Journal of Mathematical Extension
Vol. 10, No. 3, (2016), 119-135
ISSN: 1735-8299
URL: http://www.ijmex.com

A General Characterization of Additive Maps on Semiprime Rings

A. Hosseini
Kashmar Higher Education Institute

Abstract. Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $T : \mathcal{R} \to \mathcal{R}$ be a Jordan left centralizer associated with a *l-semi Hochschild 2-cocycle* $\alpha : \mathcal{R}\times\mathcal{R} \to \mathcal{R}$. Then, T is a left centralizer associated with α. Applying this main result, we prove that every Jordan generalized derivation on a 2-torsion free semiprime ring is a generalized derivation.

AMS Subject Classification: 47B47; 17B40; 16N60
Keywords and Phrases: Jordan derivation, jordan centralizer, l-semi hochschild 2-cocycle, 2-torsion free semiprime ring

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative ring with the center $Z(\mathcal{R})$, T is considered as an additive map on $\mathcal{R}$ and α is a biadditive map from $\mathcal{R} \times \mathcal{R}$ into $\mathcal{R}$. Given an integer $n \geqslant 2$, a ring $\mathcal{R}$ is said to be n-torsion free, if for $x \in \mathcal{R}$, $nx = 0$ implies that $x = 0$. We denote by $[x, y]$, the commutator $xy - yx$, for all $x, y \in \mathcal{R}$. Recall that a ring $\mathcal{R}$ is prime if for $x, y \in \mathcal{R}$, $x\mathcal{R}y = \{0\}$ implies that either $x = 0$ or $y = 0$, and is semiprime in the case that $x\mathcal{R}x = \{0\}$ implies that $x = 0$. An additive map $D : \mathcal{R} \to \mathcal{R}$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in \mathcal{R}$ and is called a Jordan derivation in the case that $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in \mathcal{R}$. Obviously, every derivation is a Jordan derivation, but the converse is not true, in general. A well-known result of Herstein [8] states that in the case that $\mathcal{R}$ is a prime

Received: December 2015; Accepted: March 2016

ring of characteristic not 2, then every Jordan derivation $D : \mathcal{R} \to \mathcal{R}$ is a derivation. A brief proof of this result has been presented in [5]. Cusack [6] has extended Herstein's result to 2-torsion free semiprime rings (see also [4] for an alternative proof). An additive map $T : \mathcal{R} \to \mathcal{R}$ is called a left (resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in \mathcal{R}$. We call T a centralizer whenever T is both a left and a right centralizer. An additive map $T : \mathcal{R} \to \mathcal{R}$ is called a Jordan left (right) centralizer when $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in \mathcal{R}$. Following some ideas from [4], Zalar [11] proved that any left (resp. right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp. right) centralizer. By using the main results of [4] and [11], Vukman [10] proved that every Jordan generalized derivation on a 2-torsion free semiprime ring is a generalized derivation. By using the main result of this paper, we offer an alternative proof for this result of Vukman.

A bi-additive map $\alpha : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is said to be a *l-semi Hochschild 2-cocycle* if

$$\alpha(xy, z) - \alpha(x, yz) + \alpha(x, y)z = 0,$$

for all $x, y, z \in \mathcal{R}$. A *l-semi Hochschild 2-cocycle* α is said to be symmetric (resp. anti symmetric) if $\alpha(x, y) = \alpha(y, x)$ (resp. $\alpha(x, y) = -\alpha(y, x)$) for all $x, y \in \mathcal{R}$. We say that an additive map $T : \mathcal{R} \to \mathcal{R}$ is a left centralizer associated with α, if there exists a bi-additive map $\alpha : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ such that $T(xy) = T(x)y + \alpha(x, y)$ holds for all $x, y \in \mathcal{R}$. Clearly, in this case we have

$$\begin{aligned}&\alpha(xy, z) - \alpha(x, yz) + \alpha(x, y)z = T(xyz) - T(xy)z - T(xyz)+\\ &T(x)yz + T(xy)z - T(x)yz = 0,\end{aligned}$$

for all $x, y, z \in \mathcal{R}$. It means that α is a *l-semi Hochschild 2-cocycle*. Let $\alpha : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a *l-semi Hochschild 2-cocycle*. An additive map $T : \mathcal{R} \to \mathcal{R}$ is said to be a Jordan left centralizer associated with α, whenever $T(x^2) = T(x)x + \alpha(x, x)$ for all $x \in \mathcal{R}$. A bi-additive map $\lambda : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called a *r-semi Hochschild 2-cocycle* if $\lambda(z, xy) - \lambda(zx, y) + z\lambda(x, y) = 0$ for all $x, y, z \in \mathcal{R}$. An additive map $T : \mathcal{R} \to \mathcal{R}$

is said to be a right centralizer associated with λ, if there exists a bi-additive map $\lambda : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ such that $T(xy) = xT(y) + \lambda(x, y)$ for all $x, y \in \mathcal{R}$. Obviously, λ is a *r-semi Hochschild 2-cocycle*.

