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P-Ideals and PMP-Ideals in Commutative Rings

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Abstract. Recently, P-ideals have been studied in C(X) by some authors. In this article we investigate P-ideals and a new concept PMP-ideal in commutative rings. We show that I is a P-ideal (resp., PMP-ideal) in R if and only if every prime ideal of R which does not contain I is a maximal (resp., minimal prime) ideal of R. Also, we characterize the largest P-ideals (resp., PMP-ideals) in commutative rings and in C(X) as well. Furthermore, we study relations between these ideals and other ideals, such as prime, maximal, pure and von Neumann regular ideals and we find that in a reduced ring P-ideals and von Neumann regular ideals coincide. Finally, we prove that C(X) is a von Neumann regular ring if and only if all of its pure ideals are P-ideals.

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1. Introduction

Throughout this paper the notation R stands for a commutative ring with unity and X stands for a topological Tychonoff space. We denote by $\text{Spec}(R)$, $\text{Max}(R)$ and $\text{Min}(R)$ the set of all prime ideals, maximal ideals and minimal prime ideals of R, respectively. Also, by $\text{Jac}(R)$ and $\text{Rad}(R)$ we mean the Jacobson radical and the prime radical of R, respectively. If $S \subseteq R$, then by $\mathcal{A}(S)$ we mean the set of all annihilators of S; briefly, we use $\mathcal{A}(a)$ instead of $\mathcal{A}(\{a\})$. For each $a \in R$, let aR, $\mathbf{M}_a$ and $\mathbf{P}_a$ be the ideal generated by a, the intersection of all maximal ideals containing a and the intersection of all minimal prime ideals containing a, respectively. If $A \subseteq R$, then we briefly use the notations

$$V(A) = \{P \in \text{Spec}(R) : A \subseteq P\} \quad , \quad D(A) = \text{Spec}(R) \setminus V(A).$$

Assuming that I is an ideal of R, the set $\{a \in R : a \in aI\}$ is denoted by $m(I)$ which is called the pure part of I. It is well-known that $m(I)$ is an ideal of R and $m(I) = \{a \in R : I + \mathcal{A}(a) = R\}$. An ideal I is said to be pure if $I = m(I)$. One can easily see that a maximal ideal M of a reduced ring R is pure if and only if $M \in \text{Min}(R)$. For more information about the pure ideals, refer to [1], [2] and [8]. The ring of all continuous functions on a topological space X is denoted by $\text{C}(X)$. By A° and $\overline{A}$ we mean the interior and the closure of a subset A of X respectively. Also if $f \in \text{C}(X)$ and $A \subseteq X$, then we define

$$Z(f) = \{x \in X : f(x) = 0\} \quad , \quad \text{Coz}(f) = X \setminus Z(f)$$

$$O_A(X) = \{f \in \text{C}(X) : A \subseteq Z^\circ(f)\} \ , \ M_A(X) = \{f \in \text{C}(X) : A \subseteq Z(f)\}.$$

In particular, if $A = \{x\}$, then we use $O_x(X)$ and $M_x(X)$ instead of $O_{\{x\}}(X)$ and $M_{\{x\}}(X)$, respectively. For undefined terms and notations, the readers is referred to [9], [11] and [15].

In Section 1, first we deal with the connection between the set of ideals of a ring R and the set of ideals contained in a fixed ideal of R. Next, we give some statements about the von Neumann regular(or briefly regular) elements and ideals, see [3] and [10], for more information about regular ideals. In the sequential, we see that, under some conditions (for example, in reduced rings) regular ideals coincide with P-ideals.

Section 2 is devoted to P-ideals and PMP-ideals in a ring R. P-ideals in $C(X)$ are introduced and studied in [14], but PMP-ideal is a new concept. In this section, we find some equivalent conditions for these notions and then we obtain some new results. For instance, we show that an ideal I of R is a P-ideal if and only if $D(I) \subseteq \mathrm{Max}(R)$; also, it is shown that an ideal I of R is a PMP-ideal if and only if $D(I) \subseteq \mathrm{Min}(R)$. Using characterization of P-ideals and PMP-ideals as intersections of prime ideals, we find that in any commutative ring R, the largest P-ideal (resp., PMP-ideal) exists.

