Generalized Weighted Weibull Distribution

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Abstract

The new class of weighted exponential (WE) distributions obtained by Gupta and Kundu (2009) by applying Azzalini's method to the exponential distribution. Kharazmi et al. (2015) extended the WE distribution to the generalized weighted exponential (GWE) distribution and studied its different properties. In this study, we generalize the weibull distribution to a new class referred to as the generalized weighted weibull (GWW) distribution with one scale parameter and two shape parameters. The GWW model is constructed in a way that is similar to the way in which the GWE is constructed. It is investigated that the new model has increasing, decreasing and upside-down bathtub shaped hazard. Several statistical and reliability properties of this new class of distribution are obtained. Estimation, Simulation and inference procedure for distribution parameters are investigated. Finally, we show that the proposed model can provide better fit than some recent classes of the extended weibull by using two real data examples.

Keywords: Weighted weibull distribution, Hazard function, Mean residual life time, Stochastic orders, Maximum likelihood estimates.

1. Introduction

Motivated by engineering applications, Weibull (1939), a Swedish physicist, suggested a distribution that has proved to be of seminal importance in reliability. The corresponding survival function is given by the equation

$$\bar{F}(x) = \exp(-\lambda x^{\beta}) \ , x > 0 \ ,$$

with parameters $\beta, \lambda > 0$. The weibull distribution is a very popular model, and has been extensively used over the past decades for modelling data in reliability, engineering and biological studies. However, the weibull distribution does not exhibit a bathtub or upside-down bathtub shaped hazard rate function and thus it cannot be used to model the complex lifetime of a system. Hence a number of extensions of the weibull distribution are introduced to overcome this shortage. For example, Almalki and Nadarajah (2012) reviewed and provided a comprehensive description of modifications of the weibull distribution.

Some recent developments in distribution theory have proposed new techniques for building distributions. Among these, the Azzalini's novel method (1985) has received a lot of attention. In the statistical literature, this technique has been used to construct new skewed distribution from a given symmetric distribution. However there is little work on the use of Azzalini's method for skewed distribution.

The now widely known skew-normal distribution is just one special case belonging to the family of distributions introduced by Azzalini (1985). For the first time Azzalini (1985) introduced a shape parameter to a normal distribution. Afterwards extensive work on introducing shape parameters for other symmetric distributions have been defined and several properties and their inference procedures have been discussed by Several authors, see for example Balakrishnan(2002), Genton (2004), Arnold and Beaver (2000a) and Nadarajah (2009).

Recently some authors effort to implement Azzalini's idea for skewed distributions, the new class of weighted exponential (WE) distribution obtained by Gupta and Kundu (2009) by implementing Azzalini's method to the exponential distribution. Shakhatreh (2012) generalized the WE distribution to the two-parameter weighted exponential distributions (TWE). Kharazmi et al. (2015) extended weighted exponential distribution to the generalized weighted exponential (GWE) distribution and studied its different properties. Several interesting properties of this distribution have been established by authors. The GWE distribution contains the above mentioned distributions as its sub-models. It was observed that the GWE distribution can provide a better fit for survival time data relative to other common distributions such as weighted exponential (WE), two parameter weighted exponential (TWE), gamma, weibull and generalized exponential (GE) distribution.

The main aim of this paper is to introduce the class of generalized weighted weibull (GWW) distribution in a way that is similar to the way in which the GWE model is proposed. In fact this way is a modification of Azzalini's method. It is investigated that the new model has increasing, decreasing and upside-down bathtub shaped hazard. We provide a comprehensive description of some mathematical properties of the GWW distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research. The proposed GWW distribution provides the GWE, WE and TWE distributions as

Its sub-models. Also, many well-known distributions can be obtained as special cases of this model.

The paper is organized as follows. In Section 2 we briefly review the WE, TWE, GWE distributions and then we define the proposed GWW distribution. Section 3 provides different representation for construction of the GWW distribution. Section 4 presents some basic statistical and reliability properties of the proposed GWW family. Section 5 gives some important theorems about the GWW distribution and related results. In Section 6, we compare the GWW and WW distributions with respect to stochastic orders information. Finally in Section 7, we discuss and study the MLEs, fisher information, simulation performance and two applications to real data for proposed model. Concluding remarks are provided in Section 8.

2. Definition and basic properties

The weighted exponential distribution was introduced in the seminal paper by Gupta and Kundu (2009). A random variable X is said to have weighted exponential distribution, denoted by $WE(\lambda,\alpha)$, if its probability density function (PDF) is given as

$$f_X(x,\alpha,\lambda)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right). \tag{1}$$

Where $x>0$, $\alpha>0$ and $\lambda>0$. Here α and λ are the shape and scale parameters, respectively. The main properties and different interpretations of this density are established by authors. A particular property that has received considerable attention is the generalized form of all life time distribution. The WE distribution was generalized to the two-parameter weighted exponential distributions $TWE(\lambda,\alpha,\alpha)$ by Shakhatreh (2012) with the following PDF,

$$f_X(x,\alpha,\lambda)=\frac{(\alpha+1)(1+2\alpha)}{2\alpha^2}\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)^2. \tag{2}$$

Where $x>0$, $\alpha>0$ and $\lambda>0$. Here α and λ are the shape and scale parameters, respectively.

Recently, Kharazmi et al. (2015). generalized WE distribution to the generalized weighted exponential distribution $GWE(\lambda,\alpha,n)$ with the following PDF

$$f_X(x,\alpha,\lambda,n)=\frac{\alpha}{B(1/\alpha,n+1)}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x})^n\ . \tag{3}$$

Where the beta function is defined in the usual way as $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ and $x>0$, $\alpha>0$, $\lambda>0$, $n\geq 1$.

The aim of this paper is to provide a similar generalization to weibull distributions for more flexibility to fitting survival data in the real application. Here, we introduce the definitions of the weighted weibull and generalized weighted weibull distribution denoted by $WW(\alpha,\beta,\lambda)$ and $WW(\alpha,\beta,\lambda,n)$, respectively.

Definition 1. A random variable X is said to have weighted weibull distribution, denoted by $WW(\alpha,\beta,\lambda)$, if its probability density function (PDF) is given as

$$f_X(x,\alpha,\beta,\lambda)=\frac{\alpha^{\beta}+1}{\alpha^{\beta}}\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(1-e^{-\lambda(\alpha x)^{\beta}}\right). \tag{4}$$

Where $x>0$, $\alpha>0, \beta>0$ and $\lambda>0$. Here α,β and λ are the shape and scale parameters, respectively.

Definition 2. A random variable X is said to have generalized weighted weibull distribution $GWW(\alpha,\beta,\lambda,n)$ with integer $n\geq 1$, shape parameters α,β and scale parameter λ, if the PDF of X is given as following

$$f_X(x,\alpha,\beta,\lambda,n)=\frac{\alpha^{\beta}}{B(1/\alpha^{\beta},n+1)}\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}(1-e^{-\lambda(\alpha x)^{\beta}})^{n} \ . \tag{5}$$

Where the beta function is defined in the usual way as $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ and $x>0$, $\alpha>0$, $\lambda>0, \beta>0$. Fig. 1 shows the shapes of $GWW(\alpha=1,\beta=1,\lambda=1,n)$ for $n=1,2,3$ and 4.

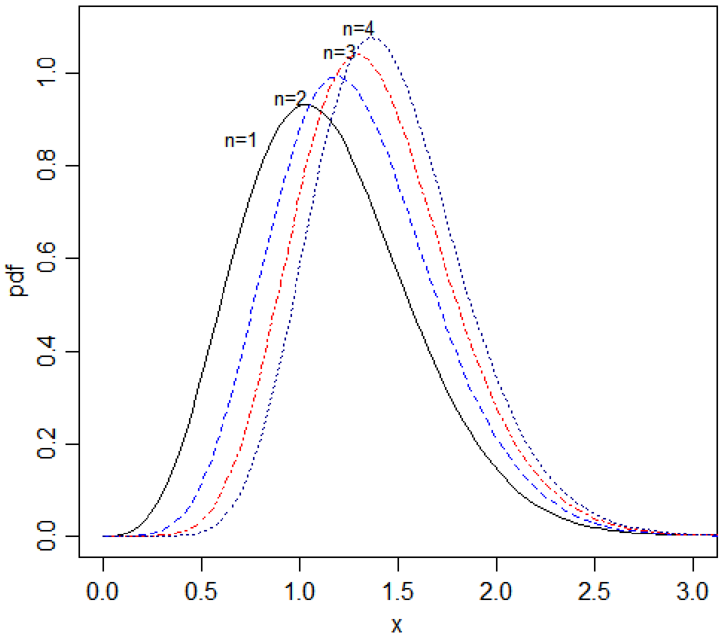


Fig. 1. Plots of the $GWW(\lambda=1,\alpha=1,\beta=2,n)$ for different values of n.

In the following, it is seen that the $GWE(\lambda,\alpha,n)$, $WE(\lambda,\alpha)$, $TWE(\lambda,\alpha,\alpha)$ and $WW(\alpha,\beta,\lambda,)$ are special cases of $GWW(\alpha,\beta,\lambda,n)$.

Case 1. When $\beta=1$, then $GWW(\alpha,\beta=1,\lambda,n)=GWE(\lambda,\alpha,n)$.
Case 2. When $n=1$, then $GWW(\alpha,\beta=1,\lambda,n=1)=WE(\lambda,\alpha)$.

