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Two-Parameter σ-C^*-Dynamical Systems and Application

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Abstract. Let $\mathcal{A}$ be a C^*-algebra and σ be a bounded linear $*$-endomorphism on $\mathcal{A}$. Introducing the notions of $*$-σ-derivations and two-parameter σ-C^*-dynamical sytems, we correspond to each so-called two-parameter σ-C^*-dynamical system a pair of σ-derivations, named as its infinitesimal generator. Using the computation formula for σ-derivations, we deal with the converse under mild conditions. Finally, as an application, we characterize each so-called two parameter σ-C^*-dynamical system on the concrete C^*-algebra $\mathcal{A} := B(\mathcal{H}) \times B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and σ is the linear $*$-endomorphism $\sigma(S, T) = (0, T)$.

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1. Introduction

Let $\mathcal{A}$ be a Banach space and $\sigma : \mathcal{A} \to \mathcal{A}$ be a bounded linear operator. A σ-one parameter group of bounded linear operators on $\mathcal{A}$ is a group homomorphism $t \mapsto \varphi_t$ from the additive group $\mathbb{R}$ of real numbers into the set $\mathbf{B}(\mathcal{A})$ of all bounded linear operators on $\mathcal{A}$ satisfying $\varphi_0 = \sigma$. The σ-one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called uniformly (resp. strongly) continuous if $\lim\limits_{t\to 0} \| \varphi_t - \sigma \| = 0$ (resp. $\lim\limits_{t\to 0} \varphi_t(a) = \sigma(a)$, for each $a \in \mathcal{A}$).

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The infinitesimal generator δ of the σ-one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is a mapping $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(a) = \lim\limits_{t\to 0} \dfrac{\varphi_t(a) - \sigma(a)}{t}$ where $D(\delta) = \{a \in \mathcal{A} : \lim\limits_{t\to 0} \dfrac{\varphi_t(a) - \sigma(a)}{t} \quad exists\}$.

If $\{\varphi_t\}_{t\in\mathbb{R}}$ is a σ-one parameter group with the generator δ, then one can easily see that

(i) $\sigma^2 = \sigma$ and for each $t \in \mathbb{R}$, $\sigma\varphi_t = \varphi_t\sigma = \varphi_t$.

(ii) for each $t \in \mathbb{R}$, φ_t is σ-bijective in the sense that $\varphi_t(\mathcal{A}) = \sigma(\mathcal{A})$ and $ker(\varphi_t) = ker(\sigma)$.

(iii) for each $a \in D(\delta)$, $\sigma\delta(a) = \delta\sigma(a) = \delta(a)$.

(iv) $\sigma(\mathcal{A})$ is a closed subspace of $\mathcal{A}$.

This notion was introduced by Janfada in 2008. We refer the reader to [7] for more details.

The classical C^*-dynamical systems are expressed by means of uniformly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator of C^*-dynamical systems are $*$-derivations which play essential role in the operator algebras.

Due to the Gelgand-Naimark-Segal representation, each non-commmutative C^*-algebra can be regarded as a C^*-subalgebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. If A is a self adjoint element in the C^*-algebra $B(\mathcal{H})$, then by Stone's theorem, ([14], Theorem 1.10.8) iA is the infinitesimal generator of a uniformly continuous group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries in $B(\mathcal{H})$, such that $u_t = e^{itA}$, and further $D(T) = i[A, T]$, as an inner $*$-derivation, is the infinitesimal generator of the uniformly continuous group of inner $*$-automorphisms $\{u_t a u_t^*\}_{t\in\mathbb{R}}$. It is now a pleasant surprise that each uniformly continuous group of $*$-automorphisms on $B(\mathcal{H})$ is of this form, i.e., it is implemented by a unitary group on $\mathcal{H}$, ([4], Theorem. 1.3.16)

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. For instance, it can be pointed to "σ-derivations" as follows.

Let $\mathcal{A}$ be a $*$-Banach algebra and σ be a $*$-linear operator on $\mathcal{A}$. It is recalled that a $*-$linear map δ from the $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathcal{A}$ is called a σ-derivation if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$, for all $a, b \in D(\delta)$. For instance, let σ be a linear $*$-endomorphism and h be an arbitrary self-adjoint element of $\mathcal{A}$. Then, the mapping $\delta_h^\sigma : \mathcal{A} \to \mathcal{A}$ defined by $\delta_h^\sigma(a) = i[h, \sigma(a)]$, where $[h, \sigma(a)]$ is the commutator $h\sigma(a) - \sigma(a)h$, is a σ-derivation which is called inner, (see [5, 9, 10, 11, 12] and references therein).

