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Generalized Inequalities for Convex Functions

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Abstract. We investigate the fundamental inequalities for convex functions on the bounded closed interval of real numbers. Using the theory of positive linear functionals, we obtain the functional forms of inequalities as generalizations of the well-known inequalities. Our consideration includes the Jensen, Jensen-Mercer, Fejér and Hermite-Hadamard inequality.

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1. Introduction

Let us remind the initial notions of convexity which refers to convex sets and functions. Let $\mathbb{X}$ be a real linear space.
A set $C \subseteq \mathbb{X}$ is said to be convex if the inclusion

$$\alpha a + \beta b \in C \tag{1}$$

holds for all points $a, b \in C$ and all coefficients $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta = 1$. The sum $\alpha a + \beta b$ fulfilling the above requirements is called a convex combination.
A function $f : C \to \mathbb{R}$ is said to be convex if the inequality

$$f(\alpha a + \beta b) \leqslant \alpha f(a) + \beta f(b) \tag{2}$$

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holds for all convex combinations $\alpha a + \beta b$ of points $a, b \in C$.

Throughout the paper, we consider convex functions on the bounded closed interval $[a, b] \subset \mathbb{R}$, where $a < b$. Each point $x \in [a, b]$ can be represented by the unique binomial convex combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b, \tag{3}$$

where numbers $\alpha = (b-x)/(b-a)$ and $\beta = (x-a)/(b-a)$ are coefficients.

Assume that we have a convex function $f : [a, b] \to \mathbb{R}$. The secant line of f passes through the corresponding graph points of a and b, and its equation is

$$f_{ab}^{\mathrm{sec}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \tag{4}$$

Let $c \in (a, b)$ be an interior point. A support line of f passing through the corresponding graph point of c is specified by the slope coefficient $\lambda \in [f'(c-), f'(c+)]$, and its equation is

$$f_c^{\mathrm{sup}}(x) = \lambda(x-c) + f(c). \tag{5}$$

The support-secant line inequality

$$f_c^{\mathrm{sup}}(x) \leqslant f(x) \leqslant f_{ab}^{\mathrm{sec}}(x) \tag{6}$$

holds for every $x \in [a, b]$. The above inequality applies to each support line at c.

2. Positive Linear Functionals on the Space of Real Functions

Let X be a nonempty set, and let $\mathbb{F} = \mathbb{F}(X)$ be a subspace of the linear space of all real functions on the domain X. We assume that the space $\mathbb{F}$ contains the unit function u defined by $u(x) = 1$ for every $x \in X$. Such space contains every real constant λ within the meaning of $\lambda = \lambda u$. The space $\mathbb{F}$ also contains every composite function $f(g)$ of a function $g \in \mathbb{F}$, and an affine function $f : \mathbb{R} \to \mathbb{R}$. Namely, using the equation $f(x) = \lambda_1 x + \lambda_2$, where λ_1 and λ_2 are real constants, we get

$$f(g) = \lambda_1 g + \lambda_2 u \in \mathbb{F}. \tag{7}$$

Let $\mathbb{L} = \mathbb{L}(\mathbb{F}(X))$ be the space of all linear functionals on the space $\mathbb{F}$.

A functional $L \in \mathbb{L}$ is said to be unital (normalized) if $L(u) = 1$. Such functional has the property $L(\lambda u) = \lambda$ for every real constant λ. We have the following equality referring to unital functionals and affine functions.

Lemma 2.1. *Let $g \in \mathbb{F}$ be a function, and let $L \in \mathbb{L}$ be a unital functional. Then each affine function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equality*

$$f(L(g)) = L(f(g)). \tag{8}$$

Proof. Using the affine equation $f(x) = \lambda_1 x + \lambda_2$, and the unital property of L, we obtain

$$f(L(g)) = \lambda_1 L(g) + \lambda_2 = L(\lambda_1 g + \lambda_2 u) = L(f(g)),$$

proving the equality in formula (8). □

A functional $L \in \mathbb{L}$ is said to be positive (nonnegative) if the inequality $L(g) \geqslant 0$ holds for every nonnegative function $g \in \mathbb{F}$. Then it follows that

$$L(g_1) \leqslant L(g_2), \tag{9}$$

for every pair of functions $g_1, g_2 \in \mathbb{F}$ satisfying $g_1(x) \leqslant g_2(x)$ for every $x \in X$.

