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Affine and Quasi-Affine Frames on Positive Half Line

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Abstract. The notion of quasi-affine frame was introduced by Ron and Shen in order to achieve shift-invariance of the discrete wavelet transform. In this paper, we have extended the notion of affine and quasi-affine frame on Euclidean space $\mathbb{R}^n$ to positive half-line. Furthermore, we establish the relation between affine and quasi-affine frames on positive half line and the preservation of frame bounds when changing an affine frame to a quasi-affine frame is also shown.

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1. Introduction

A frame differs from a Riesz basis in that it may be linearly dependent. In signal and image processing, this freedom allows the possibility of redundancy. The concept of frames, first introduced by Duffin and Schaeffer [7] in the context of non-harmonic Fourier series. Outside signal processing, frames did not seem to generate much interest until Daubechies, Grossmann, and Meyer [5] brought attention to it. They showed that Duffin and Schaeffers definition is an abstraction of a concept given by Gabor [11] for doing signal analysis.

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Discrete affine systems are obtained by applying dilations to a given shift-invariant system. Affine system is invariant under the dilaton operator, but is not shift invariant. The notion of quasi-affine frame was introduced by Ron and Shen [13] in order to achieve shift-invariance of the discrete wavelet transform. The main theorem of Ron and Shen in this direction is called "fundamental theorem of affine frames". It states that, an affine system ia an affine frame if and only if the quasi-affine system derived from it is also a frame. An operator on $L^2(\mathbb{R}^s) \times L^2(\mathbb{R}^s)$ corresponding to two affine systems was itroduced by Chui et al. [4] and they established a criterion in which this operator agrees with its analogue for two quasi-affine systems.

Although there are many results for affine frames on the real-line $\mathbb{R}$, the counterparts on positive half-line $\mathbb{R}^+$ are not yet reported. So this paper is devoted to affine systems also known as wavelet systems and quasi-affine systems on positive half-line. Farkov [8] has given general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [9] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. Shah and Debnath [15], studied Dyadic wavelet frames on a half-line using the Walsh-Fourier transform. Recently, Meenakshi et al. [12] studied NUMRA on positive half line. On the other hand, Abdullah [1] has given characterization of nonuniform wavelet sets on positive half-line. A constructive procedure for constructing tight wavelet frames on positive half-line using extension principles was recently considered by Shah in [14], in which he has pointed out a method for constructing affine frames in $L^2(\mathbb{R}^+)$. Moreover, the author has established sufficient conditions for a finite number of functions to form a tight affine frames for $L^2(\mathbb{R}^+)$.

The objective of this paper is to establish the relation between affine and quasi-affine frames on positive half line and the two frames have the same frame bounds and also establish a criterion in which the operator on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ corresponding to affine systems agrees with its analogue for quasi-affine systems.

2. Walsh-Fourier Analysis

We start this section with certain results on Walsh-Fourier analysis. We present a brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties.

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{Z}^+ = \{0, 1, ...\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and for any positive integer j, we set

$$x_j = [p^j x] (\operatorname{mod} p), \quad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{1}$$

where $x_j, \ x_{-j} \in \{0, 1, ..., p-1\}$.

Consider the addition defined on $\mathbb{R}^+$ as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2}$$

with

$$\xi_j = x_j + y_j (\operatorname{mod} p), \quad j \in \mathbb{Z} \backslash \{0\}, \tag{3}$$

where $\xi_j \in \{0, 1, 2, ..., p-1\}$ and $x_j, \ y_j$ are calculated by (1). Moreover, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & x \in \left[0, \dfrac{1}{p}\right), \\ \\ \varepsilon_p^j, & x \in \left[jp^{-1}, (j+1)p^{-1}\right), \ j = 1, 2, ..., p-1, \end{cases} \tag{4}$$

where $\varepsilon_p = \exp\left(\frac{2\pi i}{p}\right)$. The extension of the function r_0 to $\mathbb{R}^+$ is defined by the equality $r_0(x+1) = r_0(x), \ x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{\omega_m(x)\}_{m \in \mathbb{Z}^+}$ are defined by

$$\omega_0(x) = 1, \quad \omega_m(x) = \prod_{j=0}^{P} \left(r_0\left(p^j x\right)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{P} \mu_j p^j,\ \mu_j \in \{0, 1, 2, ..., p-1\},\ \mu_P \neq 0.$

For $x,\ \omega \in \mathbb{R}^+$, let

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j} \omega_j)\right), \tag{5}$$

where x_j and ω_j are calculated by (1).

