results of primitive Pythagorean triples with terms of the generalized Fibonacci and Lucas sequences. We give families of primitive Pythagorean triples whose coefficient may all be simply expressed in terms of these sequences.

## 2. Families of Primitive Pythagorean Triples Involving Terms Of Generalized Fibonacci And Lucas Sequences

Throughout this section, we denote $r^2 + 4$ and $R^2 - 4$ by $\Delta$ and $D$, respectively. We will give some results related to the sequences $\{U_n\}$, $\{V_n\}$, $\{u_n\}$ and $\{v_n\}$ for further steps.

**Theorem 2.1.** [2] *The integer solutions of* $x^2 - \Delta y^2 = 4$ *are precisely* the *pairs* $(\pm V_{2j}, \pm U_{2j})$.

**Corollary 2.2.** *The integer solutions of*

$$x^2 - \Delta y^2 = 1, \tag{5}$$

*are precisely the pairs* $\left(\pm\frac{V_{6j}}{2}, \pm\frac{U_{6j}}{2}\right)$.

**Proof.** Let $X$, $Y$ be $2x$ and $2y$, respectively. Then $X$ and $Y$ satisfy equation

$$X^2 - \Delta Y^2 = 4$$

if and only if $x$ and $y$ satisfy equation (5). From Theorem 2.1, we have $X = \pm V_{2i}$ and $Y = \pm U_{2i}$ for some nonnegative integer $i$. Now $X$ and $Y$ are both even precisely when $i$ is a multiple of 3, i.e. $i = 3j$ for some nonnegative integer $j$. □

Now, integer solutions of the other Pell equations (2) are given by Table 1 as follows:

Table 1: Integer solutions of some Pell equations

| | Pell equations | Integer solutions of them |
|---|---|---|
| 1. | $x^2 - \Delta y^2 = -4$ | $(\pm V_{2j+1}, \pm U_{2j+1})$ |
| 2. | $x^2 - \Delta y^2 = -1$ | $\left(\pm \frac{V_{6j+3}}{2}, \pm \frac{U_{6j+3}}{2}\right)$ |
| 3. | $x^2 - \Delta y^2 = -4\Delta$ | $(\pm \Delta U_{2j}, \pm V_{2j})$ |
| 4. | $x^2 - \Delta y^2 = -\Delta$ | $\left(\pm \Delta \frac{U_{6j}}{2}, \pm \frac{V_{6j}}{2}\right)$ |
| 5. | $x^2 - \Delta y^2 = 4\Delta$ | $(\pm \Delta U_{2j+1}, \pm V_{2j+1})$ |
| 6. | $x^2 - \Delta y^2 = \Delta$ | $\left(\pm \Delta \frac{U_{6j+3}}{2}, \pm \frac{V_{6j+3}}{2}\right)$ |
| 7. | $x^2 - Dy^2 = 4$ | $(\pm v_j, \pm u_j)$ |
| 8. | $x^2 - Dy^2 = 1$ | $\left(\pm \frac{v_{3j}}{2}, \pm \frac{u_{3j}}{2}\right)$ |
| 9. | $x^2 - Dy^2 = -4D$ | $(\pm D u_j, \pm v_j)$ |
| 10. | $x^2 - Dy^2 = -D$ | $\left(\pm D \frac{u_{3j}}{2}, \pm \frac{v_{3j}}{2}\right)$ |

We will look at various types of primitive Pythagorean triple families whose coefficient involving terms of the sequences $\{U_n\}$, $\{V_n\}$, $\{u_n\}$ and $\{v_n\}$. From now on, we will take odd numbers $r$ and $R$.

For $m = (\Delta - 1)/2$ and $n = (\Delta + 1)/2$, we will investigate families of primitive Pythagorean triples of the form $\Pi(*, (\Delta - 1)/2, (\Delta + 1)/2 \mid t)$ for some integer $t$. These families of primitive Pythagorean triples correspond to families of right triangles asymptotically similar to one with legs of length $\sqrt{\Delta}$ and $(\Delta - 1)/2$ and hypotenuse of length $(\Delta + 1)/2$. For odd number $r$, we get $2 \nmid \sqrt{\Delta}$, $2 \mid (\Delta - 1)/2$ and $\gcd(*, (\Delta - 1)/2, (\Delta + 1)/2) = 1$. Thus a primitive Pythagorean triple $(a, b, c)$ belongs to

$$\Pi(*, (\Delta - 1)/2, (\Delta + 1)/2 \mid t)$$

if and only if $(\Delta - 1)\, c - (\Delta + 1)\, b = 2t$, which holds if and only if

$$(\Delta - 1)\,(e'^2 + f'^2) - (\Delta + 1)\,(e'^2 - f'^2) = 4t, \qquad (6)$$

where $e'$ and $f'$ are the parameters (2).