Case 3. When $n = 2$, then $GWW(\alpha, \beta = 1, \lambda, n = 2) = TWE(\lambda, \alpha, \alpha)$.
Case 4. When $n = 1$, then $GWW(\alpha, \beta, \lambda, n = 1) = WW(\lambda, \alpha, \beta)$.

To investigate the effect of integer parameter n on the GWW distribution we plotted the $GWW(\alpha, \beta, \lambda, n)$ density for different values of n and $\alpha = 1, \beta = 2 \ \ \lambda = 1$ in Fig. 1.

3. Methods of Construction

There are different ways to construct the GWW distribution especially, the ones which are provided for the WE distribution by Gupta and Kundu (2009).

Selection Model (Modification of Azzalini's method):

The GWW distribution is constructed based on weibull distribution with scale parameter λ and shape parameter β , with density

$$f(x) = \beta\lambda x^{\beta-1}\exp(-\lambda x^{\beta}). \quad (6)$$

Where $x > 0$, $\beta > 0$ and $\lambda > 0$. We use Balakrishnan's idea, about generalized skew-normal distribution, to construct GWW model, see Balakrishnan(2002), Arnold and Beaver (2002). Let $X, X_1, X_2, \dots, X_n$ be a random sample of size $n + 1$ from the wiebull distribution with scale parameter λ and shape parameter β , then

$$X | \operatorname{Max}(X_1, X_2, \dots, X_n) \leq \alpha X \sim GWW(\alpha, \beta, \lambda, n). \quad (7)$$

This method of construction is known as a selection model. Arellano et al. (2006). In fact (7) is a modification of Azzalini's method.

Weighted Distribution:

The concept of weighted distributions is important in a wide range of statistical applications. A density function g is said to be a weighted density function corresponding to density function f with weight $w > 0$.

$$g(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad x \geq 0 \quad (8)$$

where $0 < E[w(X)] < \infty$. Patil and Rao (1977, 1978). The GWW distribution can be obtained as a special form of the weighted distribution by taking weighted function

$$W(x, \alpha, \beta, \lambda, n) = [1 - \exp(-\alpha^{\beta}\lambda x^{\beta})]^{n}.$$

Beta Family Distribution:

Form (5) can be obtained from the beta family distributions proposed by Eugene et al. (2002). Let $\bar{G}(x)$ be a survival function of a random variable with weibull distribution with parameters β and λ, then form (5) has the following distribution

$$F(x,\lambda,\alpha) = \frac{1}{B(a,b)} \int_{\bar{G}(x)^{\frac{1}{a}}}^{1} w^{a-1}(1-w)^{b-1} dw \,. \qquad (9)$$

Where $a = 1/\alpha^{\beta}$ and $b = n+1$.

Hidden Truncation Model:

The GWW distribution can be obtained as a hidden truncation model proposed by Arnold and Beaver (2002). Suppose Z and Y are two dependent random variables with the following joint density function

$$f_{Z,Y}(z,y) = n\beta^2\lambda^2 z^{2\beta-1}\, y^{\beta-1} e^{-\lambda z^{\beta}(1+y^{\beta})}(1-e^{-\lambda z^{\beta} y^{\beta}})^{n-1} \qquad (10)$$

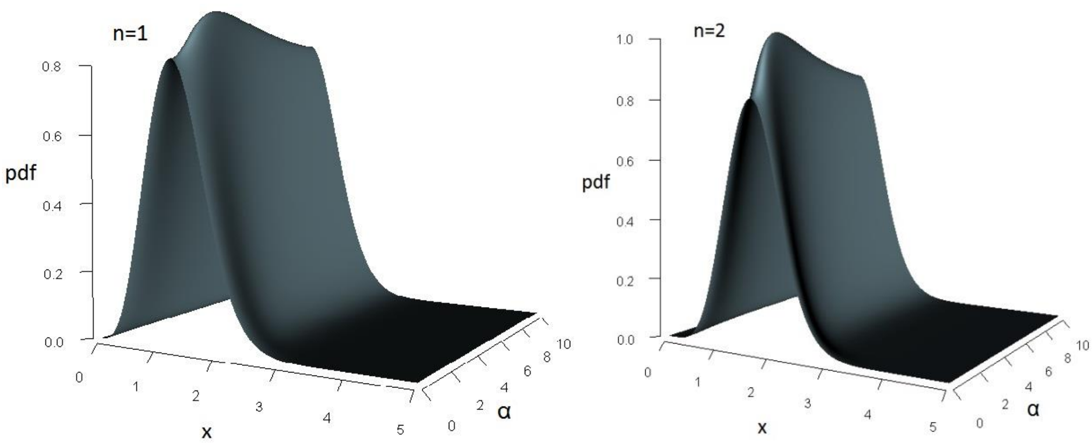
where n is a non-negative integer. It can be shown that the conditionally random variable $Z|Y \leq \alpha$ has the GWW distribution.

Additive of Independent Random Variables:

Suppose random variables $X_i\,, i = 0,1,2,\dots,n$ be independent and $X_i \sim \text{Exp}(\lambda(\alpha^{\beta} i + 1))$ then $Y = (\sum_{i=0}^{n} X_i)^{\frac{1}{\beta}}$ has PDF with form (5).This method will be study as a main theorem to generate data from the GWW distribution in section 7.

Marshall and Olkin Semi-Parametric Family:

Another way of obtaining the GWW distribution is the way that introduced by Marshall and Olkin (2007), for the semi-parametric family distributions. By composing two CDF such as $\bar{F}(x,\alpha,\lambda) = H(\bar{F}(x,\lambda),\alpha)$, where $H(t,\alpha) = [\alpha^{\beta}/B(1/\alpha^{\beta}, n+1)] \int_0^t (1-x^{\alpha^{\beta}})^n dx$ be a continuous CDF supported by [0,1] and $\bar{F}(x,\lambda) = e^{-\lambda x^{\beta}}$ represents underlying survival function.



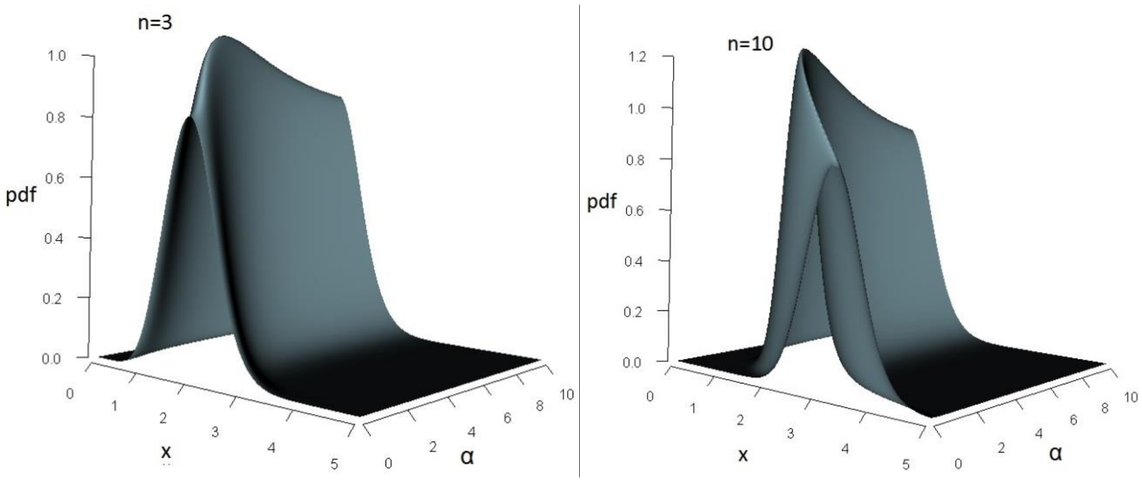


Fig. 2. 3D-Plots of the $GWW(\alpha, \beta = 2, \lambda = 1, n)$ for different values of n.

4. Statistical and reliability properties

In this section we study the several statistical and reliability properties of the GWW distribution, such as the distribution function (CDF), survival function (SF), conditional survival function (CSF), failure rate (or hazard) function (FR), moment generating function (MGF), mean residual life (MRL) time and kth moment. Firstly, we review the following lemma and theorem Kharazmi et al. (2015) in order to prove our main results.

Lemma 4.1 The following relation is fulfilled for a non-negative integer n , $\alpha > 0$ and $t \in R$

$$\sum_{j=0}^{n+1} \frac{(-1)^j \binom{n+1}{j}}{\alpha j + 1 - t} = \frac{\alpha(n+1)}{\alpha(n+1)+1-t} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha j + 1 - t}. \tag{11}$$

Theorem 4.1. The following relation is fulfilled for a non-negative integer $n, \alpha > 0$ and $t \in R$,

$$\prod_{j=0}^{n} (\alpha j + 1 - t) = \frac{n! \alpha^n}{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha j + 1 - t}} \tag{12}$$

Result 4.1. After replacing $t = 0$ in theorem 4.1 we have,

$$\frac{\alpha}{B\left(\frac{1}{\alpha}, n+1\right)} = \frac{\prod_{j=0}^{n} (j\alpha + 1)}{n! \alpha^n} = \frac{1}{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha j + 1}}. \tag{13}$$

Result 4.2. After replacing α^β instead of α in theorem 4.1 we have,

$$\frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)} = \frac{\prod_{j=0}^{n} (j\alpha^\beta + 1)}{n! \alpha^{n\beta}} = \frac{1}{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j + 1}}. \tag{14}$$

4.1. Distribution function, survival, conditional reliability and failure rate function

By using the result 4.2 the CDF of (5) can be written as

$$F_X(x,\alpha,\beta,\lambda,n) = 1 - \frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta},n+1\right)} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j+1} e^{-\lambda(\alpha^\beta j+1)x^\beta}; \quad x>0. \tag{15}$$