In Section 3, we prove that if R is a reduced ring, then I is a P-ideal if and only if I is regular and also we prove that every proper ideal I of a reduced ring R is a PMP-ideal if and only if R is a regular ring or R is a local ring(i.e., ring which has exactly one maximal ideal) with dim(R) = 1. In addition, in this section, we find an equivalent condition for a PMP-ideal to be a P-ideal.

In Proposition 1.3, in order to find a one-one correspondence between the set of prime ideals not containing a given ideal I of R and the set of prime ideals of I as a ring, we need the following lemma.

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Lemma 1.1. *Let I be an ideal of R and H be a semiprime ideal in the ring I, then H is an ideal in R.*

Proof. Suppose that $a \in H$ and $r \in R$, hence $r^2a \in I$ which implies that $(ra)^2 = (r^2a)a \in H$. This shows that $ra \in H$. $\square$

Definition 1.2. *Let I be an ideal of R. A maximal prime ideal of I is a prime ideal of I which is maximal with this property.*

In the following proposition $\mathrm{Maxp}(I)$ and $D_M(I)$ denote the set of all maximal prime ideals of I and $D(I) \cap \mathrm{Max}(R)$, respectively. For another proof of part (a) of the following proposition, see Lemma 3.1 of [13]

Proposition 1.3. *Let I be an ideal of R and φ be the mapping from $D(I)$ to $\mathrm{Spec}(I)$ with $\varphi(P) = P \cap I$. Then*

(a) *φ is an order-preserving bijection.*
(b) *H is a prime and maximal ideal of I if and only if $\varphi^{-1}(H) \in D_M(I)$. In other words we have $\varphi(D_M(I)) = \mathrm{Maxp}(I) \cap \mathrm{Max}(I)$.*

Proof. (a). It is clear that φ is well-defined. We claim that φ is onto. To see this, let $H \in \mathrm{Spec}(I)$. Clearly, $S = I \setminus H$ is a multiplicatively closed set in R and $S \cap H = \varnothing$. Hence, there exists a prime ideal P of R containing H such that $P \cap S = \varnothing$. Furthermore, it is clear that $P \cap I = H$. Now, suppose that $P, Q \in D(I)$ and $P \cap I \subseteq Q \cap I$. Therefore, $P \cap I \subseteq Q$ and $I \nsubseteq Q$ which imply that $P \subseteq Q$.
(b). Suppose that $H \in \mathrm{Maxp}(I) \cap \mathrm{Max}(I)$. By part (a), there exists $P \in D(I)$ such that $P \cap I = H$. It is sufficient to show that $P \in \mathrm{Max}(R)$. Let $a \notin P$ and $i \in I \setminus P$, hence $ai \in I \setminus P$ and then $(P + aiR) \cap I = I$. Therefore, there exist $p \in P$ and $r \in R$ such that $i = p + rai$. Thus, $i(1 - ar) = p \in P$, hence $1 - ar \in P$ and consequently $P + aR = R$. Conversely, suppose that $M = \varphi^{-1}(H) \in D_M(I)$, we must show that $H \in \mathrm{Max}(I)$. Let $a \in I \setminus H$, then $a \notin M$ and hence $M + aR = R$. Therefore, $I = IR = I(M + aR) = IM + aI \subseteq (M \cap I) + aI = H + aI$. Thus, $I = H + aI$ and we are done. □

Corollary 1.4. *Let I be an ideal of R, S be a subring of R and $I \subseteq S$. Then there exists an order preserving bijection between the set of prime ideals of R not containing I and the set of all prime ideals of S not containing I.*

Proof. By the previous proposition, the proof is clear. □

Recall that an element $a \in R$ is called a regular element whenever there exists $b \in R$ such that $a = a^2 b$. An ideal I of R is called a regular ideal if each of its elements is regular. If each member of R is regular, then we say that R is a regular ring, see [10] and [3].

The following proposition is well-known.