**Lemma 2.3.**

$$\begin{aligned}
& P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid -2\right) \\
= \; & \{(V_{6k+4}, U_{6k+4}) : k \in \mathbb{N}\} \cup \{(V_{6k+2}, U_{6k+2}) : k \in \mathbb{N}\}, \\
& P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid 2\right) \\
= \; & \left\{(V_{6k-1}, U_{6k-1}) : k \in \mathbb{Z}^{+}\right\} \cup \\
& \left\{(V_{6k-5}, U_{6k-5}) : \text{for } r > 1,\ k \in \mathbb{Z}^{+} \text{ and for } r = 1, k \in \mathbb{Z}^{+} \backslash \{1\}\right\}.
\end{aligned}$$

**Proof.** Using (4) and the parametrization in (2), we have

$$(e', f') \in P^{'}\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid \pm 2\right) \Longleftrightarrow$$

$$(\Delta-1)\left(e'^{2}+f'^{2}\right)-(\Delta+1)\left(e'^{2}-f'^{2}\right)=\pm 8$$

or

$$e'^{2}-\Delta f'^{2}=\pm 4.$$

Now we take $e'^{2}-\Delta f'^{2}=4$. From Table 1 we write

$$e'^{2}-\Delta f'^{2}=4 \Longleftrightarrow e'=V_{2j}, f'=U_{2j}. \tag{7}$$

For being valid the parametrization in (2), $e_j'=V_{2j}$ and $f_j'=U_{2j}$ must be odd, which is true if and only if $j$ is not divisible by 3. If we take $j=0$ in (7), $j=0$ is not valid since $e_0'=V_0=2$ and $f_0'=U_0=0 \notin \mathbb{Z}^{+}$. When $j=1$ in (7), we have $e_1'=V_2=r^2+2$ and $f_1'=U_2=r$. Thus $j=1$ is valid. It is easy to see that $j(j=3k+1,\, j=3k+2)$ correspond to elements $(e_j', f_j')$ of $P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid -2\right)$.
Similarly, it is seen that $j(j=3k-1,\ j=3k-3)$ correspond to elements $(e_j', f_j')$ of $P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid 2\right)$. Thus, the proof is completed. □

**Theorem 2.4.**

$$\Pi\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid 2\right)=\frac{1}{2\Delta} \times$$
$$\left\{(2\Delta U_{12k-2}, (\Delta-1) V_{12k-2}-2(\Delta+1), (\Delta+1)V_{12k-2}-2(\Delta-1)) : k \in \mathbb{Z}^{+}\right\} \cup$$
$$\left\{\begin{array}{c} (2\Delta U_{12k-10}, (\Delta-1) V_{12k-10}-2(\Delta+1), (\Delta+1)V_{12k-10}-2(\Delta-1)) : \\ \text{for } r>1, k \in \mathbb{Z}^{+} \text{ and for } r=1, k \in \mathbb{Z}^{+} \backslash \{1\} \end{array}\right\},$$

$$\Pi\left(*,(\Delta-1)/2,(\Delta+1)/2 \mid -2\right)=\frac{1}{2\Delta}\times$$
$$\{(2\Delta U_{12k+8},(\Delta-1)\,V_{12k+8}+2(\Delta+1),(\Delta+1)V_{12k+8}+2\,(\Delta-1)):k\in\mathbb{N}\}\cup$$
$$\{(2\Delta U_{12k+4},(\Delta-1)\,V_{12k+4}+2(\Delta+1),(\Delta+1)\,V_{12k+4}+2\,(\Delta-1)):k\in\mathbb{N}\}.$$

**Proof.** From (2), we have

$$\begin{aligned} a_j &= e'_j f'_j = U_{2j}, \\ b_j &= (e'^2_j - f'^2_j)/2 = (V_j^2 - U_j^2)/2, \\ c_j &= (e'^2_j + f'^2_j)/2 = (V_j^2 + U_j^2)/2. \end{aligned}$$