Also, survival function and conditional reliability are given by

$$\bar{F}_X(x,\alpha,\beta,\lambda,n) = \frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta},n+1\right)} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j+1} e^{-\lambda(\alpha^\beta j+1)x^\beta}; \quad x>0, \tag{16}$$

and

$$\bar{F}_X(x,\alpha,\beta,\lambda,n|t) = \frac{\bar{F}_X(x+t,\alpha,\beta,\lambda,n)}{\bar{F}_X(t,\lambda,\alpha,\beta,\lambda,n)} = \frac{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j+1} e^{-\lambda(\alpha^\beta j+1)(x+t)^\beta}}{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j+1} e^{-\lambda(\alpha^\beta j+1)t^\beta}}; \quad x>0, t>0, \tag{17}$$

respectively. Conditional survival function plays an important role in classifying life time distributions. From (5) and (16) it is easy to verify that the failure rate function is given by

$$h(x,\alpha,\beta,\lambda,n) = \frac{\lambda\beta x^{\beta-1} e^{-\lambda x^\beta}(1-e^{-\lambda(\alpha x)^\beta})^n}{\sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}}{\alpha^\beta j+1} e^{-\lambda(\alpha^\beta j+1)x^\beta}}; \quad x>0. \tag{18}$$

The failure rate is a key notion in reliability and survival analysis for measuring the ageing process. Understanding the shape of the failure rate is important in reliability theory, risk analysis and other disciplines. The concepts of increasing ,decreasing , bathtub shaped(first decreasing and then increasing) and upside-down bathtub shaped(first increasing and then decreasing) failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR , DFR ,BUT and UBT distributions, respectively. It is investigated that $h(x,\alpha,\beta,\lambda,n)$ is increasing for $\beta \geq 1$ and is decreasing or uni-modal (upside-down bathtub shaped) for $\beta < 1$.

4.2. On the hazard function shape

Here, we discuss the shapes of the hazard function of GWW distribution. We initially consider the pdf (5). Fig (1) indicate that GWW distribution is uni-modal and

$$\lim_{x\to 0} f_X(x,\alpha,\beta,\lambda,n) = 0 \ , \lim_{x\to\infty} f_X(x,\alpha,\beta,\lambda,n) = 0 \ .$$

The hazard rate function in equation (18) is very complex. For $\beta \geq 1$, we conclude straightforwardly that the hazard rate function is increasing. For $\beta < 1$, the shapes of the hazard function were obtained numerically. Indeed, it was verified numerically, considering the R software and the conditions of Glaser's theorem (1980). We considered several points within each region of the parametric space that characterize the shape of the hazard rate function and studied numerically their properties.

Theorem 4.2 Assume that $\beta \geq 1$, then , GWW distribution is IFR.

Proof:

(a) Let $\beta \geq 1$ and $g(x) = \frac{1}{h(x,\alpha,\beta,\lambda,n)} = \int_x^\infty \left(\frac{t}{x}\right)^{\beta-1} e^{-\lambda(t^\beta - x^\beta)} (\frac{1-e^{-\lambda(\alpha t)^\beta}}{1-e^{-\lambda(\alpha x)^\beta}})^n dt,$ now by changing variable $t^\beta - x^\beta = z$ we have

$$g(x) = \frac{1}{\beta x^{\beta-1}} \int_0^\infty e^{-\lambda z} (\frac{1 - e^{-\lambda\alpha^\beta (z+x^\beta)}}{1 - e^{-\lambda\alpha^\beta x^\beta}})^n dz$$

Sinc $g(x)$ is written as multiplicative two non-negative and decreasing function as following

$$g(x) = k(x)l(x).$$

Where $k(x) = \frac{1}{\beta x^{\beta-1}}$ and $l(x) = \int_0^\infty e^{-\lambda z} (\frac{1-e^{-\lambda\alpha^\beta (z+x^\beta)}}{1-e^{-\lambda\alpha^\beta x^\beta}})^n dz$. So, $g(x)$ is decreasing and proof is completed.

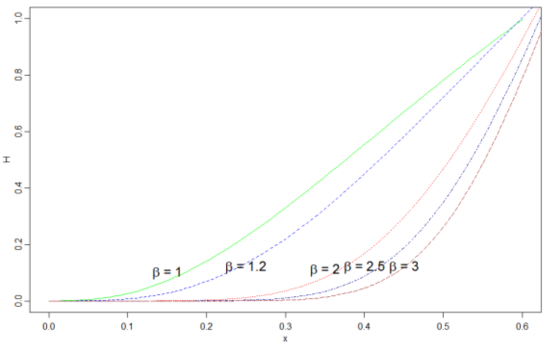
Now we discuss the hazard rate shape for $\beta < 1$. Define $\eta(x) = -\frac{f'_X(x,\alpha,\beta,\lambda,n)}{f_X(x,\alpha,\beta,\lambda,n)}$ where $f'_X(x,\alpha,\beta,\lambda,n)$ is the first derivative of the density function (5) is. Hence,

$$\eta(x) = \beta\lambda x^{\beta-1} - \frac{\beta - 1}{x} - \frac{n\beta\lambda\alpha^\beta x^{\beta-1}}{e^{\lambda\alpha^\beta x^\beta} - 1}$$

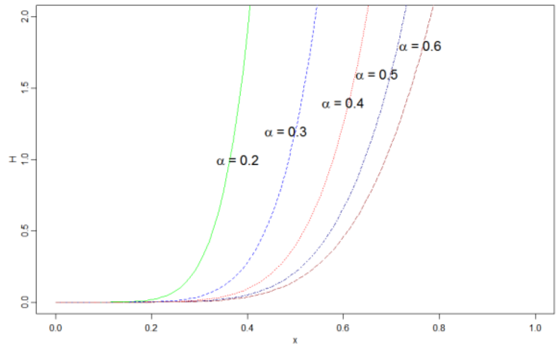
The first derivate of $\eta'(x)$ is given by

$$\eta'(x) = \beta(\beta-1)\lambda x^{\beta-2} + \frac{\beta-1}{x^2} - \frac{n\beta(\beta-1)\lambda\alpha^\beta x^{\beta-2}}{e^{\lambda\alpha^\beta x^\beta}-1} + \frac{n\beta^2(\beta-1)\lambda^2\alpha^{2\beta} x^{2(\beta-1)}}{(e^{\lambda\alpha^\beta x^\beta}-1)^2}.$$

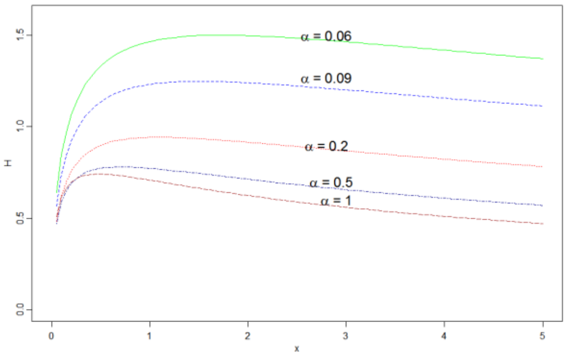
To verify that the failure rate function can be decreasing, for instance, we consider one-point parametric space $(\alpha, \beta, \lambda) = (10, 0.2, 1)$. We verify the condition $\eta'(x) \leq 0$ for $x > 0$. From Glaser (1980), we conclude that the failure rate function is decreasing (DFR). To verify that the failure rate function can be uni-modal, we consider the one point parametric space $(\alpha, \beta, \lambda) = (0.06, 0.4, 1)$, and we obtain $x_0 = 3.73$, such that $\eta'(x_0) = 0, \eta'(x) \geq 0$ for $x \in (0, x_0)$ and $\eta'(x) < 0$ for $x \in (x_0, \infty)$. Further, $lim_{x \to 0} f_X(x, \alpha, \beta, \lambda, n) = 0$. From Glaser's theorem we conclude that the hazard rate function is uni-modal (UBT). Fig. 3 shows some failure rate function shapes for some values of α , β, when $\lambda = 1$ and $n = 3$.



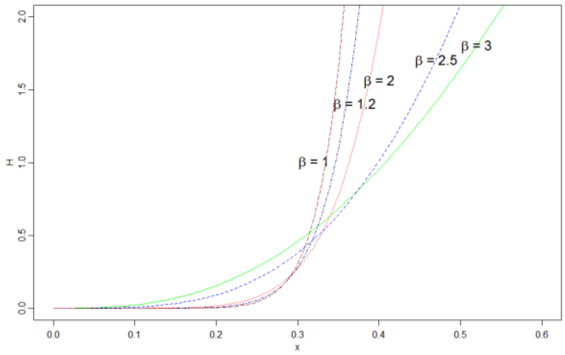
$\lambda = 1, \alpha = 2, n = 3$



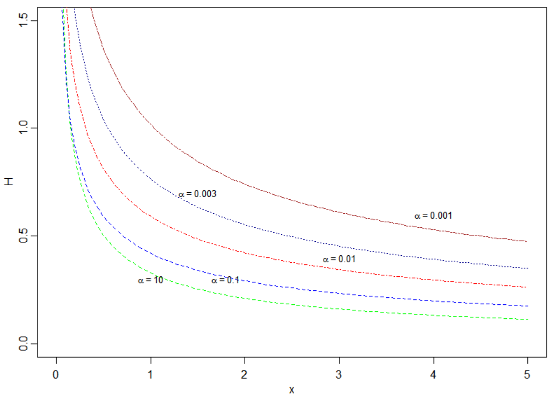
$\lambda = 1, \beta = 2, n = 3$



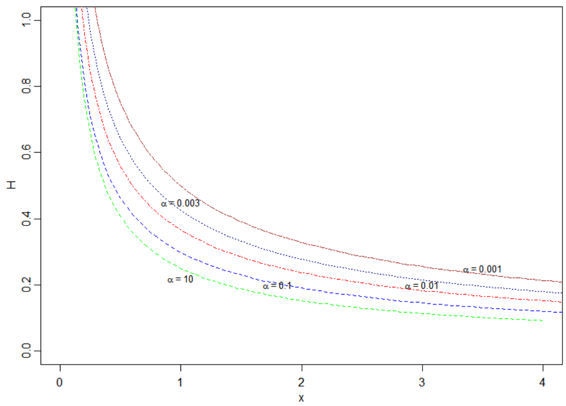
$\lambda = 1, \beta = 0.4, n = 3$



$\lambda = 1, \alpha = 0.2, n = 3$



$\lambda = 1, \beta = 0.2, n = 3$



$\lambda = 1, \beta = 0.15, n = 3$

Fig. 3. Hazard function of the GWW distribution for different values of α , β , $\lambda = 1$ and $n = 3$.