Let $T(x^2) = T(x)x + \alpha(x, x)$ for all $x \in \mathcal{R}$ (1). Replacing x by $x + y$ in (1), we get $T(xy + yx) = T(x)y + T(y)x + \alpha(x, y) + \alpha(y, x)$ for all $x, y \in \mathcal{R}$ (2). Note that if $\mathcal{R}$ is a 2-torsion free ring, then (1) and (2) are equivalent. Some authors define a Jordan left centralizer as follows: An additive map $T : \mathcal{R} \to \mathcal{R}$ is called a Jordan left centralizer if $T(xy + yx) = T(x)y + T(y)x$ holds for all $x, y \in \mathcal{R}$. With this hypothesis, we have $T(xy) - T(x)y = -(T(yx) - T(y)x)$ (3). If we define $\alpha(x, y) = T(xy) - T(x)y$ $(x, y \in \mathcal{R})$, then it follows from (3) that $\alpha(x, y) = -\alpha(y, x)$ and it means that α is anti symmetric. Suppose that $T : \mathcal{R} \to \mathcal{R}$ is a left centralizer associated with α, i.e. $T(xy) = T(x)y + \alpha(x, y)$ for all $x, y \in \mathcal{R}$. If α is anti symmetric, then we have $T(xy) - T(x)y = -T(yx) + T(y)x$ and consequently, $T(xy + yx) = T(x)y + T(y)x$ for all $x, y \in \mathcal{R}$. It means that T is a Jordan left centralizer. Therefore, If $\alpha : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ defined by $\alpha(x, y) = T(xy) - T(x)y$ is anti symmetric, then T is a Jordan left centralizer if and only if T is a left centralizer associated with the *l-semi Hochschild 2-cocycle* α.

Similar to the approach presented in [11], we prove the following main result:
Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $T : \mathcal{R} \to \mathcal{R}$ be a Jordan left centralizer associated with α, where α is a *l-semi Hochschild 2-cocycle*. Then, T is a left centralizer associated with α.

In this paper, we show that derivations, generalized derivations, σ-derivations, generalized σ-derivations, (σ, τ)-derivations and θ-centralizers are left centralizer associated with a suitable *l-semi Hochschild 2-cocycle*. This means that the aforementioned concepts can be unified and integrated together. By reviewing some papers ([1], [2], [3], and references therein) about Jordan left (θ, ϕ)-derivations, (θ, ϕ)-derivations, τ-centralizers and α-centralizers on 2-torsion free semiprime rings, it is observed that the maps like θ, ϕ, and τ are supposed as homomorphism and automorphism. We believe this assumption can reduce the generality of the topic. In this paper, therefore, we are going to present some results about Jordan σ-derivations, Jordan (σ,τ)-derivations, Jordan generalized derivations and Jordan generalized σ-derivations based on the new type of left centralizers, while τ, σ and θ are not supposed homomorphism, necessarily. It is one of the reasons that obviously proves performance and application of this type of centralizers.

2. Centralizer Associated with Semi Hochschild 2-Cocycle

Definition 2.1. *A biadditive map $\alpha : \mathcal{R}\times\mathcal{R}\to\mathcal{R}$ is said to be a* l-semi Hochschild 2-cocycle *if $\alpha(xy,z)-\alpha(x,yz)+\alpha(x,y)z=0$ for all $x,y,z\in\mathcal{R}$.*

Definition 2.2. *For a biadditive map $\alpha : \mathcal{R}\times\mathcal{R}\to\mathcal{R}$, an additive map $T:\mathcal{R}\to\mathcal{R}$ is said to be a left centralizer associated with α if $T(xy)=T(x)y+\alpha(x,y)$ for all $x,y\in\mathcal{R}$.*
As we mentioned in the introduction, such α is a l-semi Hochschild 2-cocycle.

Example 2.3. Every derivation $D:\mathcal{R}\to\mathcal{R}$ is a left centralizer associated with $\alpha:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ defined by $\alpha(x,y)=xD(y)$ for all $x,y\in\mathcal{R}$. We have

$$\begin{aligned}\alpha(xy,z)-\alpha(x,yz)+\alpha(x,y)z &= xyD(z)-xD(yz)+xD(y)z\\ &= xyD(z)-xyD(z)-xD(y)z+xD(y)z\\ &= 0.\end{aligned}$$

Hence, $D(xy)=D(x)y+xD(y)=D(x)y+\alpha(x,y)$ is a left centralizer associated with α.

Example 2.4. Suppose $D:\mathcal{R}\to\mathcal{R}$ is a derivation. Then, every D-derivation $f:\mathcal{R}\to\mathcal{R}$ is a left centralizer associated with α, if α is defined as above.

Example 2.5. Suppose $\theta:\mathcal{R}\to\mathcal{R}$ is an endomorphism. Then, $h=\theta-id$, where id is the identity mapping on $\mathcal{R}$, is a $(\theta,id)-$*derivation* as following:

$$h(xy) = \theta(xy) - xy = \theta(x)\theta(y) - \theta(x)y + \theta(x)y - xy = h(x)y + \theta(x)h(y).$$

If we define $\alpha(x, y) = \theta(x)h(y)$ for all $x, y \in \mathcal{R}$, then α is a *l-semi Hochschild 2-cocycle*. It means that h is a left centralizer associated with α.

Furthermore, suppose that $T : \mathcal{R} \to \mathcal{R}$ is a left $\theta - centralizer$, i.e. T is additive and $T(xy) = T(x)\theta(y)$ holds for all $x, y \in \mathcal{R}$. Considering $h = \theta - id$, we have

$$\begin{aligned} T(xy) &= T(x)\theta(y) = T(x)(h + id)(y) = T(x)(h(y) + y) \\ &= T(x)y + T(x)h(y). \end{aligned}$$

Defining $\alpha(x, y) = T(x)h(y)$ for all $x, y \in \mathcal{R}$, we conclude that α is a *l-semi Hochschild 2-cocycle*. Hence, T is a left centralizer associated with α.