Proposition 1.5. *Let $a \in R$, then the following statements are equivalent:*
(a) *a is a regular element.*
(b) *There exists an idempotent $e \in R$ such that $aR = eR$.*

(c) $\mathcal{A}(a)$ *is generated by an idempotent.*
(d) $\mathcal{A}^2(a)$ *is generated by an idempotent.*
(e) $\mathcal{A}(a) \oplus \mathcal{A}^2(a) = R$.
(f) $\mathcal{A}(a) \oplus aR = R$.

Lemma 1.6. *Let R be a reduced ring and $e \in R$ be an idempotent element. Then $\mathbf{P}_e = eR$. Furthermore, if* $\text{Jac}(R) = (0)$, *then $\mathbf{M}_e = \mathbf{P}_e = eR$.*

Proof. By [6, Theorem 1.4], we have $\mathbf{P}_a = \mathcal{A}^2(a)$. Hence, $\mathbf{P}_e = \mathcal{A}^2(e) = eR$. To show the second part, using [3, Theorem 2.9], we have $\mathbf{M}_a \subseteq \mathbf{P}_a$ for any $a \in R$ and so $\mathbf{M}_e = \mathbf{P}_e = eR$. $\square$

Lemma 1.7. *Let R be a reduced ring and $a, b \in R$ such that $a = b^n$ for a natural $n \geqslant 2$. Consider the following conditions:*
(a) *a is a regular element.*
(b) *aR is a semiprime ideal.*
(c) $\mathbf{P}_a = aR$.
(d) $\mathbf{M}_a = aR$.
(e) *aR is an intersection of maximal ideals.*
Then parts (a), (b) *and* (c) *are equivalent,* (a) *implies* (d) *and* (e) *and if* $\text{Jac}(R) = (\circ)$, *then all of the above conditions are equivalent.*

Proof. First we prove the implications **(a)** $\Rightarrow$ (b), (c), (d), (e). By part (b) of Proposition 1.5, there exists an idempotent element $e \in R$ such that $aR = eR$, so $\mathbf{P}_a = \mathbf{P}_e = eR = aR$. Also, if $\text{Jac}(R) = (\circ)$, then $\mathbf{M}_a = \mathbf{M}_e = eR = aR$.
(b) $\Rightarrow$ (a). By our hypothesis, we have $b \in aR$ and so there exists $c \in R$ such that $b = ac$. Clearly, $a = b^n = (ac)^n = a^2 d$ in which $d = c^n a^{n-2}$. Hence, a is a regular element.
(c) $\Rightarrow$ (b). It is clear.
Furthermore, if $\text{Jac}(R) = (\circ)$, then (d) $\Rightarrow$ (b) and (e) $\Rightarrow$ (b) are clear. $\square$

Proposition 1.8. *Let R be a reduced ring and I be an ideal of R. Consider the following conditions:*
(a) *I is a regular ideal.*
(b) *aR is a semiprime ideal for any $a \in I$.*

(c) $\mathbf{P}_a = aR$ *for any* $a \in I$.
(d) $\mathbf{M}_a = aR$ *for any* $a \in I$.
(e) aR *is an intersection of maximal ideals for any* $a \in I$.
Then parts (a), (b) *and* (c) *are equivalent,* (a) *implies* (d) *and* (e) *and if* $\mathrm{Jac}(R) = (\circ)$, *then all of the above conditions are equivalent.*

Proof. Clearly (b) implies (a) and the remainder of the proof is an immediate consequence of Lemma 1.7. □

2. P-Ideals and PMP-Ideals in Commutative Rings

In this section we study the properties P-ideals and PMP-ideals in commutative rings and we investigate the relations between these ideals.

Definition 2.1. *Let* R *be a ring and* I *be an ideal of* R. *Then* I *is called a* P*-ideal, whenever every prime ideal of the ring* I *is a maximal ideal of* I. *Also,* I *is called a* PMP*-ideal, whenever every prime ideal of the ring* I *is a maximal prime ideal of* I.