By Binet formulas of the sequences $\{U_j\}$ and $\{V_j\}$, we get

$$a_j = U_{2j}, \tag{8}$$
$$b_j = \left((\Delta-1)\,V_{2j}+2(-1)^j(\Delta+1)\right)/2\Delta, \tag{9}$$
$$c_j = \left((\Delta+1)V_{2j}+2(-1)^j\,(\Delta-1)\right)/2\Delta. \tag{10}$$

Thus, using equations (8)-(10), from Lemma 2.3, the proof is complete. □

For example, if we take $r=1$ in Theorem 2.4, we write as [4]

$$\begin{aligned} &\Pi\left(*,2,3 \mid 2\right) \\ =\ &\left\{\left(F_{12k-2},\frac{2}{5}\left(L_{12k-2}-3\right),\frac{1}{5}\left(3L_{12k-2}-4\right)\right):k\in\mathbb{Z}^+\right\}\cup \\ &\left\{\left(F_{12k-10},\frac{2}{5}\left(L_{12k-10}-3\right),\frac{1}{5}\left(3L_{12k-10}-4\right)\right):k\in\mathbb{Z}^+\backslash\left\{1\right\}\right\}, \end{aligned}$$

and

$$\begin{aligned} &\Pi\left(*,2,3 \mid -2\right) \\ =\ &\left\{\left(F_{12k+4},\frac{2}{5}\left(L_{12k+4}+3\right),\frac{1}{5}\left(3L_{12k+4}+4\right)\right):k\in\mathbb{N}\right\}\cup \\ &\left\{\left(F_{12k+8},\frac{2}{5}\left(L_{12k+8}+3\right),\frac{1}{5}\left(3L_{12k+8}+4\right)\right):k\in\mathbb{N}\right\}. \end{aligned}$$

Taking $t = \pm 2\Delta$ in (6) and using Table 1, the proofs of Lemma 2.5 and Theorem 2.6 are obtained similar to the proofs of Lemma 2.3 and Theorem 2.4, respectively.

**Lemma 2.5.**

$$\begin{aligned} & P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid 2\Delta\right) \\ = & \left\{ (\Delta U_{6k+4}, V_{6k+4}) : k \in N \right\} \cup \left\{ (\Delta U_{6k+2}, V_{6k+2}) : k \in \mathbb{N} \right\}, \\ & P'\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid -2\Delta\right) \\ = & \left\{ (\Delta U_{6k+1}, V_{6k+1}) : k \in N \right\} \cup \left\{ (\Delta U_{6k+5}, V_{6k+5}) : k \in \mathbb{N} \right\}. \end{aligned}$$

**Theorem 2.6.**

$$\Pi\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid -2\Delta\right) = \frac{1}{2} \times$$
$$\left\{ (2\Delta U_{12k+2}, (\Delta-1) V_{12k+2} + 2(\Delta+1), (\Delta+1) V_{12k+2} + 2(\Delta-1)) : k \in \mathbb{N} \right\} \cup$$
$$\left\{ (2\Delta U_{12k+10}, (\Delta-1) V_{12k+10} + 2(\Delta+1), (\Delta+1) V_{12k+10} + 2(\Delta-1)) : k \in \mathbb{N} \right\},$$

$$\Pi\left(*, (\Delta-1)/2, (\Delta+1)/2 \mid 2\Delta\right) =$$
$$\left\{ (2\Delta U_{12k+8}, (\Delta-1) V_{12k+8} - 2(\Delta+1), (\Delta+1) V_{12k+8} - 2(\Delta-1)) : k \in \mathbb{N} \right\} \cup$$
$$\left\{ (2\Delta U_{12k+4}, (\Delta-1) V_{12k+4} - 2(\Delta+1), (\Delta+1) V_{12k+4} - 2(\Delta-1)) : k \in \mathbb{N} \right\}.$$

Now, for $m = (D-1)/2$ and $n = (D+1)/2$, we will investigate families of primitive Pythagorean triples of the form $\Pi\left(*, (D-1)/2, (D+1)/2 \mid t\right)$ for some small integer $t$. For odd number $R$, we get $2 \nmid \sqrt{D}, 2 \mid (D-1)/2$ and $\gcd(*, (D-1)/2, (D+1)/2) = 1$. Thus primitive Pythagorean triple $(a, b, c)$ belongs to

$$\Pi\left(*, (D-1)/2, (D+1)/2 \mid t\right)$$

if and only if $(D-1)\, c - (D+1)\, b = 2t$, which holds if and only if

$$(D-1)\left(e'^2 + f'^2\right) - (D+1)\left(e'^2 - f'^2\right) = 4t, \tag{11}$$

where $e'$ and $f'$ are the parameters (2).

