Journal of Mathematical Extension
Vol. 10, No. 4, (2016), 35-43
ISSN: 1735-8299
URL: http://www.ijmex.com

Hypercyclic Operators on Banach Spaces

S. Panayappan*
Government Arts College

S. Meena
Bharathiar University

J. Vernold Vivin
Anna University

Abstract. In this article we obtain some new classes of hypercyclic and non-hypercyclic operators on Banach spaces.

AMS Subject Classification: 47A16; 47B99
Keywords and Phrases: Hyponormal, cohyponormal, transaloid, condition G_1, supercyclic, absolute k-paranormal, k-quasi paranormal, normaloid, convexoid and spectraloid operator

1. Introduction

A bounded linear operator $T : X \to X$ on a seperable Banach space X is hypercyclic if it has a vector with dense orbit. That is, if there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geqslant 0\}$ is dense in X. We denote the orbit of x under T by $Orb(T, x)$. An operator T on a Banach space X is said to be supercyclic if there is a vector $x \in X$ such that $\{cT^n x : n \geqslant 0, c \in \mathbf{C}\}$ is dense in X. Many fundamental results regarding the theory of hypercyclic and supercyclic operators were established

Received: November 2015; Accepted: January 2016
*Corresponding author

by C. Kitai [12]. She showed, for an operator to be hypercyclic, every connected component of its spectrum must intersect the unit circle. Herrero in [9] has shown that for any supercyclic operator (on a complex Hilbert space) there is some $r \geqslant 0$ such that every connected component of its spectrum meets the circle $|z| = r$. Examples of hypercyclic and supercylic operators arise in the classes of backward weighted shifts [15], adjoints of multiplication operator on spaces of analytic functions [11], composition operators [5], and adjoints of hyponormal operators [7]. Feldman, Miller and Miller in [7] have shown that cohyponormal operator can be hypercyclic. Bourden in [2] proved that no paranormal operator is supercyclic. In this paper we prove that transaloid and condition G_1 operators can also be hypercyclic. Also absolute k-paranormal operator and k-quasi paranormal operator which are the superclasses of paranormal operator are proved as not supercyclic in different methods.

2. Preliminaries

An operator T on a Hilbert space H is called hyponormal if $T^*T \geqslant TT^*$ and cohyponormal if T^* is hyponormal, that is, if $TT^* \geqslant T^*T$ [11]. An operator T is called paranormal if $||Tx||^2 \leqslant ||T^2x||||x||$ for all $x \in H$. Paranormal operators have been studied by many authors [8], [9] and [14]. Furuta, Ito and Yamazaki [9] have introduced the class of absolute-k-paranormal operators for $k > 0$ as generalization of paranormal operators. An operator T is said to be absolute k-paranormal if $\||T|^kTx\| \geqslant \|Tx\|^{k+1}$ for every unit vector $x \in H$. S. Mecheri [13] has studied k-quasi paranormal operator as a generalization of quasi paranormal operator. T is said to be k-quasi paranormal if $\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|$ where k is a natural number and $x \in H$. Furuta in [8] has introduced convexoid, condition G_1 and transaloid operator as generalizations of hyponormal operator. An operator T is said to be a convexoid if $\overline{W}(T) = co\sigma(T)$ where $W(T) = \{(Tx, x) : \|x\| = 1\}$ is the numerical range of T, $co\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of an operator T on a Hilbert space H. An operator T is said to be a condition G_1 operator if $\frac{1}{d(\mu,\sigma(T))} = \|(T-\mu)^{-1}\|$ for all $\mu \notin \sigma(T)$ and T is said to be a transaloid operator if $T - \mu$ is normaloid for any $\mu \in \mathbf{C}$. An operator T is said to be a normaloid if $r(T) = \|T\|$ and T is said to be spectraloid if $w(T) = r(T)$ where $w(T) = sup\{|\lambda| : \lambda \in W(T)\}$ is the numerical radius and $r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}$ is the spectral radius of an operator T [10].

3. Classes of Hypercyclic Operators on Hilbert Spaces

We begin this section by showing some general properties of a cohyponormal operator.