Example 2.6. Let $\sigma, \tau : \mathcal{R} \to \mathcal{R}$ be two endomorphisms and $d : \mathcal{R} \to \mathcal{R}$ be a σ-derivation.
(i) Every (σ, τ)-derivation $F : \mathcal{R} \to \mathcal{R}$ is a left centralizer associated with α, if α is defined by $\alpha(x, y) = F(x)(\sigma - id)(y) + \tau(x)F(y)$.
(ii) Suppose $\delta : \mathcal{R} \to \mathcal{R}$ is a generalized σ-derivation. Put $\alpha(x, y) = \delta(x)(\sigma - id)(y) + \sigma(x)d(y)$, where $d : \mathcal{R} \to \mathcal{R}$ is a σ-derivation, so δ is a left centralizer associated with α. Let $T : \mathcal{R} \to \mathcal{R}$ be a Jordan left centralizer associated with α, and define $\psi : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ by $\psi(x, y) = T(xy) - T(x)y - \alpha(x, y)$ for all $x, y \in \mathcal{R}$. The following proposition demonstrates several properties of ψ.

Proposition 2.7. *Let ψ, α and T be as above. The bi-additive map ψ satisfies the following:*
(1) ψ is anti symmetric, i.e. $\psi(x, y) = -\psi(y, x)$,
(2) ψ is a l-semi Hochschild 2-cocycle,
(3) $2\psi(xy, z) + \psi(x, y)z - \psi(y, z)x + \psi(z, x)y = 0$,
(4) If $\mathcal{R}$ is a 2-torsion free ring, then $\psi(xy, z) + \psi(xz, y) = 0$,
(5) If $\mathcal{R}$ is a 2-torsion free ring, then $\psi([x, y], z) = -\psi(x, y)z$,
(6) If $\mathcal{R}$ is a 2-torsion free ring, then $\psi(x, y)[z, w] + \psi(z, w)[x, y] = 0$,

(7) If $\mathcal{R}$ is a 2-torsion free ring, then $\psi([x,y]z,w)=0$,
(8) If $\mathcal{R}$ is a 2-torsion free ring, then $\psi(x,y)[z,w]r=0$,
for all $x,y,z,r\in\mathcal{R}$.

Proof. (1) and (2) are straightforward.
For proving (3), we have

$$\begin{aligned}
&2\psi(xy,z)+\psi(x,y)z-\psi(y,z)x+\psi(z,x)y=2\psi(xy,z)+\psi(x,y)z+\psi(z,y)x\\
&+\psi(z,x)y=T(xyz)-T(xy)z-\alpha(xy,z)-\alpha(xy,z)-T(x)yz-\alpha(x,y)z\\
&+T(zy)x-T(z)yx-\alpha(z,y)x+T(zx)y-T(z)xy-\alpha(z,x)y+T(xyz)\\
&=-T(zxy)+T(z)xy+\alpha(z,xy)-\alpha(xy,z)+T(yz)x-T(yzx)+\alpha(yz,x)\\
&+\alpha(x,yz)-\alpha(x,y)z+T(zy)x-T(z)yx-\alpha(z,y)x+T(zx)y-T(z)xy\\
&-\alpha(z,x)y=T(yzx)-T(y)zx-\alpha(y,zx)-\alpha(zx,y)+\alpha(z,xy)-\alpha(xy,z)\\
&+T(yz)x-T(yzx)+\alpha(yz,x)+\alpha(x,yz)-\alpha(x,y)z+T(zy)x-T(z)yx\\
&-\alpha(z,y)x-\alpha(z,x)y\\
&=[T(yz)-T(y)z-\alpha(y,z)+T(zy)-T(z)y-\alpha(z,y)]x\\
&-[\alpha(zx,y)-\alpha(z,xy)+\alpha(z,x)y]-[\alpha(xy,z)-\alpha(x,yz)+\alpha(x,y)z]\\
&=0.
\end{aligned}$$

(4): By using (3), we get $\psi(xy,z)=-\psi(x,y)z+\psi(y,z)x-\psi(z,x)y-\psi(xy,z)$ and $\psi(xz,y)=-\psi(x,z)y+\psi(z,y)x-\psi(y,x)z-\psi(xz,y)$. Applying the previous two equations with the fact that ψ is anti symmetric, we have $\psi(xy,z)+\psi(xz,y)=-\psi(xy,z)-\psi(xz,y)$, i.e. $2(\psi(xy,z)+\psi(xz,y))=0$. Since $\mathcal{R}$ is a 2-torsion free ring, $\psi(xy,z)+\psi(xz,y)=0$.
(5): This part is achieved from (1), (2) and (4), immediately.
(6): This part is obtained from (1) and (5).
(7): By using (2) and (5) our aim is achieved.
(8): We have $\psi(x,y)[z,w]r=-\psi(xy,[z,w]r)+\psi(x,y[z,w]r)$
$=\psi([z,w]r,xy)+\psi(x,[yz,w]r-[y,w]zr)=0-\psi([yz,w]r,x)+\psi([y,w]zr,x)$
$=0$ (see (7)). $\Box$

The following lemma has been proved by Zalar [11]. Now, we provide another proof for it.

Lemma 2.8. *Let $\mathcal{R}$ be a semiprime ring and $a\in\mathcal{R}$ be a fixed element.*

Then $a[x,y]=0$ for all $x,y\in\mathcal{R}$ if and only if there exists an ideal $\mathcal{I}$ of $\mathcal{R}$ contained in $Z(\mathcal{R})$ such that $a\in\mathcal{I}$.