Taking $t = -2$ and $2D$ in (11) and using Table 1, the proofs of Lemmas 2.7 and 2.9 and Theorems 2.8 and 2.10 are obtained similar to the proofs of Lemma 2.3 and Theorem 2.4, respectively.

**Lemma 2.7.**

$$\begin{aligned} & P'\left(*, (D-1)/2, (D+1)/2 \mid -2\right) \\ = \; & \{(v_{6k-5}, u_{6k-5}) : k \in \mathbb{Z}^+\} \cup \{(v_{6k-1}, u_{6k-1}) : k \in \mathbb{Z}^+\} \cup \\ & \left\{(v_{6k-2}, u_{6k-2}) : k \in \mathbb{Z}^+\right\} \cup \left\{(v_{6k-4}, u_{6k-4}) : k \in \mathbb{Z}^+\right\}. \end{aligned}$$

**Theorem 2.8.**

$$\begin{aligned} & \Pi\left(*, (D-1)/2, (D+1)/2 \mid -2\right) = \frac{1}{2D} \times \\ & \left\{(2Du_{12k-2}, (D-1)\, v_{12k-2} + 2(D+1), (D+1)v_{12k-2} + 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-10}, (D-1)\, v_{12k-10} + 2(D+1), (D+1)v_{12k-10} + 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-8}, (D-1)\, v_{12k-8} + 2(D+1), (D+1)v_{12k-8} + 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-4}, (D-1)\, v_{12k-4} + 2(D+1), (D+1)\, v_{12k-4} + 2\,(D-1)) : k \in \mathbb{Z}^+\right\}. \end{aligned}$$

**Lemma 2.9.**

$$\begin{aligned} & P'\left(*, (D-1)/2, (D+1)/2 \mid 2D\right) \\ = \; & \{(Du_{6k-4}, v_{6k-4}) : k \in Z^+\} \cup \{(Du_{6k-2}, v_{6k-2}) : k \in \mathbb{Z}^+\} \\ & \cup \left\{(Du_{6k-1}, v_{6k-1}) : k \in Z^+\right\} \cup \left\{(Du_{6k-5}, v_{6k-5}) : k \in \mathbb{Z}^+\right\}. \end{aligned}$$

**Theorem 2.10.**

$$\begin{aligned} & \Pi\left(*, (D-1)/2, (D+1)/2 \mid 2D\right) = \frac{1}{2} \times \\ & \left\{(2Du_{12k-2}, (D-1)\, v_{12k-2} - 2(D+1), (D+1)v_{12k-2} - 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-10}, (D-1)\, v_{12k-10} - 2(D+1), (D+1)v_{12k-10} - 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-4}, (D-1)\, v_{12k-4} - 2(D+1), (D+1)v_{12k-4} - 2\,(D-1)) : k \in \mathbb{Z}^+\right\} \cup \\ & \left\{(2Du_{12k-8}, (D-1)\, v_{12k-8} - 2(D+1), (D+1)v_{12k-8} - 2\,(D-1)) : k \in \mathbb{Z}^+\right\}. \end{aligned}$$

For $m=(\Delta-1)/2$ and $n=(\Delta+1)/2$, we will investigate families of primitive Pythagorean triples of the form

$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid t\right),$$

for some small integer $t$. These families of primitive Pythagorean triples correspond to families of right triangles asymptotically similar to one with legs of length $\sqrt{\Delta}$ and $(\Delta-1)/2$ and hypotenuse of length $(\Delta+1)/2$. For odd number $r$, we get $2 \nmid \sqrt{\Delta}$, $2 \mid (\Delta-1)/2$ and $\gcd((\Delta-1)/2,*,(\Delta+1)/2)=1$. Thus primitive Pythagorean triple $(a,b,c)$ belongs to

$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid t\right),$$

if and only if $(\Delta-1)\,c-(\Delta+1)\,a=2t$, which holds if and only if

$$(\Delta-1)\left(e^2+f^2\right)-(\Delta+1)\left(e^2-f^2\right)=2t, \tag{12}$$

where $e$ and $f$ are the parameters (1).
Using $t=\pm1$ and $\pm\Delta$ in (12) and Table 1, the proofs of the following Lemmas and Theorems are similar to the proofs of Lemma 2.3 and Theorem 2.4.