Obviously, the zero ideal is a P-ideal and PMP-ideal and also every P-ideal is a PMP-ideal; but a PMP-ideal is not a P-ideal in general. To see this, consider the reduced local ring $R = \mathbb{Z}_{2\mathbb{Z}}$, then clearly, the unique maximal ideal of R is a PMP-ideal which is not a P-ideal. In some rings such as $\mathrm{C}(X)$, these concepts coincide.
In the next proposition, we find a necessary and sufficient condition for an ideal I to be a P-ideal (resp., PMP-ideal). The first part of the following theorem is well-known in the context of $\mathrm{C}(X)$, see [14].

Theorem 2.2. *Let* R *be a ring and* I *be an ideal of* R. *Then*
(a) I *is a* P*-ideal if and only if* $D(I) \subseteq \mathrm{Max}(R)$.
(b) I *is a* PMP*-ideal if and only if* $D(I) \subseteq \mathrm{Min}(R)$.

Proof. (a $\Rightarrow$). Assume that $P \in D(I)$, then $P \setminus I \in \mathrm{Spec}(I)$. Hence, $H = P \setminus I$ is a maximal ideal of I. Now, by part (b) of Proposition 1.3, we have $P = \varphi^{-1}(H) \in \mathrm{Max}(R)$.
(a $\Leftarrow$). Suppose that $H \in \mathrm{Spec}(I)$. By part (a) of Proposition 1.3, we have $\varphi^{-1}(H) = P \in D(I)$. By our hypothesis, $P \in \mathrm{Max}(R)$ and so by

part (b) of Proposition 1.3, we have $H \in \mathrm{Maxp}(I) \setminus \mathrm{Max}(I)$.
(b). Suppose that $P \in D(I)$ and Q is a prime ideal contained in P, hence $Q \in D(I)$. Consequently, $P \setminus I \in \mathrm{Maxp}(I)$ and $Q \cap I \in \mathrm{Maxp}(I)$. Hence, $P \setminus I = Q \setminus I$ implies that $P = Q$ and consequently, $P \in \mathrm{Min}(R)$. The converse is clear. $\square$

Remark 2.3. *Let I and J be two ideals of a ring R and $I \subseteq J$. If J is a* P*-ideal (resp.,* PMP*-ideal), then J/I is a* P*-ideal (resp.,* PMP*-ideal) of the ring R/I. The converse is true if I is a* P*-ideal (resp.,* PMP*-ideal).*

We remind the reader that, for any ideal I of a ring R, the radical of I is the ideal $\sqrt{I}$ defined by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. Also I is called a semiprime ideal whenever $I = \sqrt{I}$. In the following remark, we observe that for investigating P-ideals and PMP-ideals it is enough to consider semiprime ideals.

Remark 2.4. *Let R be a ring and I be an ideal of R. Then $D(I) = D(\sqrt{I})$, hence I is a* P*-ideal (resp.,* PMP*-ideal) if and only if $\sqrt{I}$ is a* P*-ideal (resp.,* PMP*-ideal). Moreover, if $J \subseteq I$, then $D(J) \subseteq D(I)$ and consequently I is a* P*-ideal (resp.,* PMP*-ideal) if and only if every ideal contained in I is too, and this is equivalent to the fact that aR is a* P*-ideal for any $a \in I$.*

Proposition 2.5. *The sum of any family of* P*-ideals (resp.,* PMP*-ideals) of a ring R is a* P*-ideal (resp.,* PMP*-ideal).*

Proof. By the inclusion $D(\sum_{\lambda \in \Lambda} I_\lambda) \subseteq \cup_{\lambda \in \Lambda} D(I_\lambda)$, the proof is clear. $\square$