Lemma 2.2. *Let $g \in \mathbb{F}$ be a function with the image in the closed interval $I \subseteq \mathbb{R}$.*
Then each positive unital functional $L \in \mathbb{L}$ satisfies the inclusion

$$L(g) \in I. \tag{10}$$

Proof. If $I = [a, b]$, then acting with the positive and unital functional L to the image assumption

$$au \leqslant g \leqslant bu, \tag{11}$$

we get

$$a \leqslant L(g) \leqslant b. \tag{12}$$

If $I = (-\infty, b]$, we leave out the first terms of formulae (11) and (12). If $I = [a, +\infty)$, we leave out the last terms. If $I = \mathbb{R}$, then it must be $L(g) \in I$. □

The functional form of Jensen's inequality is as follows.

Lemma 2.3. *Let $g \in \mathbb{F}$ be a function with the image in the closed interval $I \subseteq \mathbb{R}$, and let $L \in \mathbb{L}$ be a positive unital functional.*
Then each continuous convex function $f : I \to \mathbb{R}$ such that $f(g) \in \mathbb{F}$ satisfies the inequality

$$f(L(g)) \leqslant L(f(g)). \tag{13}$$

In 1931, Jessen (see [7] and [8]) stated the functional form of Jensen's inequality. In 1988, Raşa (see [12]) pointed out that I must be closed, otherwise it could happen that $L(g) \notin I$, and that f must be continuous, otherwise it could happen that the inequality in formula (13) does not apply.
Some generalizations of the functional form of Jensen's inequality can be found in [10]. The book in [1] can be recommended as a concise book on functional analysis indicating the importance of positive functionals.

3. Main Results

Taking $I = [a, b]$, we can extend the inequality in formula (13) to the right side by using the secant line.

Lemma 3.1. *Let $g \in \mathbb{F}$ be a function with the image in $[a, b]$, and let $L \in \mathbb{L}$ be a positive unital functional.*
Then each continuous convex function $f : [a, b] \to \mathbb{R}$ such that $f(g) \in \mathbb{F}$ satisfies the double inequality

$$f(L(g)) \leqslant L(f(g)) \leqslant f_{ab}^{sec}(L(g)). \tag{14}$$

Proof. The point $l = L(g)$ is in $[a, b]$ by Lemma 2.2. We realize the proof in two steps depending on the position of l.
If $l \in (a, b)$, we take a support line f_l^{sup} of f at l. Applying the positive functional L to the support-secant inequality in formula (6) with $g(x)$ instead of x, we get

$$L\big(f_l^{\text{sup}}(g)\big) \leqslant L(f(g)) \leqslant L\big(f_{ab}^{\text{sec}}(g)\big).$$

By utilizing the affinity of functions f_l^{sup} and f_{ab}^{sec} according to formula (8), the above inequality takes the form

$$f_l^{\text{sup}}(L(g)) \leqslant L(f(g)) \leqslant f_{ab}^{\text{sec}}(L(g)), \tag{15}$$

where the first term

$$f_l^{\text{sup}}(L(g)) = f(L(g)).$$

If $l \in \{a, b\}$, we explore the continuity of f using a support line at a point of the open interval (a, b) that is close enough to l. Given $\varepsilon > 0$, we can find $c \in (a, b)$ so that

$$f(l) - \varepsilon < f_c^{\text{sup}}(l).$$

Combining the above inequality, and the inequality in formula (15) with the support line at c, we obtain

$$f(l) - \varepsilon < f_c^{\text{sup}}(l) \leqslant L(f(g)) \leqslant f_{ab}^{\text{sec}}(l) = f(l).$$

Letting ε to zero, we attain the equality $L(f(g)) = f(l)$. In this case, the trivial inequality $f(l) \leqslant f(l) \leqslant f(l)$ represents formula (14). $\square$

The famous integral form of Jensen's inequality (see [6]) can be generalized and extended by exploring Lemma 3.1.