Theorem 3.1. *Let T be a cohyponormal operator. Then*
(i) $(T - \mu)$ is also cohyponormal for any $\mu \in \boldsymbol{C}$.
(ii) T is a transaloid operator.
(iii) T^{-1} is also cohyponormal operator if T^{-1} exists.
(iv) T is a condition G_1 operator.

Proof. (i) Since $TT^* \geqslant T^*T$, we have

$$(T - \mu)(T - \mu)^* - (T - \mu)^*(T - \mu) = TT^* - T^*T \geqslant 0.$$

This shows $(T - \mu)$ is cohyponormal for any $\mu \in \mathbf{C}$.
(ii) By (i), $(T-\mu)$ is cohyponormal for any $\mu \in \mathbf{C}$, showing that $(T-\mu)^*$ is hyponormal.
Since every hyponormal operator is normaloid it then follows $(T - \mu)^*$ is normaloid.
Since an operator is normaloid if and only if its adjont is normaloid, showing that $(T - \mu)$ is a normaloid.
Hence T is a transaloid.
(iii) Since $TT^* \geqslant T^*T$ implies

$$T^{*^{-1}}TT^*T^{-1} \geqslant 1$$

thus

$$\begin{aligned} 1 &\geqslant TT^{*^{-1}}T^{-1}T^* \\ T^{-1}T^{*^{-1}} &\geqslant T^{*^{-1}}T^{-1} \quad \text{and so } T^{-1} \text{ is cohyponormal.} \end{aligned}$$

(iv) We have

$$\frac{1}{d(\mu, \sigma(T))} = r((T-\mu)^{-1})) \text{ for all } \mu \notin \sigma(T), \tag{1}$$

T^* is hyponormal implies

$$\|(T^* - \mu)^{-1}\| \quad = \quad r(T^* - \mu)^{-1} \text{ [For normaloid operator } r(T) = \|T\|]$$

But $\|T\| \quad = \quad \|T^*\|$ and $\quad r(T) = r(T^*)$ shows

$$\|(T-\mu)^{-1}\| \quad = \quad r(T-\mu)^{-1}. \tag{2}$$

By (1) and (2) $\|(T-\mu)^{-1}\| = \frac{1}{d(\mu,\sigma(T))}$ and so T is a condition G_1 operator. □

Remark 3.2. *Transaloid operator and condition G_1 operator can be hypercyclic.*

Proof. Cohyponormal operators belong to transaloid class of operators and convexoid operators. But cohyponormal operators can be hypercyclic under certain conditions, by theorem 4.3 [7]. Hence the proof. □

4. Classes of Non-Hypercyclic Operators on Banach Spaces

Obviously the defining condition for absolute k-paranormal operator is not specific to Hilbert space only. This leads us to general Banach spaces.

Theorem 4.1. *Absolute k-paranormal operator on a Banach space X is not hypercyclic.*

Proof. Suppose T is absolute k-paranormal operator on a Banach space X for some $k > 1$ and $x \in X$ then $\|Tx\|^{k+1} \leqslant \||T|^k Tx\|$ for every unit vector $x \in X$. We may assume $\|T\| = 1$, without loss of generality. Now,

$$\begin{aligned}
\|Tx\|^{k+1} &\leqslant \||T|^k Tx\| \\
&\leqslant \||T|^{k-1}\|\||T|Tx\| \\
&\leqslant \|T^2x\| \text{ and so} \\
\|Tx\|^{k+1} &\leqslant \|T^2x\|\|x\|^k \qquad (3)
\end{aligned}$$

Put $x = T^n x$ in (3) to get
$\|T^{n+1}x\|^{k+1} \leqslant \|T^{n+2}x\|\|T^n x\|^k$

implies

$\|T^{n+1}x\|^k\|T^{n+1}x\| \leqslant \|T^{n+2}x\|\|T^n x\|^k$.