In order to construct an extension of a C^*-dynamical system associated to σ-derivation, as its infinitesimal generator, note that each $*$-endomorphism on a C^*-algebra is norm decreasing. This specific property, provides the possibility that σ is regarded as a linear $*$-endomorphism and the desired extension is based on a class of σ-one parameter groups. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a uniformly continuous σ-one parameter group of linear $*$-endomorphisms on the C^*-algebra $\mathcal{A}$. An immediate consequence of the σ-bijective feature of $\{\varphi_t\}_{t\in\mathbb{R}}$ is that by substituting $\sigma = I$, we obtain a classical C^*-dynamical system. In 2013, the author demonstrated the mentioned extension of C^*-dynamical systems and called it a σ-C^*-dynamics which had a σ-derivation as its infinitesimal generator.

Assume that $\{\varphi_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamics on $\mathcal{A}$ with the infinitesimal generator δ. Then, the one parameter family $\{\psi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\sigma(\mathcal{A})$ defined by $\psi_t(\sigma(a)) = \varphi_t(a)$ is a C^*-dynamics and the mapping $\widetilde{\delta} : \sigma\big(D(\delta)\big) \subseteq \sigma(\mathcal{A}) \to \sigma(\mathcal{A})$ defined by $\widetilde{\delta}(\sigma(a)) = \delta(a)$ is its generator,(see [12]).

Let σ be a $*$-linear endomorphism on the C^*-algebra $\mathcal{A}$. By a σ-inner endomorphism, we mean a linear endomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi(a) = u\sigma(a)u^*$, for every $a \in \mathcal{A}$ and some unitary element $u \in \mathcal{A}$. In order to construct a σ-inner endomorphism, let c be a self-adjoint element of the C^*-algebra $\mathcal{A}$. Then, the mapping $\varphi : \mathcal{A} \to \mathcal{A}$ given by $\varphi(a) = e^{ic}\sigma(a)e^{-ic}$ is a $*$-σ-inner endomorphism.

In this paper, two parameter σ-C^*-dynamical systems are studied. In particular, we correspond to each so-called two parameter σ-C^*-dynamical systems a pair of σ-derivations, named as its infinitesimal generator. Also, using the computation formula for σ-derivations, we deal with the converse under mild conditions. More precisely, suppose that σ is an idempotent linear $*$-endomorphism and δ_j is a bounded $*$-σ-derivation on $\mathcal{A}$ in which $\delta_j\sigma = \sigma\delta_j = \delta_j$ $(j = 1, 2)$. We prove that if $\delta_1\delta_2 = \delta_2\delta_1$, then (δ_1, δ_2) induces a two parameter σ-C^*-dynamical system on $\mathcal{A}$. Finally, as an application, we characterize each so-called two parameter σ-C^*-dynamical system on the concrete C^*-algebra $\mathcal{A} := B(\mathcal{H}) \times B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and σ is the linear $*$-endomorphism $\sigma(S, T) = (0, T)$ on $\mathcal{A}$.

The reader is referred to [1, 3] and [13] for more details on Banach(resp. C^*-) algebras and to [2, 15] for more information on dynamical systems.

2. Main Results

Definition 2.1. *Let $\mathcal{A}$ be a Banach space and $\sigma : \mathcal{A} \to \mathcal{A}$ be a bounded linear operator. By a σ-two parameter group of bounded linear operators on $\mathcal{A}$, we mean a mapping $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbf{B}(\mathcal{A})$ which fulfills $\varphi_{0,0} = \sigma$ and $\varphi_{s+s',t+t'} = \varphi_{s,t}\varphi_{s',t'}$, for each $s, s', t, t' \in \mathbb{R}$.*

As in σ-one parameter case, the σ-two parameter group $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is called uniformly (resp. strongly) continuous if $\lim_{(s,t)\to(0,0)} \| \varphi_{s,t} - \sigma \| = 0$ (resp. $\lim_{(s,t)\to(0,0)} \varphi_{s,t}(a) = \sigma(a)$, for each $a \in \mathcal{A}$).

To any σ-two parameter group $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$, we associate two σ-one parameter groups $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ defined by $u_s := \varphi_{s,0}$ and $v_t := \varphi_{0,t}$. One can see that the σ-two parameter group $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is uniformly (rep. strongly) continuous if and only if so are $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$. The σ-one parameter group property implies that
$u_s v_t = \alpha_{s,0}\varphi_{0,t} = \varphi_{s,t} = \varphi_{0+s,t+0} = \varphi_{0,t}\varphi_{s,0} = v_t u_s.$

The infinitesimal generators of $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are denoted by δ_1 and δ_2, respectively. We denote the pair (δ_1, δ_2) as the infinitesimal generator of $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$.