Proof. First, note that $[xy,z]=[x,z]y+x[y,z]$ and $[x,yz]=[x,y]z+y[x,z]$ for all $x,y,z\in\mathcal{R}$. Let $a[x,y]=0$ for some fixed element a and for all x,y of $\mathcal{R}$. We have $[a,x]y[a,x]=([ay,x]-a[y,x])[a,x]=[ay,x][a,x]$ and further, $[ay[a,x],x]=[ay,x][a,x]+ay[[a,x],x]$ for all $x,y\in\mathcal{R}$. So,

$$\begin{aligned}[a,x]y[a,x]&=[ay,x][a,x]\\&=[ay[a,x],x]-ay[[a,x],x]\\&=[a([ya,x]-[y,x]a),x]-a([y[a,x],x]-[y,x][a,x])\\&=[a[ya,x]-a[y,x]a,x]-a[y[a,x],x]+a[y,x][a,x]\\&=0.\end{aligned}$$

Since $\mathcal{R}$ is semiprime, $[a,x]=0$ for all $x\in\mathcal{R}$, i.e. $a\in Z(\mathcal{R})$. Suppose $\mathcal{I}=<a>$ denotes the ideal generated by a . Hence, we have $I=\{ra+as+na+\Sigma_{i=1}^{m}r_ias_i \mid r,s,r_i,s_i\in\mathcal{R},\ n\in\mathbb{Z},i=1,2,...,m\}$. Therefore, $[ra+as+na+\Sigma_{i=1}^{m}r_ias_i,x]=0$ and thus, $a\in\mathcal{I}\subseteq Z(\mathcal{R})$. Conversely, assume that $a\in\mathcal{I}\subseteq Z(\mathcal{R})$, where $\mathcal{I}$ is a bi-ideal of $\mathcal{R}$. We will show that $a[x,y]=0$ for all $x,y\in\mathcal{R}$. Note that $(a[x,y])z(a[x,y])=([ax,y]-[a,y]x)z([ax,y]-[a,y]x)=[ax,y]z[ax,y]=0$. Since $\mathcal{R}$ is semiprime, $a[x,y]=0$ for all $x,y\in\mathcal{R}$ and our purpose is achieved. □

We are now ready for the following main result.

Theorem 2.9. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $T:\mathcal{R}\to\mathcal{R}$ be a Jordan left centralizer associated with a l-semi Hochschild 2-cocycle $\alpha:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$. Then, T is a left centralizer associated with α.*

Proof. By hypothesis,

$$T(x^2)=T(x)x+\alpha(x,x)\ for\ all\ x\in\mathcal{R}. \quad (1)$$

Replacing x by $x+y$ in (1), we get $T(x^2)+T(xy+yx)+T(y^2)=T(x)x+T(x)y+T(y)x+T(y)y+\alpha(x,x)+\alpha(x,y)+\alpha(y,x)+\alpha(y,y)$. This equation together with (1) imply that

$$T(xy+yx)=T(x)y+T(y)x+\alpha(x,y)+\alpha(y,x)\ for\ all\ x,y\in\mathcal{R}. \quad (2)$$

We replace y by $xy+yx$ in (2) to get

$$T(x(xy+yx)+(xy+yx)x)=T(x)xy+T(x)yx+T(x)yx+T(y)x^2+$$

$$\alpha(x,y)x+\alpha(y,x)x+\alpha(x,xy)+\alpha(x,yx)+\alpha(xy,x)+\alpha(yx,x). \qquad (3)$$

But this can also be calculated in a different way.

$$\begin{aligned} T(x^2y+yx^2)+2T(xyx) &= T(x)xy+\alpha(x,x)y+T(y)x^2+\alpha(x^2,y) \\ &\quad +\alpha(y,x^2)+2T(xyx)=T(x)xy+\alpha(x,x)y \\ &\quad +T(y)x^2+\alpha(x,xy)-\alpha(x,x)y+\alpha(yx,x) \\ &\quad +\alpha(y,x)x+2T(xyx). \end{aligned}$$

It means that

$$T(x^2y+yx^2)+2T(xyx)=T(x)xy+\alpha(x,x)y+T(y)x^2+\alpha(x,xy)$$

$$-\alpha(x,x)y+\alpha(yx,x)+\alpha(y,x)x+2T(xyx). \qquad (4)$$

Comparing (3) and (4), it is obtained that $2T(x)yx+\alpha(x,y)x+\alpha(x,yx)+\alpha(xy,x)=2T(xyx)$. Hence, $2T(x)yx+\alpha(x,y)x+\alpha(x,yx)+\alpha(x,yx)-\alpha(x,y)x=2T(xyx)$. It means that

$$T(xyx)=T(x)yx+\alpha(x,yx). \qquad (5)$$

Putting $x+z$ for x in (5), we have $T(x)yx+T(x)yz+T(z)yx+T(z)yz+\alpha(x,yx)+\alpha(x,yz)+\alpha(z,yx)+\alpha(z,yz)=T(xyx)+T(xyz+zyx)+T(zyz)$. The last equation together with (5) imply that

$$T(xyz+zyx)=T(x)yz+T(z)yx+\alpha(x,yz)+\alpha(z,yx). \qquad (6)$$

Put $m=T(xyzyx+yxzxy)$. We compute m in two different methods. Using (5), we have

$$m=T(x)yzyx+\alpha(x,yzyx)+T(y)xzxy+\alpha(y,xzxy) \qquad (7)$$

and by applying (6), we have

$$m = T(xy)zyx + T(yx)zxy + \alpha(xy, zyx) + \alpha(yx, zxy). \tag{8}$$

So,

$$\begin{aligned} 0 = m - m &= T(xy)zyx + T(yx)zxy + \alpha(xy, zyx) + \alpha(yx, zxy) \\ &- T(x)yzyx - \alpha(x, yzyx) - T(y)xzxy - \alpha(y, xzxy) \\ &= T(xy)zyx + T(yx)zxy + \alpha(xy, zyx) + \alpha(yx, zxy) \\ &- T(x)yzyx - T(y)xzxy - \alpha(xy, zyx) - \alpha(x, y)zyx \\ &- \alpha(yx, zxy) - \alpha(y, x)zxy. \end{aligned}$$