The previous remark follows that the largest P-ideal (resp., PMP-ideal) of R exists. We denote this largest ideal by $\mathrm{P}(R)$ (resp., $\mathrm{PMP}(R)$). It is obvious to see that if I is an ideal of R, then $I \setminus \mathrm{P}(R)$, (resp., $I \setminus \mathrm{PMP}(R)$) is the largest P-ideal (resp., PMP-ideal) contained in I. Also, J is the largest P-ideal of a ring R if and only if J is a P-ideal (resp., PMP-ideal) and R/J has no nonzero P-ideal (resp., PMP-ideal).
Here, a natural question arises: Is the largest P-ideal (resp., PMP-ideal) in a ring R (or in an ideal of R) a prime ideal? The answer is no. To see this, suppose that the topological space X has no P-point. It is enough to prove $\mathrm{PMP}(\mathrm{C}(X)) = (\circ)$. Assume that I is a nonzero ideal

of $\mathrm{C}(X)$. By our hypothesis, there exists $\circ \neq f \in I$. Thus, there exists $x \in \mathrm{Coz}(f)$. Clearly, $I \nsubseteq M_x(X)$; i.e., $M_x(X) \in D(I)$. Since x is not a P-point, we infer that $M_x(X)$ is not a minimal prime ideal and hence I is not a PMP-ideal. This example, also, shows that if I is a P-ideal (resp., PMP-ideal) and $P \in \mathrm{Min}(I)$, then P is not necessarily a P-ideal (resp., PMP-ideal).

Proposition 2.6. *Let R be a ring. Then*
(a) $\mathrm{P}(R) = \backslash_{P \in \mathrm{Min}(R)\ Max(R)} P$.
(b) $\mathrm{PMP}(R) = \backslash_{P \in \mathrm{Spec}(R)\ \mathrm{Min}(R)} P$.

Proof. (a). We show that $J_0 = \backslash_{P \in \mathrm{Min}(R)\ \mathrm{Max}(R)} P$ is a P-ideal. Clearly, $D(J_0) \subseteq \mathrm{Max}(R)$ and so by part (a) of Theorem 2.2, J_0 is a P-ideal. Now, suppose that I is a P-ideal. Thus, $D(I) \subseteq \mathrm{Max}(R)$. It follows that $\mathrm{Min}(R) \setminus \mathrm{Max}(R) \subseteq V(I)$ and so $I \subseteq \backslash_{P \in \mathrm{Min}(R)\ \mathrm{Max}(R)} P = J_0$.
(b). It is similar to the proof of part (a). □

Remark 2.7. *Let I and J be two ideals of R. Then*
(a) $IJ = (0)$ *if and only if* $D(J) \subseteq V(I)$.
(b) *If* $M_1, \cdots, M_n \in \mathrm{Max}(R)$, *then* $\mathcal{A}(\backslash_{i=1}^{n} M_i)$ *is a P-ideal.*

Corollary 2.8. *If $R/\mathcal{A}(I)$ is a regular ring, then I is a* P*-ideal.*

Proof. Since $R/\mathcal{A}(I)$ is a regular ring, it follows that $V(\mathcal{A}(I)) \subseteq \mathrm{Max}(R)$ and so by part (a) of the above remark we are done. □

The converse of Corollary 2.8 is not true. Note that if $R = \prod_{i=1}^{n} R_i$ and I_i is an ideal of R_i for every $i = 1, \cdots, n$, then $I = \prod_{i=1}^{n} I_i$ is a P-ideal of R if and only if I_i is so in R_i for every $i = 1, \cdots, n$. Now assume that $R = F \times F \times \mathbb{Z}$ where F be a field. If we let $I = (\circ) \times F \times \mathbb{Z}$ and $J = F \times (\circ) \times \mathbb{Z}$, then $K = \mathcal{A}(I \setminus J)$ is a P-ideal, but $\frac{R}{\mathcal{A}(K)}$ is not a regular ring.

The following result shows that the converse of the above corollary is true, if I is a summand.

Corollary 2.9. *Suppose that an ideal I of R is summand. Then the following statements are equivalent:*
(a) *I is a* P*-ideal.*

(b) *$R/\mathcal{A}(I)$ is a regular ring.*
(c) *I is a regular ideal.*

Proof. Since I is summand, it follows that $I \simeq R/\mathcal{A}(I)$. Thus, it suffices to show that (a) and (b) are equivalent. To see this, by our hypothesis, there exists an ideal J of R such that $R = I \oplus J$. Clearly, $D(I) = V(J) = V(\mathcal{A}(I))$ and by this fact the proof is evident. □

Corollary 2.10 *Let R be a reduced ring. If R has a maximal ideal which is a* PMP*-ideal, then every prime ideal is a minimal or maximal ideal. (i.e.,* $\dim(R) \leqslant 1$*).*

Now, we investigate some connections between annihilator ideals and P-ideals. First, we recall the following well-known fact, see [12, Lemma 11. 40].