We observe that

$$\chi\left(x,\ \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}},\ m\right) = \omega_m\left(\frac{x}{p^{n-1}}\right) \quad \forall\, x \in [0,\ p^{n-1}),\ m \in \mathbb{Z}^+.$$

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\tilde{f}(\omega) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x, \omega)}dx, \tag{6}$$

where $\chi(x, \omega)$ is given by (5).

If $f \in L^2(\mathbb{R}^+)$ and

$$J_a f(\omega) = \int_0^a f(x)\overline{\chi(x,\ \omega)}dx \quad (a < 0), \tag{7}$$

then $\tilde{f}$ is defined as limit of $J_a f$ in $L^2(\mathbb{R}^+)$ as $a \to \infty$.
The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform. It is known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2(0, 1)$. Let us denote by $\{\omega\}$ the fractional part of ω. For $l \in \mathbb{Z}^+$, we have $\chi(l, \omega) = \chi(l, \{\omega\})$.

If $x, y, \omega \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \omega) = \chi(x, \omega)\chi(y, \omega), \quad \chi(x \oplus y, \omega) = \chi(x, \omega)\overline{\chi(y, \omega)}, \tag{8}$$

Definition 2.1. *Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathbb{H}$ is called a frame for $\mathbb{H}$ if there exist constants A and B with $0 < A \leqslant B < \infty$ such that*

$$A\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2, \tag{9}$$

for all $f \in \mathbb{H}$. The largest constant A and the smallest constant B satisfying (9) are called the upper and the lower frame bound, respectively. If only the right hand side inequality holds, we say that $\{f_k\}_{k\in\mathbb{Z}}$ is a Bessel system with constant B. A frame is a tight frame if A and B can be chosen so that $A = B$ and is a normalized tight frame (NTF) if $A = B = 1$. Thus, if $\{f_k\}_{k\in\mathbb{Z}}$ is a NTF in $\mathbb{H}$, then

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2. \tag{10}$$

3. Main Results

Definition 3.1. *Let $\Psi = \{\psi_1, \psi_2, ..., \psi_L\}$ be a finite family of functions in $L^2(\mathbb{R}^+)$. The affine system generated by Ψ is the collection*

$$X(\Psi) = \{\psi_{l,j,k} : 1 \leqslant l \leqslant L, j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$$

where $\psi_{l,j,k}(x) = p^{j/2}\psi_l(p^j x \ominus k)$. The quasi-affine system generated by Ψ is

$$X^q(\Psi) = \{\tilde{\psi}_{l,j,k} : 1 \leqslant l \leqslant L, j \in \mathbb{Z}, k \in \mathbb{Z}^+\},$$

where

$$\psi^q_{l,j,k}(x) = \begin{cases} p^{j/2}\psi_l(p^j x \ominus k), & j \geqslant 0, \quad k \in \mathbb{Z}^+, \\ p^j \psi_l(p^j(x \ominus k)), & j < 0, \quad k \in \mathbb{Z}^+. \end{cases} \tag{11}$$

We say that Ψ is a set of basic wavelets of $L^2(\mathbb{R}^+)$ if the affine system $X(\Psi)$ forms an orthonormal basis for $L^2(\mathbb{R}^+)$.

Let $\Psi, \tilde{\Psi} \subset L^2(\mathbb{R}^+)$ be two finite sets in $L^2(\mathbb{R}^+)$ with the same cardinality. The operator

$$P_{\Psi,\tilde{\Psi}}(f, g) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} \langle f, \psi_{l,j,k}\rangle\langle\tilde{\psi}_{l,j,k}, g\rangle, \quad f, g \in L^2(\mathbb{R}^+), \tag{12}$$

plays a very important role in our investigation. It is bounded linear operator on $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, if both $X(\Psi)$ and $X(\tilde{\Psi})$ are affine Bessel families. Our frame analysis is based on the relation to the operator

$$P^q_{\Psi,\tilde{\Psi}}(f,g) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^+} \langle f, \psi^q_{l,j,k}\rangle \langle \tilde{\psi}^q_{l,j,k}, g\rangle, \quad f,g \in L^2(\mathbb{R}^+), \tag{13}$$

which is associated with the quasi-affine systems. The following terminology will be used.