Hence, $(T(xy)-T(x)y-\alpha(x,y))zyx+(T(yx)-T(y)x-\alpha(y,x))zxy = 0$. By the last equation and introducing a bilinear map $\psi(x, y) = T(xy) - T(x)y - \alpha(x, y)$, it can be achieved that

$$\psi(x, y)zyx + \psi(y, x)zxy = 0. \tag{9}$$

It follows from (2) that $\psi(x, y) = -\psi(y, x)$. Using this fact and equality (9), we obtain

$$\psi(x, y)z[x, y] = 0 \ for\ all\ x, y, z \in \mathcal{R}. \tag{10}$$

Replacing x by $x + u$ in (10), we get

$$\begin{aligned} 0 &= \psi(x + u, y)z[x + u, y] \\ &= (\psi(x, y) + \psi(u, y))z([x, y] + [u, y]) \\ &= \psi(x, y)z[x, y] + \psi(x, y)z[u, y] + \psi(u, y)z[x, y] + \psi(u, y)z[u, y] \\ &= 0 + \psi(x, y)z[u, y] + \psi(u, y)z[x, y] + 0. \end{aligned}$$

Therefore,

$$\psi(x, y)z[u, y] + \psi(u, y)z[x, y] = 0 \ for\ all\ x, y, z, u \in \mathcal{R}. \tag{11}$$

Using (10) and (11), we find

$$\begin{aligned} (\psi(x, y)z[u, y])\omega(\psi(x, y)z[u, y]) &= \psi(x, y)(z[u, y]\omega\psi(x, y)z)[u, y] \\ &= -\psi(u, y)z[u, y]\omega\psi(x, y)z[x, y] \\ &= 0. \end{aligned}$$

Since $\mathcal{R}$ is semiprime, we obtain

$$\psi(x,y)z[u,y]=0 \; for\ all\ x,y,z,u\in\mathcal{R}. \tag{12}$$

Replacing y by $y+v$ in (12), we arrive at

$$\psi(x,y)z[u,y]+\psi(x,y)z[u,v]+\psi(x,v)z[u,y]+\psi(x,v)z[u,v]=0,$$

this equation together with (12) imply that

$$\psi(x,y)z[u,v]=-\psi(x,v)z[u,y] \; for\ all\ x,y,z,u,v\in\mathcal{R}. \tag{13}$$

From (13) and the fact that $\psi(x,y)z[u,y]=0$, it can be concluded that

$$\begin{aligned}(\psi(x,y)z[u,v])\omega(\psi(x,y)z[u,v]) &= \psi(x,y)(z[u,v]\omega\psi(x,y)z)[u,v]\\ &= -\psi(x,v)z[u,v]\omega\psi(x,y)z[u,y]\\ &= 0.\end{aligned}$$

Now, we have

$$\begin{aligned}\psi(x,y)z[u,v]\omega\psi(x,y)z[u,v] &= \psi(x,y)(z[u,v]\omega\psi(x,y)z)[u,v]\\ &= -\psi(x,v)(z[u,v]\omega\psi(x,y)z)[u,y] \quad (see\ 13)\\ &= -\psi(x,v)z[u,v]\omega(\psi(x,y)z[u,y])\\ &= 0. \quad (see\ 12)\end{aligned}$$

Reusing the fact that $\mathcal{R}$ is semiprime, it seen that

$$\psi(x,y)z[u,v]=0 \; for\ all\ x,y,z,u,v\in\mathcal{R}. \tag{14}$$

Hence,

$$\psi(x,y)[u,v]z\psi(x,y)[u,v]=\psi(x,y)([u,v]z\psi(x,y))[u,v]=0, \quad (see\ 14)$$

and it follows from semiprimeness of $\mathcal{R}$ that

$$\psi(x,y)[u,v]=0 \; for\ all\ x,y,u,v\in\mathcal{R}. \tag{15}$$

Now, let x and y be two fixed elements of $\mathcal{R}$ and for convenience write ψ instead of $\psi(x,y)$. Using Lemma 1 we get the existence of an ideal $\mathcal{I}$

such that $\psi \in \mathcal{I} \subseteq Z(\mathcal{R})$. In particular, $y\psi, \psi y \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. This gives us

$$x.\psi^2 y = \psi^2 y.x = y\psi^2.x = y.\psi^2 x \tag{16}$$

and so, $4T(x.\psi^2 y) = 4T(y.\psi^2 x)$. Both sides of this equality will be computed using (2) and (16). Indeed, we have