4.3. Moment generating function and mean residual life time

Now let us consider different moments of the GWW(α,β,λ,n) distribution. Some of the most important features and characteristics of a distribution can be studied through its moments, such as moment generating function, the kth moment and interested reliability properties such as mean residual life time.

The moment generating function of form (5) is immediately written as

$$M_X(t)=\frac{\alpha^{\beta}}{B(1/\alpha^{\beta},n+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^{j}\frac{t^{i}}{i!}\binom{n}{j}\lambda\frac{\Gamma\left(\frac{i}{\beta}+1\right)}{\left(\lambda(\alpha^{\beta}j+1)\right)^{\frac{i}{\beta}+1}}. \tag{19}$$

Where $t\in\{t\mid M_X(t)<\infty\}$.

The kth moment and kth central moment of the GWW distribution can be derived as

$$\mu_k=E(X^k)=\frac{\alpha^{\beta}\Gamma(\frac{k}{\beta}+1)}{\lambda^{\frac{k}{\beta}}B(1/\alpha^{\beta},n+1)}\sum_{j=0}^{n}\frac{(-1)^{j}\binom{n}{j}}{(\alpha^{\beta}j+1)^{\frac{k}{\beta}+1}}. \tag{20}$$

$$\tau_k=E(X-\mu_1)^k=\frac{\alpha^{\beta}}{B(1/\alpha^{\beta},n+1)}\sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{i+j}\binom{k}{i}\binom{n}{j}\frac{\mu^{i}\Gamma\left(\frac{k-i}{\beta}+1\right)}{\lambda^{\frac{k-i}{\beta}}(\alpha^{\beta}j+1)^{\frac{k-i}{\beta}+1}} \tag{21}$$

In particular, the mean and variance are given, by

$$\mu_1=E(X)=\frac{\alpha^{\beta}\Gamma(\frac{k}{\beta}+1)}{\lambda^{\frac{1}{\beta}}B(1/\alpha^{\beta},n+1)}\sum_{j=0}^{n}\frac{(-1)^{j}\binom{n}{j}}{(\alpha^{\beta}j+1)^{\frac{1}{\beta}+1}}, \tag{22}$$

and

$$Var\ (X)=\tau_2=E(X-\mu_1)^2=\frac{\alpha^{\beta}}{B(1/\alpha^{\beta},n+1)}\sum_{i=0}^{2}\sum_{j=0}^{n}(-1)^{i+j}\binom{2}{i}\binom{n}{j}\frac{\mu^{i}\Gamma\left(\frac{2-i}{\beta}+1\right)}{\lambda^{\frac{2-i}{\beta}}(\alpha^{\beta}j+1)^{\frac{2-i}{\beta}+1}}. \tag{23}$$

Respectively. One of the well-known properties of the life time distribution is mean residual life time. For the GWW distribution it can be written as

$$m(t)=E(X-t|X>t)=\frac{\sum_{j=0}^{n}\frac{(-1)^{j}\binom{n}{j}}{(\alpha^{\beta}j+1)}\int_{t}^{\infty}\mathrm{e}^{-\lambda(\alpha^{\beta}j+1)x^{\beta}}dx}{\sum_{j=0}^{n}\frac{(-1)^{j}\binom{n}{j}}{\alpha^{\beta}j+1}e^{-\lambda\left(\alpha^{\beta}j+1\right)t^{\beta}}};\quad t>0. \tag{24}$$

By considering the behavior of hazard rate function, for $\beta\geq 1, m(t)$ is DFR and for $\beta<1$, $m(t)$ can be IFR or BUT.

4.4. Median

We can find the median (m) of GWW by solving the following equation.

$$F_X(m,\alpha,\beta,\lambda,n)=\frac{1}{2}, \tag{25}$$

Therefore, by using form (15) we have

$$\frac{\alpha^{\beta}}{B\left(\frac{1}{\alpha^{\beta}},n+1\right)}\sum_{j=0}^{n}\frac{(-1)^{j}\binom{n}{j}}{\alpha^{\beta}j+1}e^{-\lambda(\alpha^{\beta}j+1)x^{\beta}}-\frac{1}{2}=0. \tag{26}$$

There is no explicit solution for this equation, so the median m cannot be given explicitly.

5. Main results

In this section, we proposed main results about the GWW family distribution.

Theorem 5.1. The following properties are satisfied for the family of GWW distribution:

1. If $n \to \infty$, then the GWW random variable is degenerated at point 0.
2. If $\alpha \to 0$, then $f_X(x,\lambda,\alpha,\beta=1,n)$ converges to $Gamma(n+1,\lambda)$.
3. If $\alpha \to \infty$, then $f_X(x,\lambda,\alpha,\beta,n)$ converges to $W(\lambda,\beta)$
4. Let $X_1, X_2, \dots, X_n$ be a sample from exponential distribution with mean $1/\lambda$. Then $\text{Max}(X_1, X_2, \dots, X_n) \sim GWW(\alpha=1,\beta,\lambda,n-1)$.
5. Let $X, X_1, X_2, \dots, X_m$ be a sample from weibull distribution with scale parameter λ and shape parameter β , then

$$X|\,\text{Min}(X_1,X_2,\dots,X_m)\le \alpha X \sim GWW\left(m^{\frac{1}{\beta}}\alpha,\beta,\lambda,n=1\right)=WW(m^{\frac{1}{\beta}}\alpha,\beta,\lambda).$$

6. Let $Y \sim W(\lambda,\beta)$ and $X \sim GWW(\alpha,\beta,\lambda,n-1)$be independent random variable.Then conditionally random variable $X|(X\alpha \ge Y) \sim GWW(\alpha,\beta,\lambda,n)$.

Proof: The proofs of all cases are straightforward.

Theorem 5.2 (Relation between GWE and GWW distributions). Suppose the random variables $X_i\,, i=0,1,2,\dots,n$ be independent and $X_i \sim \exp(\lambda(\alpha^{\beta}i+1)$, then
(a) the random variable $Y=\sum_{i=0}^{n} X_i$ follows the $GWE\left(\lambda,\alpha^{\beta},n\right)$.
(b) the random variable $Z=\left(\sum_{i=0}^{n} X_i\right)^{\frac{1}{\beta}}$ follows the $GWW(\alpha,\beta,\lambda,n)$ distribution.

Proof:

(a) See Kharazmi et al. (2015).
(b) It is straightforward

6. Stochastic orders

In this section, we are interested in comparing the WW and GWW distributions with respect to stochastic ordering information, See shaked and Shanthikumar (2007). The comparison of two random variables is very important in reliability theory, risk analysis and other disciplines. There are many possibilities to compare random variables or their distributions, respectively, with each other. One of the most important orderings among stochastic orderings is the Likelihood ratio ordering which compares lifetimes of systems with respect to their Likelihood information. In this section we give a basic theorem for comparing the $GWW(\alpha,\beta,\lambda,n)$ and $WW\left(n^{\frac{1}{\beta}}a,\beta,\lambda\right)$ distributions. Now let us to give a quick review of required definitions of stochastic orders and notation which are used in the following theorem. Let X and Y be two random variables with distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ and density functions $f$and g. It is said that X is smaller than Y in the following expression.

1. Likelihood ratio order $(X \leq_{lr} Y)$ if $g(x)/f(x)$ is increasing in x.
2. Hazard rate order $(X \leq_{hr} Y)$if $\bar{G}(x)/\bar{F}(x)$ is increasing in x.
3. Usual stochastic order $(X \leq_{st} Y)$ if $\bar{F}(x) \leq \bar{G}(x)$.
4. Mean residual life order $(X \leq_{mrl} Y)$if $E(X - t|X > t) \leq E(Y - t|Y > t)$.

The following implications hold among these stochastic orders. $Y \leq_{lr} X \Rightarrow Y \leq_{hr} X \Rightarrow Y \leq_{st(mrl)} X$. For further results see shaked and Shanthikumar (2007).

Theorem 6.1. Let $U \sim GWW(\alpha, \beta, \lambda, n)$ and $V \sim WW\left(n^{\frac{1}{\beta}}\alpha, \beta, \lambda\right)$, then $V \leq_{lr} U$.