Definition 3.2. *Let $\Psi, \tilde{\Psi} \subset L^2(\mathbb{R}^+)$ be two finite sets in $L^2(\mathbb{R}^+)$ with the same cardinality that generate two Bessel families $X(\Psi)$ and $X(\tilde{\Psi})$. Then $\tilde{\Psi}$ is called an affine dual of Ψ, if*

$$P_{\Psi,\tilde{\Psi}}(f,g) = \langle f, g\rangle, \quad f,g \in L^2(\mathbb{R}^+),$$

and a quasi-affine dual if

$$P^q_{\Psi,\tilde{\Psi}}(f,g) = \langle f, g\rangle, \quad f,g \in L^2(\mathbb{R}^+).$$

Affine system $X(\Psi)$ are dilation-invariant in the sense that

$$X(\Psi) = D(X(\Psi)).$$

This is the reason that $P_{\Psi,\tilde{\Psi}}$ in (12) has the property

$$P_{\Psi,\tilde{\Psi}}(Df, Dg) = P_{\Psi,\tilde{\Psi}}(f,g), \quad f,g \in L^2(\mathbb{R}^+). \tag{14}$$

On the other hand, it can be seen easily from the definition of quasi-affine systems, that

$$P^q_{\Psi,\tilde{\Psi}}(T_k f, T_k g) = P^q_{\Psi,\tilde{\Psi}}(f,g), \quad f,g \in L^2(\mathbb{R}^+),\ k \in \mathbb{Z}^+.$$

Hence, $P^q_{\Psi,\tilde{\Psi}}$ is invariant with respect to multi-integer shifts of both arguments.

For $J \geqslant 0$, we let Q_J denote a complete set of representatives of $\mathbb{Z}^+/(p^J\mathbb{Z}^+)$, so that

$$\#Q_J = p^J.$$

The special choice of representatives is

$$Q_J = \mathbb{Z}^+ \cap p^J([1,2)). \tag{15}$$

We use the notations

$$P_j(f,g)=\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+}\langle f,\psi_{l,j,k}\rangle\langle\tilde{\psi}_{l,j,k},g\rangle, \tag{16}$$

and

$$P_j^q(f,g)=\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+}\langle f,\psi^q_{l,j,k}\rangle\langle\tilde{\psi}^q_{l,j,k},g\rangle, \tag{17}$$

for any $j\in\mathbb{Z}$. In particular, $P_j=P_j^q$ holds for all $j\geqslant 0$.

Lemma 3.3. *Let Ψ be a finite subset of $L^2(\mathbb{R}^+)$ and $\tilde{\Psi}$ be the dual of Ψ. Let $J>0$ be an integer. Then for all $j\geqslant -J$ and $f,g\in L^2(\mathbb{R}^+)$,*

$$P_j^q(f,g)=p^{-J}\sum_{v\in Q_J}P_j(T_vf,T_vg).$$

Proof. For any $j\geqslant 0$ the operator P_j is invariant with respect to multi-integer shifts, so that

$$P_j^q(f,g)=P_j(f,g)=P_j(T_vf,T_vg),\qquad v\in Q_J.$$

For any $-J\leqslant j<0$, we first observe that P_j is invariant with respect to shifts $k\in p^{-j}\mathbb{Z}^+$. Furthermore, $H=\mathbb{Z}^+/p^{-j}\mathbb{Z}^+$ is a normal subgroup of $G=\mathbb{Z}^+/p^J\mathbb{Z}^+$ and there is a group isomorphism of G/H and $p^{-j}\mathbb{Z}^+/p^J\mathbb{Z}^+$. Hence, $|G/H|=p^{J+j}$, and

$$\begin{aligned}p^{-J}\sum_{v\in Q_J}P_j(T_vf,T_vg)&=p^{-J}\sum_{\lambda\in Q_{-j}}\sum_{\substack{v\in Q_J\\ v-\lambda\in p^{-j}\mathbb{Z}^+}}P_j(T_vf,T_vg)\\ &=p^{j}\sum_{\lambda\in Q_{-j}}P_j(T_\lambda f,T_\lambda g).\end{aligned}$$