In the case that $\sigma = I$, the uniformly (resp. strongly) continuous σ-two parameter group $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is in fact a uniformly (resp. strongly) continuous two parameter group in the usual sense. Some applied results *about two parameter groups can be observed in [6].*

Example 2.2. Let M be a closed subspace of Hilbert space H, $M^\perp$ be the set

$$\{x \in H : < x, m >= 0 \ for \ every \ m \in M\},$$

and let $\{\psi_{s,t}\}_{s,t\in\mathbb{R}}$ be a continuous two parameter group on H. If σ is the first projection operator on M, then for $x = y + z \in M \oplus M^\perp = H$, the two parameter family $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ defined by $\varphi_{s,t}(x) = \psi_{s,t}(y)$ forms a σ-two parameter group on H with the same continuity of $\{\psi_{s,t}\}_{s,t\in\mathbb{R}}$.

The following lemma provides the sufficient and necessary condition under which the product of two σ-one parameter groups be a σ-two parameter group.

Lemma 2.3. *Let $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ be two uniformly continuous σ-one parameter groups of bounded linear operators on $\mathcal{A}$ with the generators δ_1 and δ_2, respectively. Then, the two parameter family $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ defined by $\varphi_{s,t} := u_s v_t$ forms a σ-two parameter group if and only if $\delta_1\delta_2 = \delta_2\delta_1$.*

Proof. Consider the associated uniformly continuous one parameter group $\{\widetilde{u}_s\}_{s\in\mathbb{R}}$ (resp. $\{\widetilde{v}_t\}_{t\in\mathbb{R}}$) on $\sigma(\mathcal{A})$ defined by $\widetilde{u}_s(\sigma(a)) := u_s(a)$ (resp. $\widetilde{v}_t(\sigma(a)) := v_t(a)$) with the generator $\widetilde{\delta}_1$ (resp. $\widetilde{\delta}_2$), fulfilling $\widetilde{\delta}_j(\sigma(a)) := \delta_j(a)$, $j = 1, 2$. Assume that $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter family on $\sigma(\mathcal{A})$ which is defined by $\phi_{s,t} = \widetilde{u}_s\widetilde{v}_t$. Applying the property (i) for σ-one parameter groups which stated in the first part of introduction, one can obtain that $\sigma(\widetilde{v}_t(\sigma(a)) = \widetilde{v}_t(\sigma(a))$. This fact implies that $\varphi_{s,t}(a) = \phi_{s,t}(\sigma(a))$.

If $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a σ-two parameter group, then $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter group on $\sigma(A)$ with the generator $(\widetilde{\delta}_1, \widetilde{\delta}_2)$. It follows from the part (b) of Theorem 2.2 in [6] that $\widetilde{\delta}_1\widetilde{\delta}_2 = \widetilde{\delta}_2\widetilde{\delta}_1$. But

$$\sigma(\widetilde{\delta}_j(\sigma(a))) = \sigma(\delta_j(a)) = \delta_j(a) = \widetilde{\delta}_j(\sigma(a)) \quad (j = 1, 2),$$

and consequently $\delta_1\delta_2(a) = \delta_2\delta_1(a)$, for each $a \in \mathcal{A}$.

Conversely, suppose that $\delta_1\delta_2 = \delta_2\delta_1$. Therefore

$$\widetilde{\delta}_1\widetilde{\delta}_2(\sigma(a)) = \widetilde{\delta}_1(\delta_2(a)) = \widetilde{\delta}_1\sigma(\delta_2(a)) = \delta_1\delta_2(a) = \delta_2\delta_1(a) =$$
$$\delta_2(\widetilde{\delta}_1(\sigma(a))) = \widetilde{\delta}_2\sigma(\widetilde{\delta}_1(\sigma(a))) = \widetilde{\delta}_2\widetilde{\delta}_1(\sigma(a)).$$

Using [6] (the part (b) of Theorem 2.2) once more, we conclude that $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter group on $\sigma(\mathcal{A})$ and consequently, $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a σ-two parameter group. □

The following result gives us a version of the uniqueness ([14], Theorem 1.1.3) in the setting of σ-two parameter groups.