Suppose for some $n \geqslant 0$, $\|T^{n+1}x\| > \|T^n x\|$ then

$$\frac{\|T^{n+2}x\|}{\|T^{n+1}x\|} \geqslant \frac{\|T^{n+1}x\|^k}{\|T^n x\|^k} > 1,$$

Hence, the orbit of any $x \in X$ is either decreasing in norm, or strictly increasing in norm from some index n. So, no orbit can be dense. Therefore T is not hypercyclic. □

Remark 4.2. *Suppose that $f \in X$ is such that $\|Tf\| \geqslant \|f\|$ for the absolute k-paranormal operator $T : X \to X$ then $(\|T^n f\|)$ is an increasing sequence.*

Theorem 4.3. *Absolute k-paranormal operator on a Banach space X is not supercyclic.*

Proof. Suppose T is supercyclic, then T cannot be isometry, since isometries cannot be supercyclic in a Banach space setting [1]. Since T is absolute k-paranormal, we may assume $\|T\| = 1$ without loss of generality. Then there exists a vector $x \in X$ such that $\|Tx\| < \|x\|$.
Let α be a scalar of modulus > 1 such that $\|\alpha Tx\| < \|x\|$. Set $S = \alpha T$, so that S is absolute k-paranormal, supercyclic and has norm > 1 with $\|Sx\| < \|x\|$.

Since $\|S\| > 1$ and the set of supercyclic vectors for S is dense in X, there exists a supercyclic $f \in X$ such that $\|Sf\| > \|f\|$. Now f is supercyclic means there is a subsequence (n_j) of the sequence of non negative integers and a sequence (c_j) of scalars such that $(c_j S^{n_j} f) \to x$.
By continuity we have, $(c_j S^{n_j+1} f) \to Sx$.
Since S is absolute k-paranormal and $\|Sf\| > \|f\|$ by Remark 4.2, we have $\|S^{n_j+1} f\| \geqslant \|S^{n_j} f\|$ for every j. Thus

$$\begin{aligned} \|Sx\| &= lim_j \|c_j S^{n_j+1} f\| \\ &\geqslant lim_j \|c_j S^{n_j} f\| \\ &= \|x\|, \end{aligned}$$

which is a contradiction. Hence T is not supercyclic. □

Remark 4.4. *Similar to Theorem 4.3 we observe that k-Quasi paranormal operator on a Banach space X is not supercyclic.*

Remark 4.5. *The diagram shows the inclusion relations among the classes of operators discussed in this paper. The results proved in this paper are marked as $*$.*

Acknowledgements
The first author acknowledges with thanks the financial assistance of NBHM, DAE, Mumbai(Grant No.2/48 (12)/2014/R and D II/9266).