**Lemma 2.11.**

$$\begin{aligned} P\left((\Delta-1)/2,*,(\Delta+1)/2 \mid -1\right) &= \{(V_{6k}/2, U_{6k}/2) : k\in\mathbb{Z}^+\}, \\ P\left((\Delta-1)/2,*,(\Delta+1)/2 \mid 1\right) &= \{(V_{6k+3}/2, U_{6k+3}/2) : k\in\mathbb{N}\}. \end{aligned}$$

**Theorem 2.12**

$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid -1\right)=\frac{1}{4\Delta}\times$$
$$\left\{\left((\Delta-1)\,V_{12k}+2(\Delta+1), 2\Delta U_{12k}, (\Delta+1)V_{12k}+2\,(\Delta-1)\right) : k\in\mathbb{Z}^+\right\},$$
$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid 1\right)=\frac{1}{4\Delta}\times$$
$$\left\{\left((\Delta-1)\,V_{12k+6}-2(\Delta+1), 2\Delta U_{12k+6}, (\Delta+1)V_{12k+6}-2\,(\Delta-1)\right) : k\in\mathbb{N}\right\}.$$

**Lemma 2.13**

$$\begin{aligned} P'\left((\Delta-1)/2,*,(\Delta+1)/2 \mid \Delta\right) &= \{(\Delta U_{6k}/2, V_{6k}/2) : k\in\mathbb{Z}^+\}, \\ P'\left((\Delta-1)/2,*,(\Delta+1)/2 \mid -\Delta\right) &= \{(\Delta U_{6k+3}/2, V_{6k+3}/2) : k\in\mathbb{N}\}. \end{aligned}$$

**Theorem 2.14.**

$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid \Delta\right)=\frac{1}{4}\times$$
$$\left\{\left((\Delta-1)\,V_{12k}-2(\Delta+1),2\Delta U_{12k},(\Delta+1)V_{12k}-2\,(\Delta-1)\right):k\in\mathbb{Z}^{+}\right\}$$
$$\Pi\left((\Delta-1)/2,*,(\Delta+1)/2 \mid -\Delta\right)$$
$$=\ \left\{\left((\Delta+1)V_{12k}-2\,(\Delta-1)\,,2\Delta U_{12k+6},(\Delta+1)V_{12k+6}+2\,(\Delta-1)\right):k\in\mathbb{N}\right\}.$$

Now, for $m=(D-1)/2$ and $n=(D+1)/2$, we will investigate families of primitive Pythagorean triples of the form $\Pi\left((D-1)/2,*,(D+1)/2 \mid t\right)$ for some small integer $t$. Since $R$ is odd number, we get $2\nmid\sqrt{D}, 2\mid(D-1)/2$ and $\gcd((D-1)/2,*,(D+1)/2)=1$. Thus primitive Pythagorean triple $(a,b,c)$ belongs to

$$\Pi\left((D-1)/2,*,(D+1)/2 \mid t\right)$$

if and only if

$$(D-1)\,c-(D+1)\,a=2t,$$

which holds if and only if

$$(D-1)\left(e^{2}+f^{2}\right)-(D+1)\left(e^{2}-f^{2}\right)=2t, \tag{13}$$

where $e$ and $f$ are the parameters (1).
Using $t=-1$ and $D$ in (13) and Table 1, respectively, the proofs of the following Lemmas and Theorems are obtained.

**Lemma 2.15**

$$P\left((D-1)/2,*,(D+1)/2 \mid -1\right)$$
$$=\ \left\{(v_{6k}/2,u_{6k}/2):k\in Z^{+}\right\}\cup\left\{(v_{6k+3}/2,u_{6k+3}/2):k\in\mathbb{N}\right\}.$$

**Theorem 2.16.**

$$\Pi\left((D-1)/2,*,(D+1)/2 \mid -1\right)=\frac{1}{4D}\times$$
$$\left\{\left((D-1)\,v_{12k}+2(D+1),2Du_{12k},(D+1)v_{12k}+2\,(D-1)\right):k\in\mathbb{Z}^{+}\right\}$$
$$\cup\left\{\left((D-1)\,v_{12k+6}+2(D+1),2Du_{12k+6},(D+1)v_{12k+6}+2\,(D-1)\right):k\in\mathbb{N}\right\}.$$