Lemma 2.4. *Let $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ and $\{\psi_{s,t}\}_{s,t\in\mathbb{R}}$ be two uniformly continuous σ-two parameter groups of bounded linear operators on $\mathcal{A}$ with the same generator (δ_1, δ_2). Then, $\varphi_{s,t} = \psi_{s,t}$.*

Proof. Consider the associated uniformly continuous two parameter group $\{\widetilde{\varphi}_{s,t}\}_{s,t\in\mathbb{R}}$ (resp. $\{\widetilde{\psi}_{s,t}\}_{s,t\in\mathbb{R}}$) on $\sigma(\mathcal{A})$ defined by $\widetilde{\varphi}_{s,t}(\sigma(a)) := \varphi_{s,t}(a)$ (resp. $\widetilde{\psi}_{s,t}(\sigma(a)) := \psi_{s,t}(a)$) with the generator $(\widetilde{\delta}_1, \widetilde{\delta}_2)$ fulfilling $\widetilde{\delta}_j(\sigma(a)) := \delta_j(a)$, $j = 1, 2$. Since $\widetilde{\delta}_1$ is the generator for the uniformly continuous one parameter groups $\{\widetilde{\varphi}_{s,0}\}_{s\in\mathbb{R}}$ and $\{\widetilde{\psi}_{s,0}\}_{s\in\mathbb{R}}$, hence, by the uniqueness ([14], Theorem 1.1.3) it follows that $\widetilde{\varphi}_{s,0} = \widetilde{\psi}_{s,0}$. A similar argument for $\widetilde{\delta}_2$ implies the equality of $\widetilde{\varphi}_{0,t}$ and $\widetilde{\psi}_{0,t}$. Furthermore, since $\widetilde{\varphi}_{s,t}(\sigma(a)) \in \sigma(\mathcal{A})$ and σ is idempotent, thus, $\sigma(\widetilde{\varphi}_{s,t}(\sigma(a))) = \widetilde{\varphi}_{s,t}(\sigma(a))$, for all $s, t \in \mathbb{R}$ and therefore

$$\begin{aligned}
\varphi_{s,t}(a) &= \varphi_{s,0}(\varphi_{0,t}(a)) \\
&= \widetilde{\varphi}_{s,0}\sigma(\widetilde{\varphi}_{0,t}(\sigma(a))) \\
&= \widetilde{\psi}_{s,0}\sigma(\widetilde{\varphi}_{0,t}(\sigma(a))) \\
&= \widetilde{\psi}_{s,0}(\widetilde{\varphi}_{0,t}(\sigma(a))) \\
&= \widetilde{\psi}_{s,0}(\widetilde{\psi}_{0,t}(\sigma(a))) \\
&= \widetilde{\psi}_{s,0}\widetilde{\psi}_{0,t}(\sigma(a)) \\
&= \widetilde{\psi}_{s,t}(\sigma(a)) \\
&= \psi_{s,t}(a). \quad \square
\end{aligned}$$

From now on, $\mathcal{A}$ is a C^*-algebra and σ is a $*$-linear endomorphism on $\mathcal{A}$.

Definition 2.5. *A two parameter σ-C^*-dynamical system, is a uniformly continuous σ-two parameter group $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ of linear $*$-endomorphisms on the C^*-algebra $\mathcal{A}$.*

According the notations which mentioned in Definition 2.1, to any two parameter σ-C^*-dynamical system $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$, we associate two σ-C^*-dynamical systems $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ defined by $u_s := \varphi_{s,0}$ and $v_t := \varphi_{0,t}$.

The infinitesimal generators of $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are denoted by δ_1 and δ_2, respectively. We denote the pair (δ_1, δ_2) as the infinitesimal generator of $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$.

Theorem 2.6. *Let $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ be a two parameter σ-C^*-dynamical system on $\mathcal{A}$ with the generator (δ_1, δ_2). Then, δ_j is an everywhere defined bounded $*$-σ-derivation, $j = 1, 2$.*

Proof. First note that, since $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are uniformly continuous, so δ_j $(j = 1, 2)$ is an everywhere defined bounded operators by Theorem 1.1.2 of [14]. Consider the σ-one parameter groups $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ associated to $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$. Let $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \lim_{s\to 0} \frac{u_s(ab) - \sigma(ab)}{s} &= \lim_{s\to 0} \frac{u_s(a)u_s(b) - \sigma(a)\sigma(b)}{s} \\ &= \lim_{s\to 0} \frac{(u_s(a) - \sigma(a))\sigma(b)}{s} + \lim_{s\to 0} \frac{u_s(a)(u_s(b) - \sigma(b))}{s} \\ &= \delta_1(a)\sigma(b) + \sigma(a)\delta_1(b). \end{aligned}$$

Therefore, $ab \in D(\delta_1)$ and $\delta_1(ab) = \delta_1(a)\sigma(b) + \sigma(a)\delta_1(b)$.