$$4T(x.\psi^2 y) = 4T(y.\psi^2 x).$$

$$\Rightarrow 2T(x\psi^2 y + \psi^2 yx) = 2T(y\psi^2 x + \psi^2 xy).$$

$$\begin{aligned}\Rightarrow & 2T(x)\psi^2 y + 2\alpha(x, \psi^2 y) + 2T(\psi^2 y)x + 2\alpha(\psi^2 y, x) \\ &= 2T(y)\psi^2 x + 2\alpha(y, \psi^2 x) + 2T(\psi^2 x)y + 2\alpha(\psi^2 x, y).\end{aligned}$$

$$\begin{aligned}\Rightarrow & 2T(x)\psi^2 y + 2\alpha(x, \psi^2 y) + T(\psi^2 y + y\psi^2)x + 2\alpha(\psi^2 y, x) \\ &= 2T(y)\psi^2 x + 2\alpha(y, \psi^2 x) + T(\psi^2 x + x\psi^2)y + 2\alpha(\psi^2 x, y).\end{aligned}$$

$$\begin{aligned}\Rightarrow & 2T(x)\psi^2 y + 2\alpha(x, \psi^2 y) + T(\psi^2)yx + \alpha(\psi^2, y)x + T(y)\psi^2 x + \alpha(y, \psi^2)x \\ &+ 2\alpha(\psi^2 y, x) = 2T(y)\psi^2 x + 2\alpha(y, \psi^2 x) + T(\psi)\psi xy + \alpha(\psi, \psi)xy \\ &+ \alpha(\psi^2, x)y + T(x)\psi^2 y + \alpha(x, \psi^2)y + 2\alpha(\psi^2 x, y).\end{aligned}$$

$$\begin{aligned}\Rightarrow & T(x)\psi^2 y + 2\alpha(x, \psi^2 y) + T(\psi)\psi yx + \alpha(\psi, \psi)yx + \alpha(\psi^2, y)x + \alpha(y, \psi^2)x \\ &+ 2\alpha(\psi^2 y, x) = T(y)\psi^2 x + 2\alpha(y, \psi^2 x) + T(\psi)\psi xy + \alpha(\psi, \psi)xy \\ &+ \alpha(\psi^2, x)y + \alpha(x, \psi^2)y + 2\alpha(\psi^2 x, y).\end{aligned}$$

Since $\psi yx = x\psi y = \psi xy$, we obtain

$$\begin{aligned}& T(x)\psi^2 y + 2\alpha(x, \psi^2 y) + \alpha(\psi, \psi)yx + \alpha(\psi^2, y)x + \alpha(y, \psi^2)x + 2\alpha(\psi^2 y, x) = \\ & T(y)\psi^2 x + 2\alpha(y, \psi^2 x) + \alpha(\psi, \psi)xy + \alpha(\psi^2, x)y + \alpha(x, \psi^2)y + 2\alpha(\psi^2 x, y).\end{aligned} \tag{17}$$

Rearranging (17) we get

$$T(y)x\psi^2 = T(x)\psi^2 y + 2\alpha(x,\psi^2 y) + \alpha(\psi,\psi)yx + \alpha(\psi^2,y)x + \alpha(y,\psi^2)x + 2\alpha(\psi^2 y,x) - 2\alpha(y,\psi^2 x) - \alpha(\psi,\psi)xy - \alpha(\psi^2,x)y - \alpha(x,\psi^2)y - 2\alpha(\psi^2 x,y). \tag{18}$$

On the other hand, we also have

$$4T(xy\psi^2) = 4T(x\psi.y\psi)$$

$$\Rightarrow 2T(xy\psi^2 + \psi^2 xy) = 2T(x\psi y\psi + y\psi x\psi)$$

$$\begin{aligned}\Rightarrow & 2T(xy)\psi^2 + 2T(\psi^2)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy)\\ &= 2T(\psi x)\psi y + 2T(\psi y)\psi x + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x)\end{aligned}$$

$$\begin{aligned}\Rightarrow & 2T(xy)\psi^2 + 2T(\psi)\psi xy + 2\alpha(\psi,\psi)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy)\\ &= 2T(\psi x)\psi y + 2T(\psi y)\psi x + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x)\end{aligned}$$

$$\begin{aligned}\Rightarrow & 2T(xy)\psi^2 + 2T(\psi)\psi xy + 2\alpha(\psi,\psi)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy)\\ &= T(\psi x + x\psi)\psi y + T(y\psi + \psi y)\psi x + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x)\end{aligned}$$

$$\begin{aligned}\Rightarrow & 2T(xy)\psi^2 + 2T(\psi)\psi xy + 2\alpha(\psi,\psi)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy)\\ &= T(x)\psi^2 y + \alpha(x,\psi)\psi y + \alpha(\psi,x)\psi y + T(\psi)\psi xy + T(y)\psi^2 x\\ &\quad + \alpha(y,\psi)\psi x + T(\psi)\psi xy + \alpha(\psi,y)\psi x\\ &\quad + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x)\end{aligned}$$

$$\Rightarrow 2T(xy)\psi^2 + 2\alpha(\psi,\psi)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy) = T(x)\psi^2 y + \alpha(x,\psi)\psi y + \alpha(\psi,x)\psi y + T(y)x\psi^2 + \alpha(y,\psi)\psi x + \alpha(\psi,y)\psi x + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x) \tag{19}$$