Theorem 3.2. *Let X be a measurable set respecting a positive measure μ so that $\mu(X)$ is the positive number. Let $g : X \to \mathbb{R}$ be a μ-integrable function with the image in $[a, b]$, and let $h : X \to \mathbb{R}$ be a positive μ-integrable function.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\Big(\frac{\int_X gh\,d\mu}{\int_X h\,d\mu}\Big) \leqslant \frac{\int_X f(g)h\,d\mu}{\int_X h\,d\mu} \leqslant \frac{\int_X (b-g)h\,d\mu}{\int_X (b-a)h\,d\mu} f(a) + \frac{\int_X (g-a)h\,d\mu}{\int_X (b-a)h\,d\mu} f(b). \tag{16}$$

Proof. Let $\mathbb{F}$ be the space of all μ-integrable functions over the domain X. The convex function f is bounded on $[a, b]$, and may be discontinued only at endpoints a or b. Therefore, the composition $f(g)$ is bounded and measurable, and as such is μ-integrable over X.
We define the integrating linear functional

$$L(q) = L(q; h) = \frac{\int_X qh\,d\mu}{\int_X h\,d\mu}, \tag{17}$$

for every function $q \in \mathbb{F}$. The functional L is positive and unital. Applying the functional L to the given functions g and f, we have the following. The first term of formula (16) is equal to $f(L(g))$, the second term is equal to $L(f(g))$, and the third term is equal to $f_{ab}^{\sec}(L(g))$. As regards the third term, using the equation of the secant line, we obtain

$$\begin{aligned} f_{ab}^{\sec}(L(g)) &= \frac{b - L(g)}{b - a} f(a) + \frac{L(g) - a}{b - a} f(b) \\ &= \frac{\int_X (b-g)h\,d\mu}{\int_X (b-a)h\,d\mu} f(a) + \frac{\int_X (g-a)h\,d\mu}{\int_X (b-a)h\,d\mu} f(b). \end{aligned}$$

If we suppose that the function f is continuous, then formula (16) fits into the frame of formula (14).
Let us verify that the inequality in formula (16) applies to a convex function which is not continuous at endpoints. We observe the position of the point

$$l = L(g) = \frac{\int_X gh\,d\mu}{\int_X h\,d\mu}. \tag{18}$$

If $l \in (a, b)$, then we may utilize the continuous extension $\widetilde{f}$ of $f/(a, b)$ to $[a, b]$ in formula (16). Applying the left-hand side of the inequality in formula (14) to the continuous function $\widetilde{f}$, and using the inequality $\widetilde{f} \leqslant f \leqslant f_{ab}^{\rm sec}$, we obtain

$$\begin{aligned} f(L(g)) &= \widetilde{f}(L(g)) \leqslant L(\widetilde{f}(g)) \\ &\leqslant L(f(g)) \leqslant L(f_{ab}^{\rm sec}(g)) = f_{ab}^{\rm sec}(L(g)). \end{aligned}$$

Thus, formula (16) applies to f in this case.
If $l \in \{a,b\}$, then either $g(x) - l \geqslant 0$ or $g(x) - l \leqslant 0$ for every $x \in X$. By rearranging formula (18) to the integral equation

$$\int_X (g-l)h\, d\mu = 0,$$

it follows that $g(x) = l$ for almost every $x \in X$, and therefore $f(g(x)) = f(l)$ for almost every $x \in X$. The trivial inequality $f(l) \leqslant f(l) \leqslant f(l)$ represents formula (16) in this case.
Respecting all considerations, we may conclude that the inequality in formula (16) applies to any convex function f. □

Using $h(x) = 1$ in formula (16), we get the extended integral form of Jensen's inequality,

$$f\Big(\frac{\int_X g\, d\mu}{\mu(X)}\Big) \leqslant \frac{\int_X f(g)\, d\mu}{\mu(X)} \leqslant \frac{\int_X (b-g)\, d\mu}{(b-a)\mu(X)} f(a) + \frac{\int_X (g-a)h\, d\mu}{(b-a)\mu(X)} f(b). \tag{19}$$

We now return to the inequality in formula (14). Relying on the fact that the point $a+b-x$ belongs to the interval $[a,b]$ if $x \in [a,b]$, we have the following version of Lemma 3.1.

Lemma 3.3. *Let $g \in \mathbb{F}$ be a function with the image in $[a,b]$, and let $L \in \mathbb{L}$ be a positive unital functional.*

Then each continuous convex function $f : [a,b] \to \mathbb{R}$ such that $f(g) \in \mathbb{F}$ satisfies the double inequality

$$f\big(a+b-L(g)\big) \leqslant f_{\{a,b\}}^{sec}\big(a+b-L(g)\big) \leqslant f(a) + f(b) - L(f(g)). \tag{20}$$

Proof. The point $a+b-g(x)$ belongs to $[a,b]$ for every $x \in X$, and so the point $a+b-L(g)$ also belongs to $[a,b]$ by Lemma 2.2.