Lemma 2.11. *Let R be a reduced ring and I be an ideal of R, then* $\mathcal{A}(I) = \bigcap_{P \in \mathrm{Min}(R) \cap D(I)} P = \bigcap_{P \in D(I)} P$.

Proposition 2.12. *Let R be a ring and I be an ideal of R.*
(a) *If $\mathcal{A}(I)$ is the intersection of finitely many maximal ideals, then I is a* P*-ideal.*
(b) *If R is reduced and I is a* P*-ideal, then $\mathcal{A}(I)$ is the intersection of a family of maximal ideals.*

Proof. (a). Suppose that $\mathcal{A}(I) = \backslash_{i=1}^{n} M_i$ where $M_i \in \mathrm{Max}(R)$ for any $i = 1, \cdots, n$. Let $P \in D(I)$, since $\mathcal{A}(I) = \backslash_{i=1}^{n} M_i \subseteq P$, there exists $1 \leqslant i \leqslant n$, such that $M_i \subseteq P$. Hence by the maximality of M_i, it follows that $P = M_i$. Therefore, by part (a) of Theorem 2.2, I is a P-ideal.
The proof of part (b) is clear, by the above lemma. □

Corollary 2.13. *Suppose that R is a semilocal (i.e., ring which has only finitely many maximal ideals) reduced ring, then I is a* P*-ideal if and only if $\mathcal{A}(I)$ is the intersection of finitely many maximal ideals.*

The converse of part **(a)** of Proposition 2.12 is not true in general (even if $\mathcal{A}(I)$ is also an intersection of finitely many minimal prime ideal). For instance assume that I is a nonzero ideal of the ring $\mathbb{Z}$. Then $\mathcal{A}(I) = (\circ)$ is a minimal prime ideal and also is the intersection of infinitely many maximal ideals; while I is not a P-ideal.

3. Von Neumann Regularity, Pure Ideals and P-Ideals (PMP-Ideals)

In this section we observe that in every reduced ring, P-ideals and regular ideals coincide. We also show that every P-ideal in a reduced ring is a z°-ideal. Finally, we prove that an ideal I in a reduced ring is a P-ideal if and only if it is a pure PMP-ideal.

Proposition 3.1. *For a reduced ring R the following statements are equivalent:*
(a) *R is a regular ring.*
(b) *Every ideal I of R is a* P*-ideal and $\frac{R}{I}$ is a regular ring.*
(c) *There exists an ideal I such that I is a* P*-ideal and $\frac{R}{I}$ is a regular ring.*

Proof. It is evident. □

Proposition 3.2. *Let R be a ring, $a \in R$ and $S = \{a^n : n \in \mathbb{N}_0\}$. Then the ideal aR is a* P*-ideal if and only if* $\mathrm{Spec}(S^{-1}R) = \mathrm{Max}(S^{-1}R)$.

Proof. Since there exists an order isomorphism between $D(aR)$ and $\mathrm{Spec}(S^{-1}R)$, the proof is obvious. □

Let $P \in \mathrm{Spec}(R)$, we define $O(P) = \{a \in R : \mathcal{A}(a) \nsubseteq P\}$. The following theorem shows that this concept is closely related to the concept of pure ideal.