**Lemma 2.17.**

$$\begin{aligned} & P\left((D-1)/2, *, (D+1)/2 \mid D\right) \\ = \ & \{(Du_{6k}/2, v_{6k}/2) : k \in Z^+\} \cup \{(Du_{6k+3}/2, Dv_{6k+3}/2) : k \in \mathbb{N}\}. \end{aligned}$$

**Theorem 2.18.**

$$\begin{aligned} & \Pi\left((D-1)/2, *, (D+1)/2 \mid D\right) = \frac{1}{4} \times \\ & \left\{\left((D-1)\, v_{12k} - 2(D+1), 2Du_{12k}, (D+1)v_{12k} - 2\,(D-1)\right) : k \in \mathbb{Z}^+\right\} \\ & \cup \left\{\left((D-1)\, v_{12k+6} - 2(D+1), 2Du_{12k+6}, (D+1)v_{12k+6} - 2\,(D-1)\right) : k \in \mathbb{N}\right\}. \end{aligned}$$

For $m = r$ and $n = 2$, the families of primitive Pythagorean triples correspond to families of right triangles asymptotically similar to one with legs of length $r$ and 2 and hypotenuse of length $\sqrt{\Delta}$.
Consider the families of primitive Pythagorean triples of the form $\Pi\left(r, 2, * \mid t\right)$, where $t$ is a small integer. From the parametrization in (1), every primitive Pythagorean triple $(a, b, c)$ in $\Pi\left(r, 2, * \mid t\right)$ satisfy the equation $rb - 2a = t$, which implies that the parameters $e$ and $f$ satisfies the equation

$$2f^2 + 2ref - 2e^2 = t. \tag{14}$$

Since the left side is even, the right side must also be even. Hence, considering $t = \pm 2$ and $\pm 2\Delta$ in (14), respectively, the following desired results are given.

**Lemma 2.19.**

$$\begin{aligned} P(r, 2, * \quad & \mid \quad -2) = \\ \{(U_{6k-3}, U_{6k-4}) \quad & : \quad k \in \mathbb{Z}^+\} \cup \{(U_{6k+1}, U_{6k}) : k \in \mathbb{Z}^+\}, \\ P(r, 2, * \quad & \mid \quad 2) = \\ \{(U_{6k-2}, U_{6k-3}) \quad & : \quad k \in \mathbb{Z}^+\} \cup \{(U_{6k}, U_{6k-1}) : k \in \mathbb{Z}^+\}. \end{aligned}$$

**Theorem 2.20.**

$$\begin{aligned} \Pi\left(r, 2, * \mid -2\right) \quad & = \quad \frac{1}{\Delta} \times \\ \{(rV_{12k-7} + 4, 2(V_{12k-7} - r), \Delta U_{12k-7}) \quad & : \quad k \in \mathbb{Z}^+\} \cup \\ \{(rV_{12k+1} + 4, 2(V_{12k+1} - r), \Delta U_{12k+1}) \quad & : \quad k \in \mathbb{Z}^+\}, \end{aligned}$$

$$\begin{aligned} \Pi\,(r,2,*\mid 2) &= \frac{1}{\Delta}\times \\ \{(rV_{12k-5}-4, 2(V_{12k-5}+r), \Delta U_{12k-5}) &: k\in\mathbb{Z}^+\}\cup \\ \{(rV_{12k-1}-4, 2(V_{12k-1}+r), \Delta U_{12k-1}) &: k\in\mathbb{Z}^+\}. \end{aligned}$$

**Lemma 2.21.**

$$\begin{aligned} P(r,2,* \mid 2\Delta) &= \{(V_{6k-3}, V_{6k-4}) : k\in\mathbb{Z}^+\} \\ \cup\{(V_{6k+1}, V_{6k}) &: \text{for } r>1,\ k\in\mathbb{N} \text{ and for } r=1, k\in\mathbb{Z}^+\}, \\ P(r,2,* \mid -2\Delta) &= \{(V_{6k-2}, V_{6k-3}) : k\in\mathbb{Z}^+\}\cup\{(V_{6k}, V_{6k-1}) : k\in\mathbb{Z}^+\}. \end{aligned}$$