By inserting the definition (11) of $\psi^q_{l,j,k}$ we see that the last expression is equal to $P_j^q(f,g)$. $\square$

Lemma 3.4. *Let $\Psi=\{\psi_1,\psi_2,...,\psi_L\}\subset L^2(\mathbb{R}^+)$ and let $f\in L^2(\mathbb{R}^+)$ be a function with compact support. Then*

$$\lim_{N\to\infty}\sum_{j<0} P_j^q(D^N f, D^N f) = 0, \tag{18}$$

and

$$\lim_{N\to\infty} p^{-N}\sum_{j<-N}\sum_{v\in Q_N} P_j(T_v f, T_v f) = 0, \tag{19}$$

where the special choice of representatives (15) is used in (19) for any $N \in \mathbb{N}$.

Proof. Let Ω denote the support of f and let us first choose $N_0 > 0$ so large that $D^{-N}\Omega$ is contained in a ball of radius $1/2$ around the origin for all $N \geqslant N_0$. In order to prove (18), we consider

$$\begin{aligned}
\sum_{j<0} P_j^q(D^N f, D^N f) &= \sum_{j<0} p^j \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+} \left|\langle f, D^{-N}T_k D^j\psi_l\rangle\right|^2 \\
&= \sum_{j<0} p^j \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+} \left|\langle T_{p^N k} f, D^{j-N}\psi_l\rangle\right|^2 \\
&\leqslant \sum_{j<0} p^j \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+} \|f\|^2 \int_{\Omega+p^N k} \left|D^{j-N}\psi_l(x)\right|^2 dx \\
&= \|f\|^2 \sum_{j<0} p^j \sum_{k\in\mathbb{Z}^+} \int_{p^{j-N}\Omega+p^j k} \sum_{l=1}^{L} |\psi_l(x)|^2\, dx. \qquad (20)
\end{aligned}$$

By our previous choice of N_0, we obtain that

$$\sum_{j<0} P_j^q(D^N f, D^N f) \leqslant \|f\|^2 \int_{\mathbb{R}^+} w_N(x) \sum_{l=1}^{L} |\psi_l(x)|^2\, dx,$$

holds for all $N \geqslant N_0$, where

$$w_N(x) = \sum_{j<0} p^j \chi_{p^j(\mathbb{Z}^+ + p^{-N}\Omega)}(x), \quad x \in \mathbb{R}^+. \tag{21}$$

Since

$$w_N(x) \leqslant \sum_{j<0} p^j = \frac{p}{p-1}, \quad N \geqslant N_0,$$

and since $\sum_{l=1}^{L} |\psi_l|^2 \in L^1(\mathbb{R}^+)$, the dominated convergence theorem can be applied to the above integral. It thus suffices to show that

$$\lim_{N\to\infty} w_N(x) = 0 \quad \text{for all } x \in U = \mathbb{R}^+ \backslash \bigcup_{j<0} p^j \mathbb{Z}^+.$$

This last assertion can be shown as follows. By the compactness of Ω the sequence of numbers

$$r_j = \sup\{\|p^j y\| : y \in \Omega\},$$

tends to zero as $j \to -\infty$. If we fix $x \in U$, then all terms in (21) which satisfy

$$r_{j-N} < \operatorname{dist}(x, p^j \mathbb{Z}^+) = d_j(x), \quad j < 0,$$

vanish. In other words,

$$w_N(x) \leqslant \sum_{\substack{j<0 \\ r_{j-N} \geqslant d_j(x)}} p^j \to 0 \ \text{ as } \ N \to \infty.$$

In the second relation (19), we let $N \geqslant N_0$ and use similar transformations as in (20) in order to obtain

$$\begin{aligned} p^{-N} &\sum_{v\in Q_N} \sum_{j<-N} P_j(T_v f, T_v f) \\ &\leqslant \|f\|^2 p^{-N} \sum_{v\in Q_N} \sum_{j<-N} \sum_{k\in\mathbb{Z}^+} \int_{p^j(\Omega+v)+k} \sum_{l=1}^{L} |\psi_l(x)|^2 dx. \end{aligned}$$