Using (18) and (19), we have

$$\begin{aligned}2T(xy)\psi^2 + 2\alpha(\psi,\psi)xy + 2\alpha(xy,\psi^2) + 2\alpha(\psi^2,xy) = T(x)\psi^2 y + \alpha(x,\psi)\psi y\\ + \alpha(\psi,x)\psi y + \alpha(y,\psi)\psi x + \alpha(\psi,y)\psi x + 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x)\\ + T(x)\psi^2 y + 2\alpha(x,\psi^2 y) + \alpha(\psi,\psi)yx + \alpha(\psi^2,y)x + \alpha(y,\psi^2)x\\ + 2\alpha(\psi^2 y,x) - 2\alpha(y,\psi^2 x) - \alpha(\psi,\psi)xy - \alpha(\psi^2,x)y\\ - \alpha(x,\psi^2)y - 2\alpha(\psi^2 x,y)\end{aligned}$$

Hence,

$$\begin{aligned}2T(xy)\psi^2 - 2T(x)y\psi^2 &= \alpha(x,\psi)\psi y + \alpha(\psi,x)\psi y + \alpha(y,\psi)\psi x + \alpha(\psi,y)\psi x\\ &+ 2\alpha(\psi x,\psi y) + 2\alpha(\psi y,\psi x) + 2\alpha(x,\psi^2 y)\\ &+ \alpha(\psi,\psi)yx + \alpha(\psi^2,y)x + \alpha(y,\psi^2)x + 2\alpha(\psi^2 y,x)\\ &- 2\alpha(y,\psi^2 x) - \alpha(\psi,\psi)xy - \alpha(\psi^2,x)y - \alpha(x,\psi^2)y\\ &- 2\alpha(\psi^2 x,y) - 2\alpha(\psi,\psi)xy - 2\alpha(xy,\psi^2)\\ &- 2\alpha(\psi^2,xy) \qquad (\star)\end{aligned}$$

Our next task is to prove the equation bellow:

$$2(T(xy) - T(x)y - \alpha(x,y))\psi^2 = 0$$

In order to prove the previous equation we need the following relations:

$$\begin{aligned}\alpha(\psi,x)\psi y &= \alpha(\psi,x\psi)y - \alpha(\psi x,\psi)y = \alpha(\psi,\psi x)y - \alpha(\psi x,\psi)y\\ &= \alpha(\psi^2,x)y + \alpha(\psi,\psi)xy - \alpha(\psi x,\psi)y \qquad (i)\end{aligned}$$

$$\begin{aligned}2\alpha(\psi x,\psi y) &= \alpha(\psi x + x\psi,\psi y) = \alpha(\psi x,\psi y) + \alpha(x\psi,\psi y)\\ &= \alpha(\psi^2 x,y) + \alpha(\psi x,\psi)y + \alpha(x\psi,\psi y)\\ &= \alpha(\psi^2 x,y) + \alpha(x,\psi^2)y - \alpha(x,\psi)\psi y + \alpha(x\psi,\psi y) \qquad (ii)\end{aligned}$$

$$\begin{aligned}\alpha(\psi,y)\psi x &= \alpha(\psi,y\psi x) - \alpha(\psi y,\psi x) = \alpha(\psi,\psi xy) - \alpha(\psi y,\psi x)\\ &= \alpha(\psi^2,xy) + \alpha(\psi,\psi)xy - \alpha(\psi y,\psi x) \qquad (iii)\end{aligned}$$

$$2\alpha(\psi y,\psi x)=2\alpha(y\psi,\psi x)=2\alpha(y,\psi^2x)-2\alpha(y,\psi)\psi x \qquad (iv)$$

$$\begin{aligned}\alpha(\psi,\psi)yx&=\alpha(\psi,\psi yx)-\alpha(\psi^2,yx)=\alpha(\psi,\psi xy)-\alpha(\psi^2,yx)\\&=\alpha(\psi^2,xy)+\alpha(\psi,\psi)xy-\alpha(\psi^2,yx)\\&=\alpha(\psi^2,xy)+\alpha(\psi,\psi)xy-\alpha(\psi^2y,x)-\alpha(\psi^2,y)x \qquad (v)\end{aligned}$$

$$\alpha(\psi^2y,x)=\alpha(y\psi^2,x)=\alpha(y,\psi^2x)-\alpha(y,\psi^2)x \qquad (vi)$$

$$-2\alpha(xy,\psi^2)=-2\alpha(x,y\psi^2)+2\alpha(x,y)\psi^2 \qquad (vii)$$

$$-\alpha(\psi^2x,y)=-\alpha(\psi x,\psi y)+\alpha(\psi x,\psi)y \qquad (viii)$$

By using the above eight relations, the equation $(\star)$ turns into

$$2(T(xy)-T(x)y-\alpha(x,y))\psi^2=0.$$

Since $\mathcal{R}$ is a 2-torsion free ring, the above equation reduces to $(T(xy)-T(x)y-\alpha(x,y))\psi^2=0$, and this means that $\psi^3=0$. Hence, $\psi^2\mathcal{R}\psi^2=\psi^4\mathcal{R}=\psi\psi^3\mathcal{R}=\{0\}$. Thus, the semiprimeness of $\mathcal{R}$ implies that $\psi^2=0$. Furthermore, we have $\psi\mathcal{R}\psi=\psi^2\mathcal{R}=\{0\}$. Reusing the fact that $\mathcal{R}$ is a semiprime ring, it is concluded that $\psi=0$. This is exactly what we had to prove. $\square$

The Previous theorem implies the following corollaries.

Corollary 2.10. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\sigma,T:\mathcal{R}\to\mathcal{R}$ be additive maps such that $T(x^2)=T(x)\sigma(x)$ for all $x\in\mathcal{R}$. If*

$T(xy)\sigma(z) - T(xy)z - T(x)\sigma(yz) + T(x)\sigma(y)z = 0$ for all $x, y, z \in \mathcal{R}$, then $T(xy) = T(x)\sigma(y)$ for all $x, y \in \mathcal{R}$.