**Theorem 2.22.**

$$\begin{aligned} \Pi\,(r,2,*\mid 2\Delta) &= \{(rV_{12k-7}-4, 2(V_{12k-7}+r), rV_{12k-7}+2V_{12k-8}) : k\in\mathbb{Z}^+\} \\ &\cup\{(rV_{12k+1}-4, 2(V_{12k+1}+r), rV_{12k+1}+2V_{12k}) : \\ &\text{for } r>1,\ k\in\mathbb{N} \text{ and for } r=1,\ k\in\mathbb{Z}^+\}, \end{aligned}$$

$$\begin{aligned} \Pi\,(r,2,*\mid -2\Delta) &= \\ \{(rV_{12k-5}+4, 2(V_{12k-5}-r), rV_{12k-5}+2V_{12k-6}) &: k\in\mathbb{Z}^+\} \\ \cup\{(rV_{12k-1}+4, 2(V_{12k-1}-r), rV_{12k-1}+2V_{12k-2}) &: k\in\mathbb{Z}^+\} \end{aligned}$$

For $m=2$ and $n=r$, we consider the families of primitive Pythagorean triples of the form $\Pi\,(2,r,*\mid t)$, where $t$ is a small integer. From (2), every primitive Pythagorean triple $(a,b,c)$ in $\Pi\,(2,r,*\mid t)$ satisfies the equation $2b-ra=t$, which implies that the parameters $e'$ and $f'$ satisfies the equation

$$e'^2 - re'f' - f'^2 = t. \tag{15}$$

Multiplying (15) by 4 and completing the square yields

$$(2e'^2 - rf'^2)^2 - \Delta f'^2 = 4t. \tag{16}$$

If we take $w' = 2e' - rf'$ and $t=\pm1, \pm\Delta$ in (16), respectively, the proofs of Lemmas 2.23 and 2.25 and Theorems 2.24 and 2.26 are completed.

**Lemma 2.23**

$$\begin{aligned} P'(2,r,*\mid -1) &= (U_{6k+2},U_{6k+1}) : \text{for } r>1, k\in\mathbb{N} \text{ and for } r=1, k\in\mathbb{Z}^+\}, \\ P'(2,r,*\mid 1) &= \{(U_{6k-1},U_{6k-2}) : k\in\mathbb{Z}^+\}. \end{aligned}$$

**Theorem 2.24.**

$$\Pi(2,r,*\mid -1) = \frac{1}{2\Delta}\times\{(2(V_{12k+3}+r), rV_{12k+3}-4, \Delta U_{12k+3}) :$$
$$\text{for } r>1,\ k\in\mathbb{N} \text{ and for } r=1,\ k\in\mathbb{Z}^+\Big\},$$
$$\Pi(2,r,*\mid 1) = \frac{1}{2\Delta}\{(2(V_{12k-3}-r), rV_{12k-3}+4, \Delta U_{12k-3}) : k\in\mathbb{Z}^+\}.$$

**Lemma 2.25.**

$$\begin{aligned} P'(2,r,*\mid \Delta) &= \{(V_{6k+2},V_{6k+1}) : k\in\mathbb{N}\}, \\ P'(2,r,*\mid -\Delta) &= \{(V_{6k-1},V_{6k-2}) : k\in\mathbb{Z}^+\}. \end{aligned}$$

**Theorem 2.26.**

$$\begin{aligned} \Pi(2,r,*\mid \Delta) &= \frac{1}{2}\times\{(2(V_{12k+3}-r), rV_{12k+3}+4, rV_{12k+3}+2V_{12k+2}) : k\in\mathbb{N}\}, \\ \Pi(2,r,*\mid -\Delta) &= \{(2(V_{12k-3}+r), rV_{12k-3}-4, rV_{12k-3}+2V_{12k-4}) : k\in\mathbb{Z}^+\}. \end{aligned}$$

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# References

[1] C. V. Eynden, *Elementary number theory (second edition)*, McGraw Hill, 2001.

[2] E. Kılıç and N. Ömür, Conics characterizing the generalized Fibonacci and Lucas sequences with indices in arithmetic progressions, *Ars Combinatoria*, 94 (2010), 459-464.

[3] D. Terr, Some interesting infinite families of primitive Pythagorean triples, *Fib. Quart.*, 12 (1) (2012), 68-81.

[4] S. Vajda, *Fibonacci&Lucas numbers, and the Golden section*, Chichester:Ellis Horwood Ltd., 1989.

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