We now define

$$v_N(x) = p^{-N} \sum_{v\in Q_N} \sum_{j<-N} \sum_{k\in\mathbb{Z}^+} \chi_{p^j(\Omega+v)+k}(x), \quad x \in \mathbb{R}^+. \tag{22}$$

Our remaining task is to show that v_N is uniformly bounded in N, as N tends to infinity, and converges to zero pointwise a.e. We will even show that v_N tends to zero uniformly. Recall that Q_N is assumed to be of the form (15) and hence

$$p^{-N}(\Omega+v)\subset S=(1/2,3/2)\quad \text{for all } N\geqslant N_0,\ v\in Q_N.$$

In order to resolve the various summations in (22), we fix j and k and first observe that

$$\sum_{v\in Q_N}\chi_{p^j(\Omega+v)+k}\leqslant c_1\chi_{p^{j+N}S+k},$$

holds for all $N\geqslant N_0,\ j<-N$ and $k\in\mathbb{Z}^+$ and the constant $c_1>0$ only depends on the compact set Ω. Next the properties of p and the set S where the special choice of Q_N enters, gives the estimate

$$\sum_{j<-N}\chi_{p^{j+N}S+k}\leqslant c_2\chi_{B_r(0)+k},$$

for all $N\geqslant N_0,\ k\in\mathbb{Z}^+$, where the constant $c_2,r>0$ only depends on p. Finally, summation over $k\in\mathbb{Z}^+$ gives another contant $c_3>0$ which only depends on r (hence on p) such that

$$\sum_{k\in\mathbb{Z}^+}\chi_{B_r(0)+k}\leqslant c_3.$$

Combining all these relations yields

$$v_N(x)\leqslant c_1c_2c_3p^{-N},\quad x\in\mathbb{R}^+.$$

and hence the uniform convergence of v_N to zero. This completes the proof of the lemma. $\square$

Theorem 3.5. *Let Ψ be a finite subset of $L^2(\mathbb{R}^+)$. Then*

(a) $X(\Psi)$ is a Bessel family if and only if $X^q(\Psi)$ is a Bessel family. Furthermore, their exact upper bounds are equal.

(b) $X(\Psi)$ is an affine frame if and only if $X^q(\Psi)$ is a quasi-affine frame. Furthermore, their lower and upper exact bounds are equal.

Proof. (a) Let $\Psi = \tilde{\Psi}$. All summands of $P_{\Psi,\Psi}$ and $P^q_{\Psi,\Psi}$ are nonnegative. If Ψ generates an affine Bessel family with upper bound $B \geqslant 0$, then

$$\begin{aligned} P^q_{\Psi,\Psi}(f,f) = \lim_{J\to\infty} \sum_{j\geqslant -J} P^q_j(f,f) \\ = \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} \sum_{j\geqslant -J} P_j(T_v f, T_v f) \\ \leqslant \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} P_{\Psi,\Psi}(T_v f, T_v f) \\ \leqslant \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} B\|T_v f\|^2 = B\|f\|^2, \end{aligned}$$

holds for all $f \in L^2(\mathbb{R}^+)$. Here, we used the translation invariance of the norm. We have thus shown that the quasi-affine frame $X^q(\Psi)$ is also a Bessel family with the same upper bound B.

Conversely, let us assume that $X^q(\Psi)$ is a Bessel family with upper bound $B_q \geqslant 0$. Let us further assume that there exists an $f \in L^2(\mathbb{R}^+)$, such that

$$\|f\| = 1 \quad \text{and} \quad P_{\Psi,\Psi}(f,f) > B_q.$$

Then by the dilation invariance of $X(\Psi)$, we can find $N \in \mathbb{N}$ such that

$$\sum_{j=-N}^{\infty} P_j(f,f) = \sum_{j=0}^{\infty} P_j(D^N f, D^N f) > B_q.$$

But this contradicts the definition of B_q, since

$$P^q_{\Psi,\Psi}(D^N f, D^N f) \geqslant \sum_{j=0}^{\infty} P^q_j(D^N f, D^N f) = \sum_{j=0}^{\infty} P_j(D^N f, D^N f),$$

and the dilation D is an isometry. Thus, we can conclude that $X(\Psi)$ must be a Bessel family with upper bound B_q.