Furthermore,

$$\varphi_{s,0}(a^*) - \sigma(a^*) = \big(\varphi_{s,0}(a) - \sigma(a)\big)^*$$

and since the conjugation operation is norm continuous, so

$$\lim_{s\to 0} \frac{u_s(a^*) - \sigma(a^*)}{s} = \lim_{s\to 0} (\frac{u_s(a) - \sigma(a)}{s})^* = \delta_1(a)^*.$$

Therefore $a^* \in D(\delta_1)$ and $\delta_1(a^*) = \delta_1(a)^*$ which shows that δ_1 is a $*$-σ-derivation. A similar argument can be stated for δ_2. $\square$

We are going to establish some conditions making the converse of the above theorem be held. More precisely, we like to investigate some restrictions under which a pair of bounded $*$-σ-derivations induces a two parameter σ-C^*-dynamical system. To show this, we need the following useful lemma which can be found in [10].

Lemma 2.7. *Suppose that $\sigma : \mathcal{A} \to \mathcal{A}$ is an idempotent linear operator and δ is a $\sigma-$ derivation such that $\delta\sigma = \sigma\delta = \delta$. Then,*

$$\delta^n(ab) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}(\sigma(a))\delta^k(\sigma(b)), \qquad (a, b \in \mathcal{A} \text{ and } n \in \mathbb{N}).$$

Theorem 2.8. *Let σ be an idempotent linear $*$-endomorphism and δ_j be a bounded $*$-σ-derivation on $\mathcal{A}$ fulfilling $\delta_j\sigma = \sigma\delta_j = \delta_j$ $(j = 1, 2)$. If moreover, $\delta_1\delta_2 = \delta_2\delta_1$, then (δ_1, δ_2) induces a two-parameter σ-C^*-dynamical system on $\mathcal{A}$.*

Proof. For each $s, t \in \mathbb{R}$ and $a \in \mathcal{A}$, define $\varphi_{s,t}(a) = e^{s\delta_1 + t\delta_2}(\sigma(a))$. Since $\delta_1\delta_2 = \delta_2\delta_1$, it follows from Lemma **??** that $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a σ-two parameter group. Also, similar the method as stated in the proof of Theorem 1.2.1 of [14], it can be shown that the associated σ-one parameter groups $\{u_s\}_{s\in\mathbb{R}}$ and $\{v_t\}_{t\in\mathbb{R}}$ are uniformly continuous with the generators δ_1 and δ_2, respectively. Hence, $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is uniformly continuous. Finally, for each $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
u_s(ab) &= e^{s\delta_1}(\sigma(ab)) \\
&= \sum_{k=0}^{\infty} \frac{s^k}{k!}\delta_1^k(\sigma(ab)) \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k} \frac{s^{(k-r)+r}}{k!}\frac{k!}{r!(k-r)!}\delta_1^r(\sigma(a))\delta_1^{k-r}(\sigma(b)) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{s^m.s^n}{n!m!}\delta_1^n(\sigma(a))\delta_1^m(\sigma(b)) \\
&= \sum_{n=0}^{\infty} \frac{s^n}{n!}\delta_1^n(\sigma(a)). \sum_{m=0}^{\infty} \frac{s^m}{m!}\delta_1^m(\sigma(b)) \\
&= u_s(a).u_s(b).
\end{aligned}$$

That is u_s (and similarly v_t) is an endomorphism. Therefore,

$$\begin{aligned}\varphi_{s,t}(ab) &= u_s v_t(ab) \\ &= u_s\big(v_t(a)v_t(b)\big) \\ &= u_s(v_t(a)).u_s(v_t(b)) \\ &= \varphi_{s,t}(a).\varphi_{s,t}(b).\end{aligned}$$

Which means $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two-parameter σ-C^*-dynamical system on $\mathcal{A}$ with the generator (δ_1, δ_2). $\square$

The following theorem investigates the relationship between inner $*$-σ-derivations and two parameter σ-C^*-dynamical systems of $*$-σ-inner endomorphisms.