Since $f \leqslant f_{\{a,b\}}^{\rm sec}$, the left-hand side of the inequality in formula (20) is valid. Applying the affinity of $f_{\{a,b\}}^{\rm sec}$, and using the right-hand side of the inequality in formula (14), we get

$$f_{\{a,b\}}^{\rm sec}\big(a+b-L(g)\big) = f(a) + f(b) - f_{\{a,b\}}^{\rm sec}(L(g)) \leqslant f(a) + f(b) - L(f(g))$$

proving the right-hand side of the inequality in formula (20). □

The geometric presentation of the inequalities in formulae (14) and (20) can be seen in Figure 1. The black dots above the point l represent the terms of the inequality in formula (14), and black dots above the point $a+b-l$ represent the terms of the inequality in formula (20). The shaded triangles are congruent.

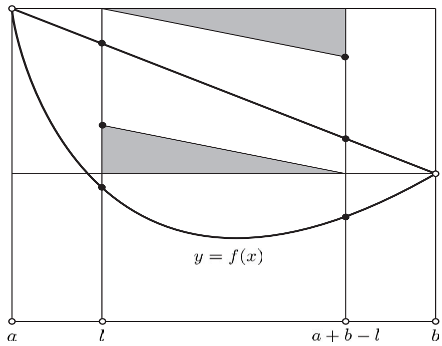

Figure 1. Geometric image of inequalities in formulae (14) and (20)

Using Lemma 3.3, we can generalize and refine the Jensen-Mercer inequality (see [5] and [9]). The point evaluations $g(x_i)$ and $h(x_i)$ will be shortened by g_i and h_i, respectively.

Corollary 3.4. *Let $g : X \to \mathbb{R}$ be a function with the image in $[a,b]$, and let $h : X \to \mathbb{R}$ be a positive function. Let $x_1, \ldots, x_n \in X$ be points.*
Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\Big(a+b-\frac{\sum_{i=1}^n g_i h_i}{\sum_{i=1}^n h_i}\Big) &\leqslant \frac{\sum_{i=1}^n (g_i-a)h_i}{(b-a)\sum_{i=1}^n h_i} f(a) + \frac{\sum_{i=1}^n (b-g_i)h_i}{(b-a)\sum_{i=1}^n h_i} f(b) \\ &\leqslant f(a)+f(b)-\frac{\sum_{i=1}^n f(g_i)h_i}{\sum_{i=1}^n h_i}. \end{aligned} \tag{21}$$

Proof. We use $\mathbb{F}$ as the space of all real functions on the domain X, and take the summarizing linear functional defined by

$$L(q) = L(q;h) = \frac{\sum_{i=1}^n q_i h_i}{\sum_{i=1}^n h_i}, \tag{22}$$

for every $q \in \mathbb{F}$. To verify the inequality in formula (21), we can reuse the proof of Theorem 3.2 relying on Lemma 3.3. □

If $X = [a,b]$, then using points $x_i = g_i$ and coefficients $\lambda_i = h_i / \sum_{i=1}^n h_i$ in formula (21), we obtain the refinement of the Jensen-Mercer inequality,

$$\begin{aligned} f\Big(a+b-\sum_{i=1}^{n}\lambda_i x_i\Big) &\leqslant \frac{\sum_{i=1}^{n}\lambda_i x_i - a}{b-a}f(a)+\frac{b-\sum_{i=1}^{n}\lambda_i x_i}{b-a}f(b) \\ &\leqslant f(a)+f(b)-\sum_{i=1}^{n}\lambda_i f(x_i). \end{aligned} \tag{23}$$

Namely, the Jensen-Mercer inequality was originally made up of the first and last term of the above inequality.