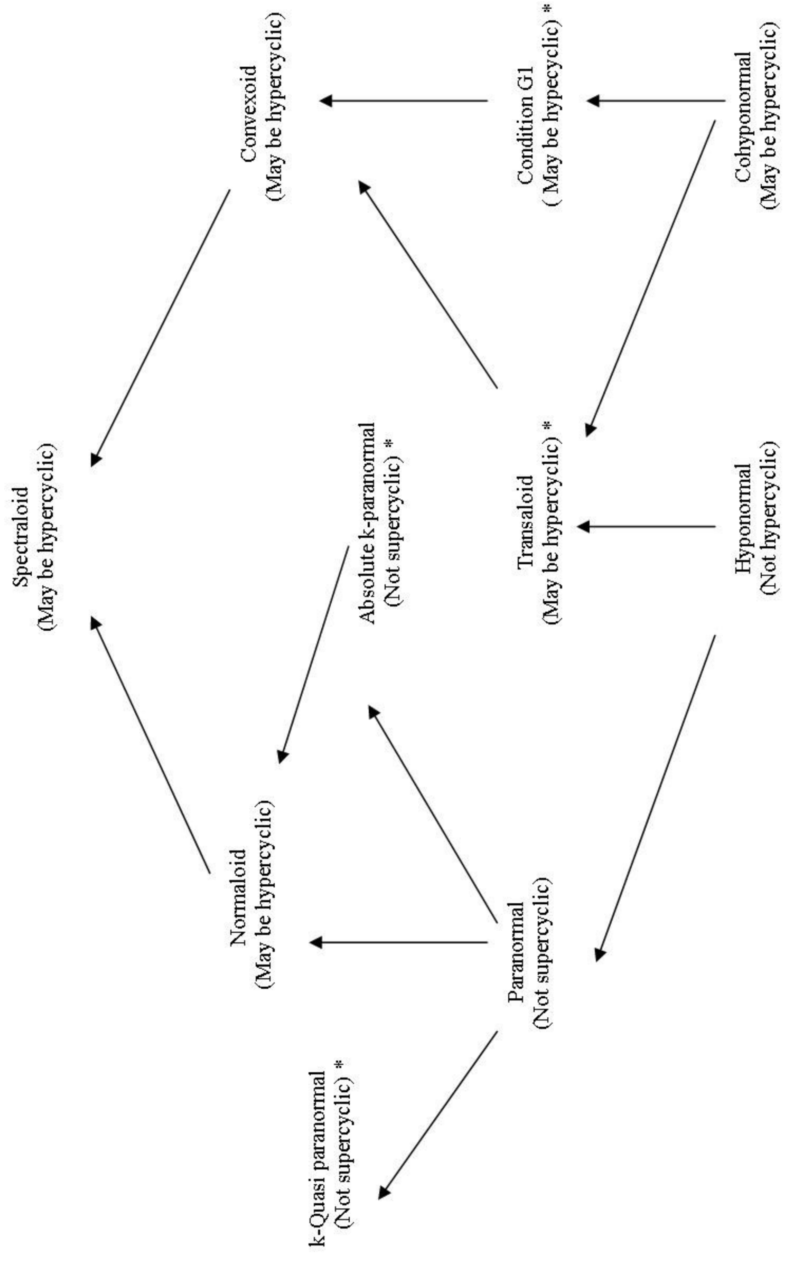

Figure 1. Inclusion relations among various classes of operators

References

[1] S. L. Ansari and P. S. Bourden, Some properties of cyclic operators, *Acta Math* (Szeged), 63 (1997), 195-207.

[2] P. S. Bourden, Orbits of hyponormal operators, *Michigan Math. J.*, 44 (2) (1997), 345-353.

[3] P. S. Bourden and J. H. Shapiro, Cyclic phenomena for composition operators, *Mem. Amer. Math. Soc.,* 125, No. 596, (1997).

[4] N. S. Feldman, T. L. Miller, and V. G. Miller, Hypercyclic and supercyclic cohyponormal operators, *Acta Sci, Math* (Szeged), 68 (2002), 303-328.

[5] T. Furuta, On the class of paranormal operators, *Proc. Japan. Acad*, 43 (1967), 594-598.

[6] T. Furuta, M. Ito, and T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several related classes, *Scientiae Mathematicae*, 1 (1998), 389-403.

[7] T. Furuta and Z. Takeda, A Charecterization of spectraloid operators and its application, *Proc. Japan. Acad*, 43 (1967), 599-604.

[8] G. Godefroy and J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds, *J. Funct. Anal*, 98 (1991), 229-269.

[9] D. A. Herrero, Limits of hypercyclic and supercyclic operators, *J. Funct. Anal*, 99 (1991), 179-190.

[10] H. M. Hilden and L. J. Wallen, Some cyclic and non cyclic vectors of certain operators, *Ind. Univ. Math. J.*, 23 (1974), 557-565.

[11] G. Karl, Alfred Peris, *Linear Chaos*, Springer-Verlag, (2011).

[12] C. Kitai, *Invariant Closed Sets for Linear Operators*, Dissertation, Univ. of Toronto, (1982).

[13] S. Mecheri, Bishop's property (β) and Riesz idempotent for k-quasi-paranormal operators, *Banach. J. Math. Anal*, 6 (1) (2012), 147-154.

[14] S. Panayappan and T. Veluchamy, Paranormal composition operators, *Indian. J. pure appl. math.*, 24 (1993), 257-262.

[15] H. N. Salas, Hypercyclic weighted shifts, *Trans AMS.*, 347 (1995), 993-1004.

S. Panayappan
Professor of Mathematics
Post Graduate and Research Department of Mathematics
Government Arts College (Autonomous),
Coimbatore-641 018, Tamil Nadu, India.
E-mail: panayappan@gmail.com

S. Meena
Research scholar
Research and Developement Centre
Bharathiar University
Coimbatore-641 046, Tamil Nadu, India.
E-mail: meenaprem14@gmail.com

J. Vernold Vivin
Department of Mathematics,
Assistant Professor of Mathematics
University College of Engineering Nagercoil
(Anna University Constituent College)
Nagercoil-629 004, Tamil Nadu, India.
E-mail: vernoldvivin@yahoo.in