Theorem 2.9. *Let h_1 and h_2 be self-adjoint elements in the C^*-algebra $\mathcal{A}$ satisfying $h_1h_2 = h_2h_1$. If σ is a linear $*$-endomorphism such that $\sigma(h_j) = h_j$ $(j = 1, 2)$, then the pair $(\delta^\sigma_{h_1}, \delta^\sigma_{h_2})$ of inner $*$-σ-derivations induces the two parameter*
σ-C^*-dynamical system $\varphi_{s,t}(a) = e^{i(sh_1+th_2)}\sigma(a)e^{-i(sh_1+th_2)}$ of $*$-σ-inner endomorphisms.

Proof. By using induction on the aforementioned assumption $\sigma(h_j) = h_j$, we obtain that $\sigma({h_j}^n) = {h_j}^n$ for each $n \in \mathbb{N}$ and $j = 1, 2$.

Taking $u_{s,t} := e^{i(sh_1+th_2)}$, it follows that for each $s, t \in \mathbb{R}$, $u_{s,t}$ is a unitary element of $\mathcal{A}$. Also, $h_1h_2 = h_2h_1$, so by Lemma 2.3 we obtain $\{u_{s,t}\}_{s,t\in\mathbb{R}}$ is a uniformly continuous two parameter groups of unitaries in $\mathcal{A}$.

Furthermore, the continuity feature of σ implies that for each $s', t' \in \mathbb{R}$,

$$\sigma(u_{s',t'}a) = \sigma(e^{i(s'h_1+t'h_2)}a) = e^{is'h_1}\sigma(e^{it'h_2}a) = e^{i(s'h_1+t'h_2)}\sigma(a) = u_{s',t'}\sigma(a)$$

and similarly, $\sigma(au^*_{s',t'}) = \sigma(a)u^*_{s',t'}$.

Hence, $\varphi_{s,t}$ is a $*$-σ-inner endomorphism satisfying $\varphi_{0,0} = \sigma$ and

$$\begin{aligned}\varphi_{s,t}(\varphi_{s',t'}(a)) &= u_{s,t}\sigma(u_{s',t'}\sigma(a)u^*_{s',t'})u^*_{s,t} \\ &= u_{s,t}.u_{s',t'}\sigma(\sigma(a)u^*_{s',t'})u^*_{s,t} \\ &= u_{s+s',t+t'}\sigma^2(a)u^*_{s',t'}.u^*_{s,t} \\ &= u_{s+s',t+t'}\sigma(a)u_{s+s',t+t'} \\ &= \varphi_{s+s',t+t'}(a).\end{aligned}$$

Moreover, $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is uniformly continuous since

$$\begin{aligned}\| \varphi_{s,t}(a)-\sigma(a) \| &= \| u_{s,t}\sigma(a)u^*_{s,t}-\sigma(a) \| \\ &= \| (u_{s,t}\sigma(a)-\sigma(a)u_{s,t})u^*_{s,t} \| \\ &\leqslant \| u_{s,t}\sigma(a)-\sigma(a)u_{s,t} \| \\ &\leqslant \| u_{s,t}\sigma(a)-\sigma(a) \| + \| \sigma(a)-\sigma(a)u_{s,t} \| \\ &\leqslant 2 \| u_{s,t}-I \| \| \sigma \| \| a \|\end{aligned}$$

and consequently

$$\| \varphi_{s,t}-\sigma \| \leqslant 2 \| u_{s,t}-I \| \| \sigma \| .$$

Finally, applying the L'Hopital rule we have

$$\begin{aligned}\delta_1(a) &= \lim_{s\to 0}\frac{e^{ish_1}\sigma(a)e^{-ish_1}-\sigma(a)}{s} \\ &= \lim_{s\to 0}(ih_1e^{ish_1}\sigma(a)e^{-ish_1}-ie^{ish_1}\sigma(a)h_1e^{-ish_1}) \\ &= i(h_1\sigma(a)-\sigma(a)h_1).\end{aligned}$$

Therefore, δ_1 (and similarly δ_2) is an inner σ-derivation. □

In this step, we apply the C^*-algebra $B(\mathcal{H})$ to construct the new C^*-algebra $\mathcal{A} := B(\mathcal{H})\times B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. For this aim, suppose that $\mathcal{A}_j$ $(j=1,2)$ is a C^*-algebra. It is easy to observe that, $\mathcal{A} := \mathcal{A}_1\times\mathcal{A}_2$ is also a C^*-algebra by regarding the following algebraic structure

(i) $(a,b)+(c,d)=(a+c,b+d)$,

(ii) $\lambda(a,b)=(\lambda a,\lambda b)$

(iii) $(a,b).(c,d)=(ac,bd)$, $(a,b)^*=(a^*,b^*)$

(iv) $\| (a,b) \|_{\mathcal{A}}=\max\{\| a \|_{\mathcal{A}_1}, \| b \|_{\mathcal{A}_2}\}$.