Proof:

It is sufficient to show that the ratio $\frac{f_U(x)}{f_V(x)} = \frac{\frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)}\lambda\beta x^{\beta-1}e^{-\lambda x^\beta}(1-e^{-\lambda(\alpha x)^\beta})^n}{\frac{n\alpha^\beta+1}{n\alpha^\beta}\lambda\beta x^{\beta-1}e^{-\lambda x^\beta}(1-e^{-\lambda n(\alpha x)^\beta})}$ is increasing in x. By direct calculation we have

$$\frac{d}{dx}\left(\frac{f_U(x)}{f_V(x)}\right) = \frac{\frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)}}{\frac{n\alpha^\beta+1}{n\alpha^\beta}} \frac{n\lambda\beta\alpha^\beta x^{\beta-1}(1-e^{-\lambda(\alpha x)^\beta})^{n-1}[e^{-\lambda(\alpha x)^\beta}\left(1-e^{-\lambda n(\alpha x)^\beta}\right)-e^{-\lambda n(\alpha x)^\beta}(1-e^{-\lambda(\alpha x)^\beta})]}{(1-e^{-\lambda n(\alpha x)^\beta})^2} \geq 0,$$

so the proof is complete.

Result 6.1. $V \leq_{lr} U \Rightarrow V \leq_{hr} U \Rightarrow V \leq_{st(mrl)} U$.

7. Estimation , Fisher Information Matrix , Simulation and Application

In this section, first, we discuss the maximum likelihood estimation, fisher information matrix and simulation performance for the GWW distribution and then we show that the proposed model can provide better fit than some recent classes of extended weibull by using two real data examples. Section7.1 gives procedures for maximum-likelihood estimation of the GWW distribution. Section7.2 devoted to the computing of the fisher information matrix. Section7.3 assesses the performance of the MLEs in terms of biases, mean-squared errors, coverage probabilities, and coverage lengths by means of a simulation study. Finally in section 7.4 we discuss the application of GWW distribution.

7.1. Maximum likelihood estimation

Let $X_1, X_2, \dots, X_m$ be a random sample from the distribution with density (5). The likelihood function based on observed values $x_1, x_2, \dots, x_m$ is given by

$$L(\alpha,\beta,\lambda,\underline{\boldsymbol{x}}) = \left(\frac{\alpha^{\beta}}{B\left(\frac{1}{\alpha^{\beta}},n+1\right)}\right)^{m} (\lambda\beta)^{m}\left(\prod_{i=1}^{m} x_i\right)^{\beta-1} e^{-\lambda\sum_{i=1}^{m} x_i{}^{\beta}} \left(\prod_{i=1}^{m}\left(1-e^{-\lambda(\alpha x_i)^{\beta}}\right)\right)^{n}. \tag{27}$$

Therefore, the log-likelihood function is written as

$$l(\alpha,\beta,\lambda,\underline{\boldsymbol{x}}) = m\ln\frac{\alpha^{\beta}}{B\left(\frac{1}{\alpha^{\beta}},n+1\right)} + m\ln\lambda + m\ln\beta + (\beta-1)\sum_{i=1}^{m} lnx_i - \lambda\sum_{i=1}^{m} x_i{}^{\beta} + n\sum_{i=1}^{m}\ln\left(1-e^{-\lambda(\alpha x_i)^{\beta}}\right). \tag{28}$$

The associated gradients are found as follows

$$\frac{\partial\ell}{\partial\alpha} = m\frac{\partial\left(\ln\frac{\alpha^{\beta}}{B\left(\frac{1}{\alpha^{\beta}},n+1\right)}\right)}{\partial\alpha} + n\lambda\beta\alpha^{\beta-1}\sum_{i=1}^{m}\frac{x_i{}^{\beta}e^{-\lambda(\alpha x_i)^{\beta}}}{1-e^{-\lambda(\alpha x_i)^{\beta}}} = 0,$$

$$\frac{\partial\ell}{\partial\beta} = m\frac{\partial\left(\ln\frac{\alpha^{\beta}}{B\left(\frac{1}{\alpha^{\beta}},n+1\right)}\right)}{\partial\beta} + \frac{m}{\beta} + \sum_{i=1}^{m} lnx_i - \lambda\sum_{i=1}^{m}(lnx_i)x_i{}^{\beta}$$
$$+n\lambda\sum_{i=1}^{m}\frac{(\alpha x_i)^{\beta}e^{-\lambda(\alpha x_i)^{\beta}}\ln(\alpha x_i)}{1-e^{-\lambda(\alpha x_i)^{\beta}}} = 0,$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_i{}^{\beta} + n\alpha^{\beta}\sum_{i=1}^{m}\frac{x_i{}^{\beta}e^{-\lambda(\alpha x_i)^{\beta}}}{1-e^{-\lambda(\alpha x_i)^{\beta}}} = 0,$$

Due to the non-linearity of these equations the MLEs of parameters can be obtained numerically. In this paper we use the optim function from the statistical software R (R Development Core Team, 2011) to solve these equations. The default method for optim is a derivative-free optimization routine called the Nelder-Mead simplex algorithm (Nelder and Mead, 1965). This algorithm requires initial values. For some functions, particularly functions with many minimums or maximums, the initial values have a great impact on the converged point. Here we use the Method of Moment Estimation (MME) to specify initial values.

7.2. Fisher Information Matrix

To obtain the asymptotic variance and covariance of the maximum likelihood estimators, the local Fisher information matrix must be found. The second partial derivatives of (27) are given below:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = m \frac{\partial^2 \left(\ln \frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)} \right)}{\partial \alpha^2} + n\lambda\beta(\beta-1)\alpha^{\beta-2} \sum_{i=1}^{m} \frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}} + n\lambda\beta\alpha^{\beta-1} \sum_{i=1}^{m} \frac{\partial}{\partial \alpha} \left(\frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}}\right)$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = m \frac{\partial^2 \left(\ln \frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)} \right)}{\partial \beta^2} - \frac{m}{\beta^2} - \lambda \sum_{i=1}^{m} (ln x_i)^2 {x_i}^\beta + n\lambda \sum_{i=1}^{m} \frac{\partial}{\partial \beta} \frac{(\alpha x_i)^\beta e^{-\lambda(\alpha x_i)^\beta} \ln(\alpha x_i)}{1-e^{-\lambda(\alpha x_i)^\beta}}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{m}{\lambda^2} + n\alpha^\beta \sum_{i=1}^{m} \frac{\partial \ell}{\partial \lambda} \frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}},$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = m \frac{\partial}{\partial \alpha} \frac{\partial \left(\ln \frac{\alpha^\beta}{B\left(\frac{1}{\alpha^\beta}, n+1\right)} \right)}{\partial \beta} + n\lambda \sum_{i=1}^{m} \frac{\partial}{\partial \alpha} \frac{(\alpha x_i)^\beta e^{-\lambda(\alpha x_i)^\beta} \ln(\alpha x_i)}{1-e^{-\lambda(\alpha x_i)^\beta}}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = n\beta\alpha^{\beta-1} \sum_{i=1}^{m} \frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}} + n\alpha^\beta \sum_{i=1}^{m} \frac{\partial}{\partial \alpha} \left(\frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}}\right)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \lambda} = -\sum_{i=1}^{m} {x_i}^\beta \ln(x_i) + n\alpha^\beta ln\alpha \sum_{i=1}^{m} \frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}} + n\alpha^\beta \sum_{i=1}^{m} \frac{\partial}{\partial \beta} \left(\frac{{x_i}^\beta e^{-\lambda(\alpha x_i)^\beta}}{1-e^{-\lambda(\alpha x_i)^\beta}}\right)$$

So the Fisher information matrix is given by

$$I = \begin{bmatrix} -E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda}\right) \\ -E\left(\frac{\partial^2 \ell}{\partial \beta \partial \alpha}\right) & -E\left(\frac{\partial^2 \ell}{\partial \beta^2}\right) & -E\left(\frac{\partial^2 \ell}{\partial \beta \partial \lambda}\right) \\ -E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha}\right) & -E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \beta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) \end{bmatrix}$$

Therefore,

$$I^{-1} = \begin{bmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\lambda}, \hat{\alpha}) & \mathrm{cov}(\hat{\lambda}, \hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix}$$

It can be shown that the GWW family satisfies the regularity conditions. Hence, the MLE vector $(\hat{\alpha},\hat{\beta},\hat{\lambda})^T$ is consistent and $\sqrt{m}(\hat{\alpha}-a,\hat{\beta}-\beta,\hat{\lambda}-\lambda)^T$ is asymptotically normal with mean vector **0** and the variance- covariance matrix I^{-1};

$$\sqrt{m}(\hat{\alpha}-a,\hat{\beta}-\beta,\hat{\lambda}-\lambda)^T \xrightarrow{D} N_3(\mathbf{0},I^{-1}).$$

7.3 Simulation

Here, we assess the performance of the maximum likelihood estimates given by (28) (under the case λ is known) with respect to sample size m for the $GWW(\alpha,\beta,\lambda=1,n=3)$ distribution. The assessment of the performance is based on a simulation study by using the Monte Carlo as follows:

1. generate ten thousand samples of size m for the $GWW(\alpha,\beta,\lambda=1,n=3)$ distribution by using theorem 5.2;
2. compute the maximum likelihood estimates for the ten thousand samples, say $\hat{\alpha}_i$ and $\hat{\beta}_i$ for $i=1,2,\dots,10000$;
3. Compute $\widehat{I_i^{11}}$, the (1.1) element of $I^{-1}(\hat{\alpha}_i,\hat{\beta}_i)$, $\widehat{I_i^{22}}$ and the (2.2) element of $I^{-1}(\hat{\alpha}_i,\hat{\beta}_i)$ for the ten thousand samples;
4. compute the biases, mean squared errors, coverage probabilities and coverage lengths given by

$$bias_\alpha(m)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\alpha}_i-\alpha),$$

$$bias_\beta(m)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\beta}_i-\beta),$$

$$MSE_\alpha(m)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\alpha}_i-\alpha)^2,$$

$$MSE_\beta(m)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\beta}_i-\beta)^2,$$

$$CP_\alpha(m)=\frac{1}{10000}1\sum_{i=1}^{10000}I\left\{\hat{\alpha}_i-1.965.\sqrt{\widehat{I_i^{11}}}\le\alpha\le\hat{\alpha}_i+1.965.\sqrt{\widehat{I_i^{11}}}\right\},$$

$$CP_\beta(m)=\frac{1}{10000}\sum_{i=1}^{10000}I\left\{\hat{\beta}_i-1.965.\sqrt{\widehat{I_i^{22}}}\le\beta\le\hat{\beta}_i+1.965.\sqrt{\widehat{I_i^{22}}}\right\},$$

$$CL_\alpha(m)=\frac{2\,.\,1.965}{10000}\sum_{i=1}^{10000}\sqrt{\widehat{I_i^{11}}},$$

and

$$CL_{\beta}(m) = \frac{2\,.1.965}{10000} \sum_{i=1}^{10000} \sqrt{\widehat{I_i^{22}}}\,.$$

Where $I\{\,.\}$ denotes the indicator function.