Proof. Note that $T(x^2) = T(x)\sigma(x) = T(x)(x + \sigma(x) - x) = T(x)x + T(x)(\sigma(x) - x)$ for all $x \in \mathcal{R}$. By defining $\alpha(x, y) = T(x)(\sigma(y) - y)$ and using the hypothesis, we have

$$\begin{aligned}\alpha(xy, z) - \alpha(x, yz) + \alpha(x, y)z &= T(xy)(\sigma(z) - z) - T(x)(\sigma(yz) - yz)\\ &\quad + T(x)(\sigma(y) - y)z\\ &= T(xy)\sigma(z) - T(xy)z - T(x)\sigma(yz)\\ &\quad + T(x)yz + T(x)\sigma(y)z - T(x)yz\\ &= T(xy)\sigma(z) - T(xy)z - T(x)\sigma(yz) + T(x)\sigma(y)z\\ &= 0.\end{aligned}$$

It means that α is a *l-semi Hochschild 2-cocycle*. By using Theorem 2.9, it is obtained that

$$\begin{aligned}T(xy) &= T(x)y + \alpha(x, y) = T(x)y + T(x)(\sigma(y) - y)\\ &= T(x)y + T(x)\sigma(y) - T(x)y\\ &= T(x)\sigma(y)\end{aligned}$$

for all $x, y \in \mathcal{R}$. □

Corollary 2.11. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\delta : \mathcal{R} \to \mathcal{R}$ be a Jordan generalized σ-derivation, i.e. $\delta(x^2) = \delta(x)\sigma(x) + \sigma(x)d(x)$ for each $x \in \mathcal{R}$ and some σ-derivation d on $\mathcal{R}$. Assume that $T = \delta - d$ has the following property:*

$$T(xy)\sigma(z) - T(xy)z - T(x)\sigma(yz) + T(x)\sigma(y)z = 0 \text{ for all } x, y, z \in \mathcal{R}.$$

Then, δ is a generalized σ-derivation.

Proof. We have

$$\begin{aligned}T(x^2) &= \delta(x^2) - d(x^2)\\ &= \delta(x)\sigma(x) + \sigma(x)d(x) - d(x)\sigma(x) - \sigma(x)d(x)\\ &= (\delta(x) - d(x))\sigma(x)\\ &= T(x)\sigma(x)\end{aligned}$$

for all $x \in \mathcal{R}$. According to the previous corollary, $T(xy) = T(x)\sigma(y)$ for all $x, y \in \mathcal{R}$. Hence,

$$\begin{aligned}\delta(xy) &= T(xy) + d(xy) = T(x)\sigma(y) + d(x)\sigma(y) + \sigma(x)d(y)\\ &= \delta(x)\sigma(y) + \sigma(x)d(y)\end{aligned}$$

for all $x, y \in \mathcal{R}$.

The following corollary has been proved by Vukman in [10]. Now, we present an alternative proof for it.

Corollary 2.12. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\delta : \mathcal{R} \to \mathcal{R}$ be a Jordan generalized derivation. In this case δ is a generalized derivation.*

Proof. We have the relation $\delta(x^2) = \delta(x)x + xd(x)$ for all $x \in \mathcal{R}$, where d is a Jordan derivation of $\mathcal{R}$. It follows from Theorem 1 of [4] that d is a derivation. The proof is completed by substituting $\sigma = id$ in Corollary 2.11. $\square$

Acknowledgements
The author is greatly indebted to the referee(s) for his/her valuable suggestions and careful reading of the paper.

References

[1] S. Ali and C. Haetinger, Jordan α-centralizers in rings and some applications, *Bol. Soc. Paran. Mat.,* 26 (2008), 71-80.

[2] M. Ashraf, N. Rehman, Sh. Ali, and M. R. Mozumder, On generalized (θ, ϕ)-derivations in semiprime rings with involution, *Math. Slovaca*, 62 (3) (2012), 451-460.

[3] M. Ashraf, On left (θ, φ)-derivations of prime rings, *Archivum Mathematicum*, 41 (2) (2005), 157-166.

[4] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc*, 140 (4) (1988), 1003-1006.

[5] M. Brešar and J. Vukman, Jordan derivations on prime rings, *Bull. Austral. Math. Soc*, 3 (1988), 321-322.

[6] J. Cusack, Jordan derivations on rings, *Proc. Amer. Math. Soc.*, 53 (1975), 1104-1110.

[7] G. Dales, et al., *Introduction to Banach Algebras, Operators and Harmonic Analysis,* Cambridge University press, 2002.

[8] I. N. Herstein, Jordan derivations of prime rings, *Proc. Amer. Math. Soc.*, 8 (1957), 1104-1110.

[9] J. Vukman and I. Kosi-Ulbl, Centralizers on rings and algebras, *Bull. Austral. Math. Soc.*, 71 (2005), 225-234.

[10] J. Vukman, A note on generalized derivations of semiprime rings, *Taiwanese Journal of Mathematics*, 11 (2) (2007), 367-370.

[11] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolin*, 32 (4) (1991), 609-614.

Amin Hosseini
Department of Mathematics
Assistant Professor of Mathematics
Kashmar Higher Education Institute
Kashmar, Iran
E-mail: A.Hosseini@mshdiau.ac.ir