Now, consider $\mathcal{A}_j$ $(j=1,2)$ as the concrete C^*-algebra $B(\mathcal{H})$, and define $\sigma:\mathcal{A}\to\mathcal{A}$ by $\sigma(S,T):=(0,T)$. Trivially, σ is an idempotent norm decreasing linear $*$-endomorphism on $\mathcal{A}$. We are going to characterize each so-called two parameter σ-C^*-dynamical system on $\mathcal{A}$. Before this, we need the following useful representation for bounded $*$-derivations on $B(\mathcal{H})$ which is demonstrated in Lemma 1.3 of [4].

Lemma 2.10. *Let $\mathcal{H}$ be a Hilbert space and d be a bounded $*$-derivation on $B(\mathcal{H})$. Then, there exists a self-adjoint operators A in $B(\mathcal{H})$ such that $d(T) = i(AT - TA)$, for all $T \in B(\mathcal{H})$,.*

We are ready to state and prove the following main result.

Theorem 2.11. *Let $\mathcal{H}$ be a Hilbert space. The following assertions are equivalent.*

(i) $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter σ-C^*-dynamical system on $\mathcal{A} := B(\mathcal{H})\times B(\mathcal{H})$.

(ii) There exists a uniformly continuous two parameter group $\{u_{s,t}\}_{s,t\in\mathbb{R}}$ of unitary elements in $\mathcal{A}$ satisfying $\sigma(u_{s,t}) = u_{s,t}$ and for each $T \in B(\mathcal{H})$, $\varphi_{s,t}(S,T) = u_{s,t}\sigma(S,T)u^*_{s,t}$.

Proof. Suppose that $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter σ-C^*-dynamical system on $\mathcal{A}$ with the generator (δ_1, δ_2). Then, for each $S, T \in B(\mathcal{H})$, there exists a pair $(S', T') \in \mathcal{A}$ such that $\delta_j(S,T) = (S', T')$, $j = 1, 2$. But, $\delta_j(\sigma(S,T)) = \delta_j(S,T) = \sigma(\delta_j(S,T))$ and therefore, $S' = 0$ and $\delta_j(0,T) = \delta_j(S,T) = (0,T')$. Define $d_j : B(\mathcal{H}) \to B(\mathcal{H})$ by $d_j(T) := T'$. Hence, for each $S, T \in B(\mathcal{H})$, $\delta_j(S,T) = (0, d_j(T))$. Trivially, d_j $(j = 1, 2)$ is a $*$-linear mapping. Also, by Theorem 2.6, δ_j $(j = 1, 2)$ is an everywhere defined bounded $*$-σ-derivation. Then, for each $T_1, T_2 \in B(\mathcal{H})$ we have

$$\begin{aligned}
(0, d_j(T_1T_2)) &= \delta_j(0, T_1T_2)\\
&= \delta_j((0,T_1).(0,T_2))\\
&= \delta_j(0,T_1).\sigma(0,T_2) + \sigma(0,T_1).\delta_j(0,T_2)\\
&= (0, d_j(T_1)).(0,T_2) + (0,T_1).(0, d_j(T_2))\\
&= (0, d_j(T_1).T_2 + T_1.d_j(T_2))
\end{aligned}$$

which means that d_j $(j = 1, 2)$ is an everywhere defined bounded $*$-derivation. It follows from Lemma 2.10 that, there exist self-adjoint operators A_1 and A_2 in $B(\mathcal{H})$ such that for each $T \in B(\mathcal{H})$, $d_j(T) =$

$i(A_jT - TA_j)$, $j = 1, 2$.

Therefore, for each $S, T \in B(\mathcal{H})$,

$$\begin{aligned}\delta_j(S,T) &= (0, i(A_jT - TA_j)) \\ &= (0, iA_j)(0,T) - (0,T)(0,iA_j) \\ &= i\Big((0,A_j)\sigma(S,T) - \sigma(S,T)(0,A_j)\Big) \\ &= i[(0,A_j), \sigma(S,T)].\end{aligned}$$

That is δ_j $(j = 1, 2)$ is the inner $*$-σ-derivation $\delta^{\sigma}_{(0,A_j)}$. Further, $\sigma(0, A_j) = (0, A_j)$, $j = 1, 2$.