We repeated these steps for $m = 10,11,\ldots,100$, $\alpha = 1$ and $\beta = 1$, so computing above quantities. Figs. 4–7 show how the biases, the mean squared errors, the coverage probabilities and the coverage lengths vary with respect to m. Figure 4 shows how the two biases vary with respect to m. The biases for each parameter decrease to zero as $m \to \infty$. Figure 5 shows how the two mean-squared errors vary with respect to m. The mean-squared errors for each parameter decrease to zero as $m \to \infty$. Figure 6 shows how the two coverage probabilities vary with respect to m. The red line corresponds to the nominal coverage probability of 0.95 . The coverage probabilities for α and β appear to have reached the nominal level at $m = 100$ Figure 7 shows how the two coverage lengths vary with respect to $m \to \infty$. The coverage lengths for each parameter decrease to zero as $m \to \infty$. The reported observations are for only one choice for (α, β) namely that $(\alpha, \beta) = (1, 1)$. But the results were similar for other choices.

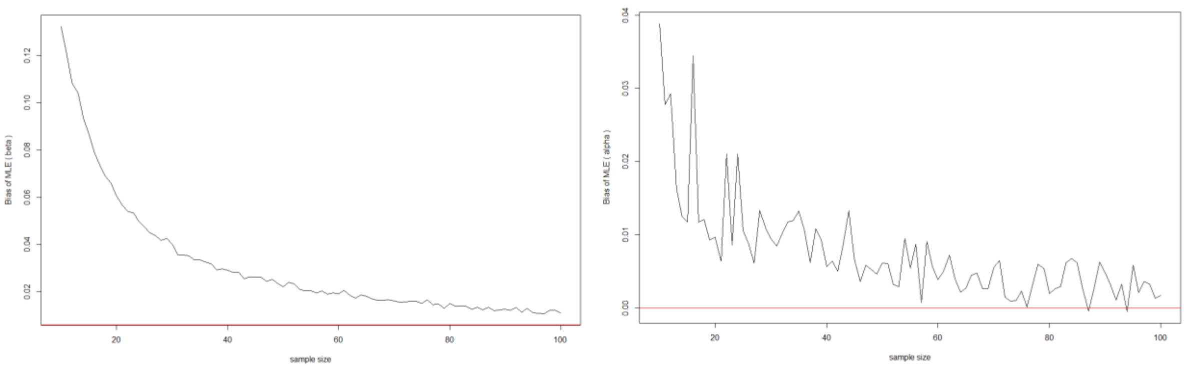


Fig. 4. Biases of the MLEs of (α, β) versus $m = 10,11,\ldots,100$.

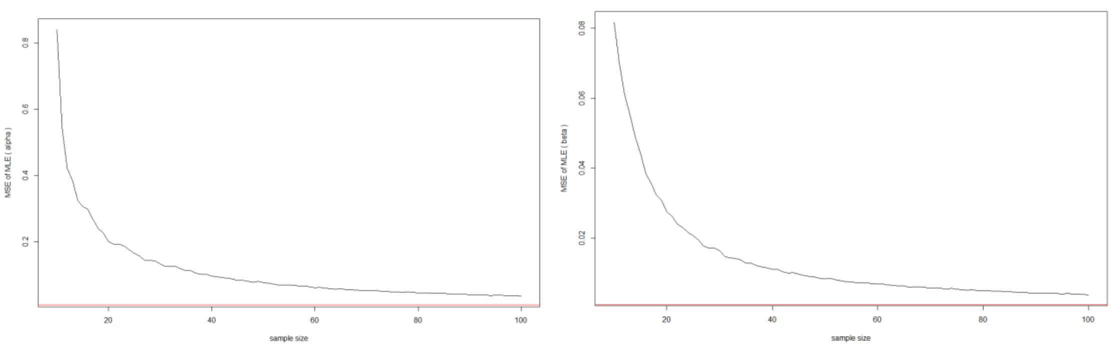


Fig. 5. Mean-squared errors of the MLEs of (α, β) versus $m = 10,11,\ldots,100$.

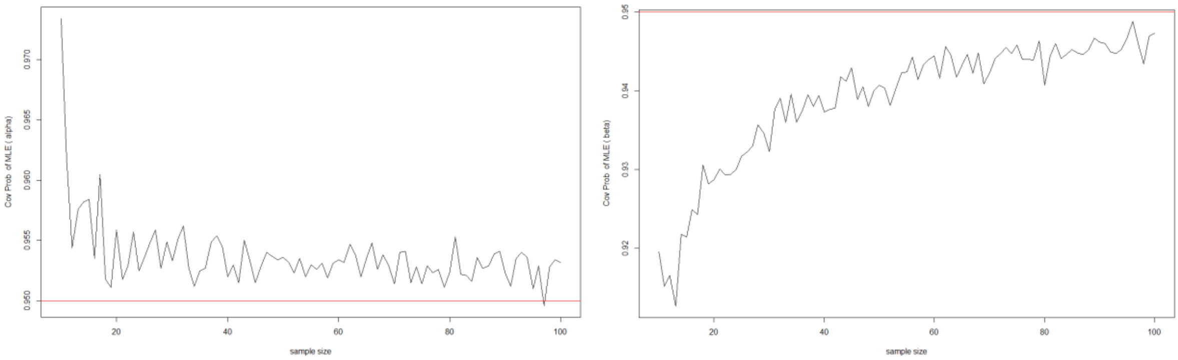


Fig. 6. Coverage probabilities of the MLEs of (α, β) versus $m = 10,11, \ldots ,100$.

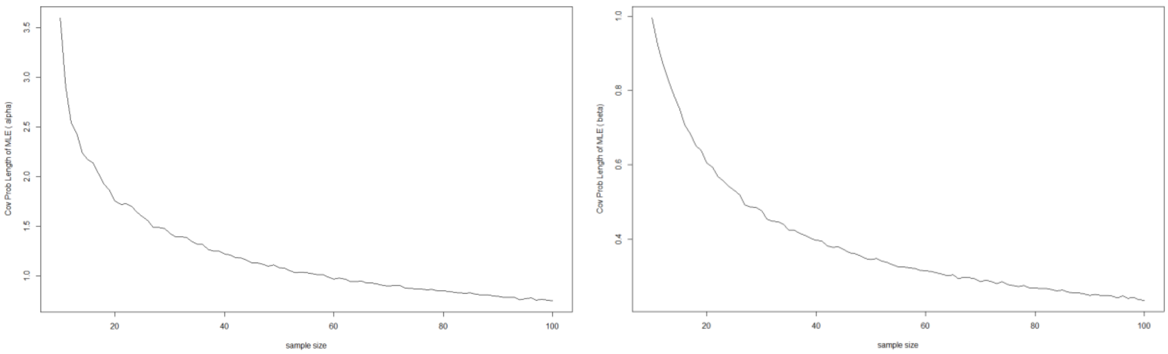


Fig. 7. Coverage lengths of the MLEs of (α, β) versus $m = 10,11, \ldots ,100$.

7.4. Data analysis and applications

In this section, we illustrate the usefulness of the GWW distribution. We fit this distribution to two data sets and compare the results with the beta weibull-Geometric (BWG), weibull-Geometric (WG) and weibull (W) with respective densities

$$f_{BWG}(x) = \frac{(1-p)^b \alpha \beta^\alpha x^{\alpha-1} e^{-b(\beta x)^\alpha} (1-e^{-(\beta x)^\alpha})^{a-1}}{B(a,b)(1-pe^{-(\beta x)^\alpha})^{a+b}}, \quad x \geq 0$$

$$f_{WG}(x) = (1-p)\alpha\beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha} \left(1-pe^{-(\beta x)^\alpha}\right)^{-2}, \quad x \geq 0$$

$$f_W(x) = \beta\lambda^\beta x^{\beta-1} e^{-(\lambda x)^\beta}, \quad x \geq 0$$

For more details, see Bidram et al. (2011). To investigate the advantage of proposed distribution and its sub-models, WE, TWE, GWE, weibull (W) and weighted weibull (WW) we consider two real data sets.