4. Inequalities Depending on Integral Means

Let X be a measurable set of some measure space respecting a positive measure μ so that $\mu(X)$ is the positive number. Let $g : X \to \mathbb{R}$ be a μ-integrable function, and let $h : X \to \mathbb{R}$ be a positive μ-integrable function. The μ-integral arithmetic mean of g respecting h can be defined by the number

$$\frac{\int_X gh\,d\mu}{\int_X h\,d\mu}. \tag{24}$$

Taking $h = 1$, we get the μ-integral arithmetic mean of g as $\int_X g\,d\mu/\mu(X)$. Using the measure ν on X defined by $\nu(S) = \int_S h\,d\mu$ for every μ-measurable set $S \subseteq X$, we have that

$$\frac{\int_X g\,d\nu}{\nu(X)} = \frac{\int_X gh\,d\mu}{\int_X h\,d\mu}. \tag{25}$$

If X is the set of real numbers, and g is the identity function on X, then the number in formula (24) represents the μ-barycenter of X respecting h. Taking $h = 1$, we get the μ-barycenter of the set X as $\int_X x\,d\mu/\mu(X)$.

The inequality in formula (16) can be directed to the Fejér (see [2]) and Hermite-Hadamard (see [4] and [3]) inequality. In this regard, we have the following corollary of Theorem 3.2.

Corollary 4.1. *Let X be a measurable set respecting a positive measure μ so that $\mu(X)$ is the positive number. Let $g : X \to \mathbb{R}$ be a μ-integrable function with the image in $[a, b]$, and let $h : X \to \mathbb{R}$ be a positive μ-integrable function such that*

$$\frac{\int_X gh\,d\mu}{\int_X h\,d\mu} = \frac{\int_X g\,d\mu}{\mu(X)}. \tag{26}$$

Then each convex function $f:[a,b]\rightarrow\mathbb{R}$ satisfies the double inequality

$$f\Big(\frac{\int_X g\,d\mu}{\mu(X)}\Big)\leqslant\frac{\int_X f(g)h\,d\mu}{\int_X h\,d\mu}\leqslant\frac{\int_X (b-g)\,d\mu}{(b-a)\mu(X)}f(a)+\frac{\int_X (g-a)h\,d\mu}{(b-a)\mu(X)}f(b). \tag{27}$$

Using the Lebesgue measure on the interval $X=[a,b]$, and taking the identity function $g(x)=x$, the condition in formula (26) yields

$$\frac{\int_a^b xh\,dx}{\int_a^b h\,dx}=\frac{\int_a^b x\,dx}{b-a}=\frac{a+b}{2}, \tag{28}$$

and so the inequality in formula (27) turns into the Fejér inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{\int_a^b fh\,dx}{\int_a^b h\,dx}\leqslant\frac{f(a)+f(b)}{2}. \tag{29}$$

Fejér was originally used a positive integrable function $h(x)$ satisfying the equation $h(x)=h(a+b-x)$ that represents the symmetry with the center at the midpoint $(a+b)/2$. Namely, as a consequence of this symmetry we have

$$\frac{\int_a^b xh\,dx}{\int_a^b h\,dx}=\frac{\int_a^b\left(x-\frac{a+b}{2}\right)h\,dx}{\int_a^b h\,dx}+\frac{\int_a^b\frac{a+b}{2}h\,dx}{\int_a^b h\,dx}=\frac{a+b}{2},$$

because

$$\int_a^b\left(x-\frac{a+b}{2}\right)h\,dx=0.$$

Putting the unit function $h=1$ in formula (29), we get the Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big)\leqslant\frac{\int_a^b f\,dx}{b-a}\leqslant\frac{f(a)+f(b)}{2}. \tag{30}$$

The above discussion indicates that the Fejér inequality depends on the barycenter of the interval $[a,b]$ respecting the function h, and that the Hermite-Hadamard inequality depends on the barycenter of the interval $[a,b]$.
The refinements of the Hermite-Hadamard inequality in formula (30), as well as the refinements of some standard means were obtained in [11].
In the light of applications of Corollary 4.1, the section will be completed by the following symmetrical form of Theorem 3.2.

Corollary 4.2. Let X be a measurable set respecting a positive measure μ so that $\mu(X)$ is the positive number. Let $g:X\rightarrow\mathbb{R}$ be a μ-integrable function with the image in $[a,b]$, let $h : X \to \mathbb{R}$ be a positive μ-integrable function, and let $\alpha a + \beta b$ be the convex combination such that

$$\frac{\int_X gh\,d\mu}{\int_X h\,d\mu} = \alpha a + \beta b. \tag{31}$$

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leqslant \frac{\int_X f(g)h\,d\mu}{\int_X h\,d\mu} \leqslant \alpha f(a) + \beta f(b). \tag{32}$$

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