It remains to show that $A_1A_2 = A_2A_1$. For this aim, since δ_1 and δ_2 are the infinitesimal generators of $\{\varphi_{s,0}\}_{s\in\mathbb{R}}$ and $\{\varphi_{0,t}\}_{t\in\mathbb{R}}$, respectively, so by Lemma 2.3 we have $\delta_1\delta_2 = \delta_2\delta_1$. This means that for each $S, T \in B(\mathcal{H})$ we have $\delta^{\sigma}_{(0,A_1)}(\delta^{\sigma}_{(0,A_2)}(S,T)) = \delta^{\sigma}_{(0,A_2)}(\delta^{\sigma}_{(0,A_1)}(S,T))$ which follows that

$$-A_1A_2T + A_1TA_2 + A_2TA_1 - TA_2A_1 = -A_2A_1T + A_2TA_1 + A_1TA_2 - TA_1A_2.$$

So, $(A_1A_2 - A_2A_1)T = T(A_1A_2 - A_2A_1)$, for all $T \in B(\mathcal{H})$. But the center of $B(\mathcal{H})$ is $\mathbb{C}I$, hence, there exists an element $\lambda \in \mathbb{C}$ for which $A_1A_2 - A_2A_1 = \lambda I$. Since B_1 is self-adjoint and λI is in the center of $B(\mathcal{H})$, by Exercise 4.6.34 of [8], $\lambda I = 0$. Thus, $\lambda = 0$ and consequently $A_1A_2 = A_2A_1$. Whence, $(0, A_1).(0, A_2) = (0, A_2).(0, A_1)$. Applying Theorem 2.9, we conclude that $(\delta^{\sigma}_{(0,A_1)}, \delta^{\sigma}_{(0,A_2)})$ induces the two parameter σ-C^*-dynamical system $\varphi_{s,t}(S,T) = e^{i(0,sA_1+tA_2)}\sigma(S,T)e^{-i(0,sA_1+tA_2)}$ of $*$-σ-inner endomorphisms. Take $u_{s,t} := e^{i(0,sA_1+tA_2)}$. Applying the same method as mentioned in the proof of Theorem **??**, we observe that $\sigma(u_{s,t}) = u_{s,t}$. Also, since A_j $(j = 1, 2)$ is a self-adjoint element in $B(\mathcal{H})$, so $(0, A_j)$ is a self-adjoint element in $\mathcal{A}$. By Stone's Theorem ([14], Theorem 1.10.8) $\{u_{s,0}\}_{s\in\mathbb{R}}$ and $\{u_{0,t}\}_{t\in\mathbb{R}}$ are one parameter group of unitary elements in $\mathcal{A}$ with the infinitesimal generators $i(0, A_1)$ and $i(0, A_2)$, respectively. Moreover, following the method as stated in the proof of Theorem 1.2.1 of [14], it can be shown that $\{u_{s,0}\}_{s\in\mathbb{R}}$ and $\{u_{0,t}\}_{t\in\mathbb{R}}$ are uniformly continuous one parameter groups and the fact that $(0, A_1).(0, A_2) = (0, A_2).(0, A_1)$ implies by Lemma 2.3 that $\{u_{s,t}\}_{s,t\in\mathbb{R}}$ is a uniformly continuous two parameter group and $\varphi_{s,t}(S,T) = u_{s,t}\sigma(S,T)u^*_{s,t}$ which completes the proof.

Conversely, let $\{u_{s,t}\}_{s,t\in\mathbb{R}}$ be a uniformly continuous two parameter group of unitary elements in $\mathcal{A}$ and $\varphi_{s,t}(S,T) = u_{s,t}\sigma(S,T)u^*_{s,t}$. Therefore $\{u_{s,0}\}_{s\in\mathbb{R}}$ and $\{u_{0,t}\}_{t\in\mathbb{R}}$ are uniformly continuous one parameter groups of unitaries in $\mathcal{A}$. Applying Stone's theorem once more, we obtain that there are self-adjoint elements A_1, A_2, B_1 and B_2 in $B(\mathcal{H})$ such that $u_{s,0} = e^{is(B_1,A_1)}$, and $u_{0,t} = e^{it(B_2,A_2)}$. Consequently, $u_{s,t} = e^{is(B_1,A_1)+it(B_2,A_2)}$ and
$\varphi_{s,t}(S,T) = e^{is(B_1,A_1)+it(B_2,A_2)}\sigma(S,T)e^{-is(B_1,A_1)-it(B_2,A_2)}$. Finally, using the method as stated in the proof of Theorem 2.9, one can conclude that $\{\varphi_{s,t}\}_{s,t\in\mathbb{R}}$ is a two parameter σ-C^*-dynamical system on $\mathcal{A}$. □

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