First data set (fatigue life data):

The first data set is given by Birnbaum and Saunders (1969) on the fatigue life of 6061-T6 aluminum coupons cut the direction of rolling and oscillated at 18 cycles per second. The data set consists of 102 observations with maximum stress per cycle 26,000 psi. The data are given below:

233 258 268 276 290 310 312 315 318 321 321 329 335 336 338 338 342 342 342 344 349 350 350 351 351 352 352 356 358 358 360 362 363 366 367 370 370 372 372 374 375 376 379 379 380 382 389 389 395 396 400 400 400 403 404 406 408 408 410 412 414 416 416 416 420 422 423 426 428 432 432 433 433 437 438 439 439 443 445 445 452 456 456 460 464 466 468 470 470 473 474 476 476 486 488 489 490 491 503 517 540 560.

Before analyzing this data set, we use the scaled-TTT plot to verifiy our model validity, see Aarset (1987). It allows to identify the shape of hazard function graphically. We provide the empirical scaled-TTT plot of above data set. Fig. 8. Shows the scaled-TTT plot is concave. It indicates that the hazard function is increasing; therefore it verifies our model validity.

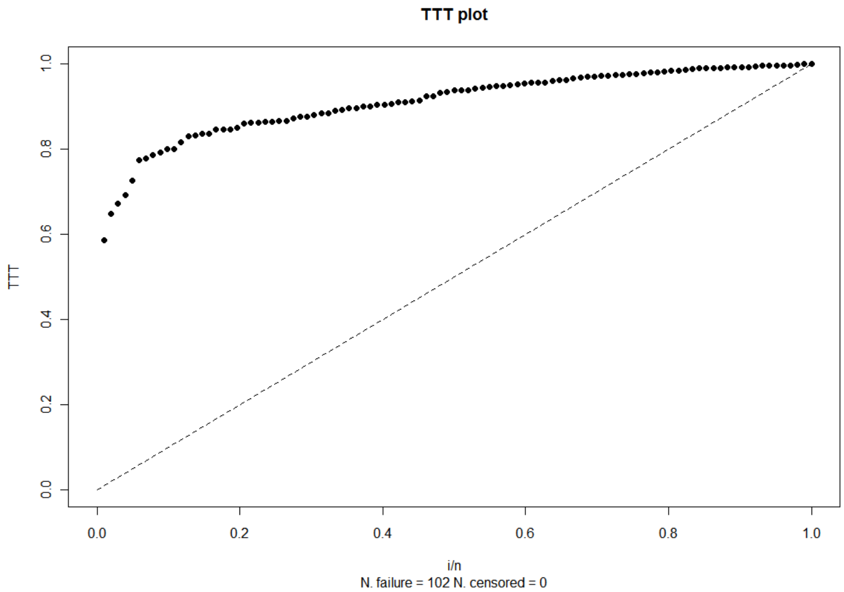


Fig.8. Scaled-TTT plot of the first data set.

Table 1 shows the MLEs of parameters, Kolmogorov–Smirnov (K–S) distance between the empirical distribution and the fitted model, its corresponding P-value , log-likelihood and Akaike information criterion (AIC) for the first data set. We fit the GWW distribution to the data and compare it with the BWG, WG and weibull densities. For more details see Bidram et al. (2011). We can see for this data set the model $GWW(\alpha,\beta,\lambda,n=10)$ provides the best fit among above models included in data analysis. The relative histogram fitted $GWW(\alpha,\beta,\lambda,n=10)$, BWG, WG and weibull (W) PDFs for fatigue life data are plotted in Fig 9(a). Fig 9(b) shows the empirical and fitted survival functions $GWW(\alpha,\beta,\lambda,n=10)$ for fatigue life data.

Table1. The MLEs of parameters for fatigue life data.

Model	MLEs of parameters	Log-likelihood	AIC	K-S test (P-value)
$\boldsymbol{GWW(\alpha,\beta,\lambda,n=10)}$	$\boldsymbol{(\hat{\alpha},\hat{\beta},\hat{\lambda})=(0.002,1.94,9.52e^{-05})}$	**− 565.96**	**1137.9**	**0.044 (0.98)**
$BWG\ (a,b,\alpha,\beta,p)$	$(\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{p})=(1.63,0.90,7.28,0.002,0.71)$	− 565.79	1141.6	0.051 (0.93)
$WG\ (\alpha,\beta,p)$	$(\hat{\alpha},\hat{\beta},\hat{p})=(9.66\ ,0.002,0.846)$	− 570.25	1146.5	0.140 (0.033)
$W\ (\beta,\lambda)$	$(\hat{\beta},\hat{\lambda})=(7.0075\ ,0.0023\ )$	− 567.80	1142.4	0.118 (0.103)

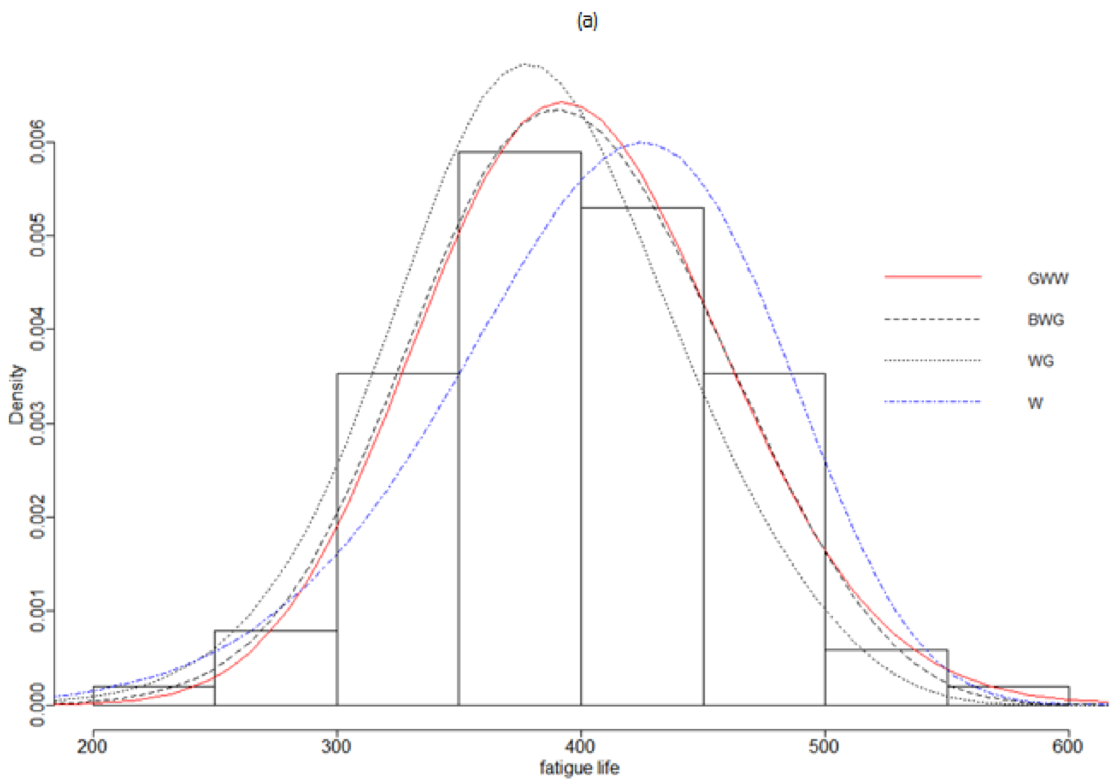


Figure 9(a). Fitted densities plots for the first data set

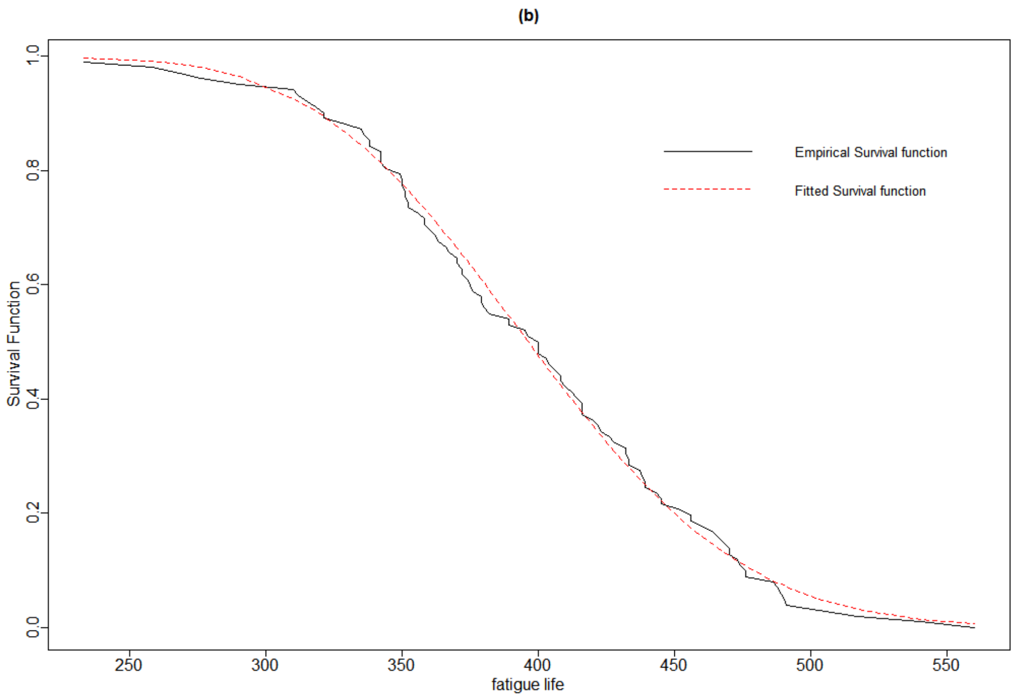


Fig 9(b). Empirical and fitted survival functions for the first data set: $GWW(\alpha, \beta, \lambda, n = 10)$.