(b) It only remains to consider the lower frame bounds A and A_q. The proof follows the same argument as the proof of (a). The only differences are the use of Lemma 3.4 at certain places e.g., the relation $A_q \geqslant A$ follows by using (19) in the third line of

$$\begin{aligned} P^q_{\Psi,\Psi}(f,f) &= \lim_{J\to\infty} \sum_{j\geqslant -J} P^q_j(f,f) \\ &= \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} \sum_{j\geqslant -J} P_j(T_v f, T_v f) \\ &= \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} \sum_{j\in\mathbb{Z}} P_j(T_v f, T_v f) \\ &= \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} P_{\Psi,\Psi}(T_v f, T_v f) \\ &\geqslant \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} A\|T_v f\|^2 = A\|f\|^2, \end{aligned}$$

for all $f \in L^2(\mathbb{R}^+)$ which have compact support. Since this is the dense subset of $L^2(\mathbb{R}^+)$, the relation $P^q_{\Psi,\Psi}(f,f) \geqslant A\|f\|^2$ holds for all $f \in L^2(\mathbb{R}^+)$. The opposite relation $A_q \leqslant A$ is shown by assuming the contrary, so that

$$P_{\Psi,\Psi}(f,f) \leqslant A_q - \varepsilon \quad \text{for some } f \in L^2(\mathbb{R}^+), \quad \|f\| = 1,$$

and some $\varepsilon > 0$. Without loss of generality, we can assume that f has compact support. The dilation invariance of the operator $P_{\Psi,\Psi}(f,f)$ gives

$$P_{\Psi,\Psi}(D^N f, D^N f) \leqslant A_q - \varepsilon \quad \text{for all } N \in \mathbb{N}.$$

By (18), there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} P^q_{\Psi,\Psi}(D^N f, D^N f) &< \sum_{j=0}^{\infty} P^q_j(D^N f, D^N f) + \frac{\varepsilon}{2} = \sum_{j=0}^{\infty} P_j(D^N f, D^N f) + \frac{\varepsilon}{2} \\ &\leqslant P_{\Psi,\Psi}(D^N f, D^N f) + \frac{\varepsilon}{2} \leqslant A_q - \frac{\varepsilon}{2}, \end{aligned}$$

which contradicts with the definition of the lower frame bound A_q of $X^q(\Psi)$. $\square$

Theorem 3.6. *Let Ψ be a finite subset of $L^2(\mathbb{R}^+)$ and $\tilde{\Psi}$ be the dual of Ψ. ssume that Ψ and $\tilde{\Psi}$ generate two affine Bessel families. Then $P_{\Psi,\tilde{\Psi}}$ translation-invariant if and only if*

$$P_{\Psi,\tilde{\Psi}} = P^q_{\Psi,\tilde{\Psi}}. \tag{23}$$

Proof. First we assume that $P_{\Psi,\tilde{\Psi}}$ is translation-invariant. Then, as in the proof of theorem 3.5, we have

$$P^q_{\Psi,\tilde{\Psi}}(f,g) = \lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} P_{\Psi,\tilde{\Psi}}(T_v f, T_v g)$$

for all $f, g \in L^2(\mathbb{R}^+)$ with compact support. Since $P_{\Psi,\tilde{\Psi}}$ is assumed to be translation-invariant, the right-hand side equals

$$\lim_{J\to\infty} p^{-J} \sum_{v\in Q_J} P_{\Psi,\tilde{\Psi}}(f,g) = P_{\Psi,\tilde{\Psi}}(f,g)$$

The equality extends to all functions in $L^2(\mathbb{R}^+)$ by density and boundedness of both operator. On the other hand, as a consequence of (23), the operator $P_{\Psi,\tilde{\Psi}}$ is invariant with respect to shifts T_k, $k \in \mathbb{Z}^+$. Its dilation invariance implies that it is invariant with respect to shifts

$$T_x, \quad x \in \bigcup_{j\in\mathbb{Z}} p^j \mathbb{Z}^+.$$

This union of sets is dense in $\mathbb{R}^+$. Since translation is a continuous operation on $L^2(\mathbb{R}^+)$ and $P_{\Psi,\tilde{\Psi}}$ is bounded, it is invariant with respect to any shift T_x, $x \in \mathbb{R}^+$. $\square$

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