Second data set (Strength data) :

The data represent the strength measured in GPA , Badar and Priest (1982), for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. For illustrative purposes, we are considering the single fibers data set of 10 mm in gauge lengths with sample size 63. The data are presented below:

1.901 2.132 2.203 2.228 2.257 2.350 2.361 2.396 2.397 2.445 2.454 2.474 2.518 2.522 2.525 2.532 2.575 2.614 2.616 2.618 2.624 2.659 2.675 2.738 2.740 2.856 2.917 2.928 2.937 2.937 2.977 2.996 3.030 3.125 3.139 3.145 3.220 3.223 3.235 3.243 3.264 3.272 3.294 3.332 3.346 3.377 3.408 3.435 3.493 3.501 3.537 3.554 3.562 3.628 3.852 3.871 3.886 3.971 4.024 4.027 4.225 4.395 5.020.

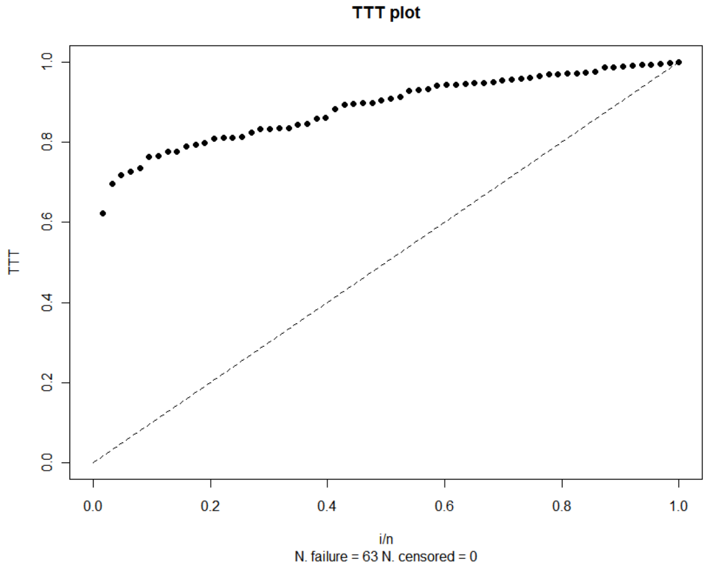


Fig.10. Scaled-TTT plot of the second data set .

Fig. 10. Shows the scaled-TTT plot for second data set is concave.

The results of the MLEs of parameters and (K–S) distance are reported in Table 2. Analysis of Table 2 shows that the model $GWW(\alpha,\beta,\lambda,n=18)$ provides the best fit among other models all those used here to fit data set. Fig 11(a) shows the relative histogram and the fitted $GWW(\alpha,\beta,\lambda,n=18)$, BWG, WG and weibull (W) PDFs for Strength data. The empirical (black line) and fitted survival (red line) functions $GWW(\alpha,\beta,\lambda,n=18)$ for Strength data are plotted in Fig 11(b).

Table2. The MLEs of parameters for Strength data set.

Model	MLEs of parameters	Log-likelihood	AIC	K-S test (P-value)
$\boldsymbol{GWW(\alpha,\beta,\lambda,n=18)}$	$(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\lambda}})=(\mathbf{1.88},\mathbf{3.28},\mathbf{0.031})$	$\mathbf{-55.782}$	$\mathbf{117.6}$	$\mathbf{0.066}$ $(\mathbf{0.95})$
$BWG\ (a,b,\alpha,\beta,p)$	$(\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{p})=(3.27,0.65,21.29,0.123,0.047)$	-55.7819	121.6	0.067 (0.94)
$WG\ (\alpha,\beta,p)$	$(\hat{\alpha},\hat{\beta},\hat{p})=(8.35\ ,0.215,0.973)$	-57.499	120.9	0.0833 (0.7093)
$W\ (\beta,\lambda)$	$(\hat{\beta},\hat{\lambda})=(5.049\ ,0.3016\ )$	-61.95	127.9	0.0876 (0.7192)

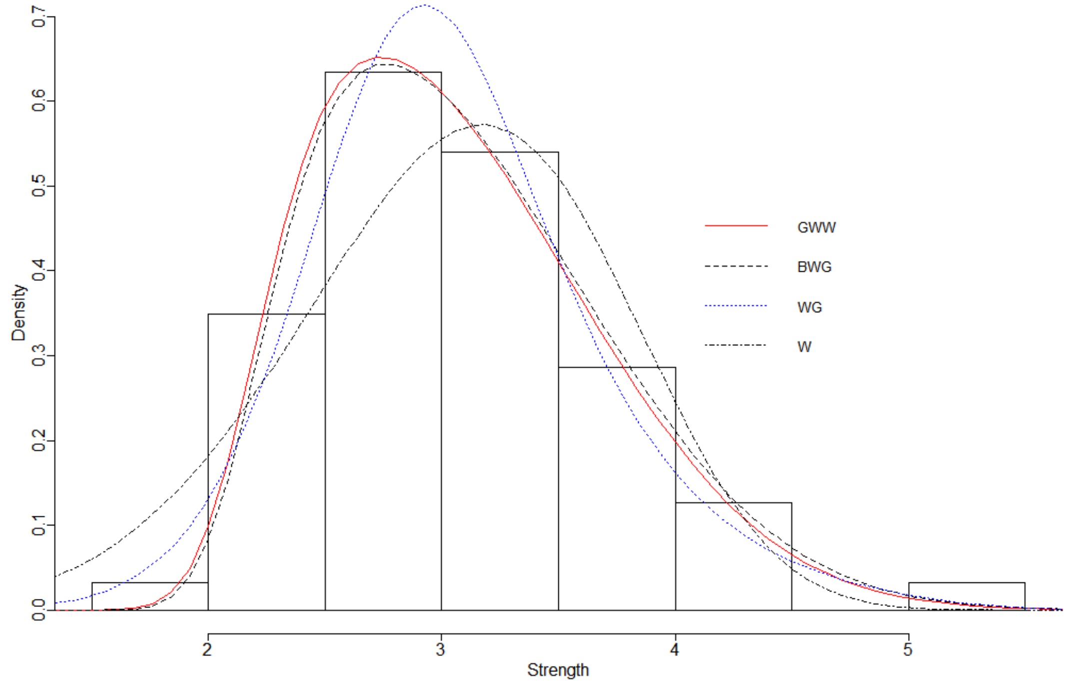


Fig 11(a). The fitted PDFs and the relative histogram for the Strength data.

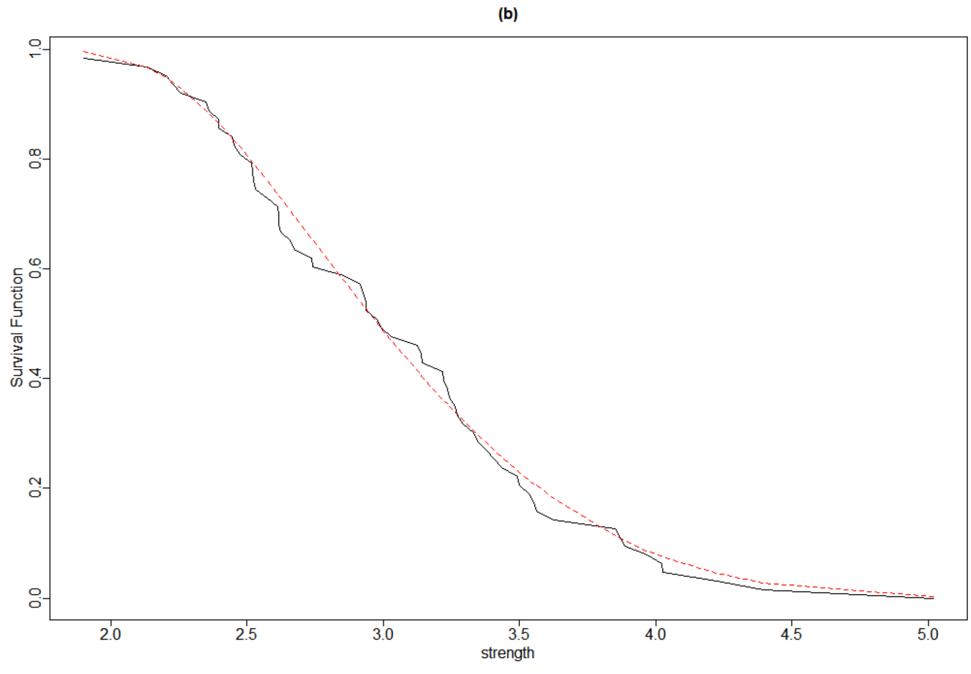


Fig 11(b). Empirical and fittrd survival functions for Strength data: $GWW(\alpha, \beta, \lambda, n = 18)$.

8. Conclusions

In this paper, we have proposed a generalized weigthed weibull distribution denoted by GWW. The proposed distribution generalizes the WE , TWE , Weibull (W) and GWE distributions and contains these distributions as its sub-models. The GWW model is constructed in a way that is similar to the way in which the GWE is constructed. It is investigated that the new model has increasing, decreasing and upside-down bathtub shaped hazard. It is expected that this generalization will be widely applicable in reliability theory, risk analysis and other disciplines. Two applications of the GWW distribution to real data sets are provided to illustrate that this distribution provides a better fit than its sub-models and some other recent extentions of the weibull distribution.

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