Theorem 3.3. *Suppose that R is a ring, $Q \in \mathrm{Spec}(R)$ and $\mathcal{B} = \{P \in \mathrm{Min}(R) : P \subseteq Q\}$. Then*
(a) $m(Q) \subseteq O(Q) \subseteq \backslash_{P\in\mathcal{B}}P$.
(b) *If Q is a pure ideal, then $Q \in \mathrm{Min}(R)$.*
(c) *If Q is a maximal ideal, then $m(Q) = O(Q)$.*
Furthermore, if R is reduced, then
(d) $O(Q) = \backslash_{P\in\mathcal{B}}P$.
(e) *If $Q \in \mathrm{Max}(R)$, then $m(Q) = O(Q) = \backslash_{P\in\mathcal{B}}P$.*

(f) *If $Q \in \mathrm{Max}(R)$, then Q is a pure ideal if and only if $Q \in \mathrm{Min}(R)$.*

Proof. (a). Let $a \in m(Q)$, then there exists $q \in Q$ such that $a = aq$, hence $a(1-q) = \circ$. Therefore, $\mathcal{A}(a) \nsubseteq Q$ and so $a \in O(Q)$. Now, suppose that $a \in O(Q)$, hence $\mathcal{A}(a) \nsubseteq P$ for any $P \in \mathcal{B}$. This implies that $a \in P$ for any $P \in \mathcal{B}$, and consequently $a \in \backslash_{P \in \mathcal{B}} P$.
(b). By part (a), it is clear.
(c). Suppose that $Q \in \mathrm{Max}(R)$ and $a \in O(Q)$. Clearly

$$a \in O(Q) \Leftrightarrow \mathcal{A}(a) \nsubseteq Q \Leftrightarrow Q + \mathcal{A}(a) = R \Leftrightarrow a \in m(Q).$$

(d). Let $a \in \backslash_{P \in \mathcal{B}} P$ and $S = R \backslash Q$. It is clear that $\frac{a}{1} \in \mathrm{Rad}(S^{-1}R)$. This implies that there exists a natural number n such that $(\frac{a}{1})^n = \circ$. Hence there exists $s \in S$ such that $sa^n = \circ$. Therefore, $\mathcal{A}(a) = \mathcal{A}(a^n) \nsubseteq Q$ and so $a \in O(Q)$.
(e). and (f) are obvious. □

The next proposition is a counterpart of Theorem 2.4 in [3], which we use it in the sequel.

Proposition 3.4. *An element $a \in R$ is regular if and only if for every $M \in \mathrm{Max}(R)$ with $a \in M$, we have $a \in m(M)$.*

Recall that an ideal in a ring R is called z-ideal (resp., z°-ideal) whenever $\mathbf{M}_a \subseteq I$ (resp., $\mathbf{P}_a \subseteq I$) for any $a \in I$. For more details and examples of z-ideals and z°-ideals in reduced commutative rings and in $\mathrm{C}(X)$ the reader is referred to [4], [6] and [7]. In the following theorem, we show that in reduced rings, regular ideals and P-ideals coincide.

Theorem 3.5. *Let R be a reduced ring and I is an ideal of R. Consider the following conditions:*
(a) *I is a P-ideal.*
(b) *I is a regular ideal.*
(c) *aR is a semiprime ideal for any $a \in I$.*
(d) *$\mathbf{P}_a = aR$ for any $a \in I$.*
(e) *aR is a z°-ideal for any $a \in I$.*
(f) *$\mathbf{M}_a = aR$ for any $a \in I$.*
(g) *aR is an intersection of maximal ideals for any $a \in I$.*

(h) *aR is a z-ideal for any $a \in I$.*
Then parts **(a)**, **(b)**, **(c)**, **(d)** *and* **(e)** *are equivalent, and if* $\mathrm{Jac}(R) = (\circ)$, *then all of the above conditions are equivalent.*

Proof. By Proposition 1.8 and definitions of z-ideal and z°-ideal, it suffices to prove that (a) and (b) are equivalent.
(a) $\Rightarrow$ (b). Suppose that $M \in \mathrm{Max}(R)$ and $a \in I \setminus M$. By Proposition 3.4, it suffices to show that $a \in m(M)$. On the other hand, by Theorem 3.3, we have $m(M) = O(M) = \bigcap\{P \in \mathrm{Min}(R) : P \subseteq M\}$. Thus, it is enough to show that $a \in \bigcap\{P \in \mathrm{Min}(R) : P \subseteq M\}$. Let $P \in \mathrm{Min}(R)$ and $P \subseteq M$, we must show that $a \in P$. This is clear, for on the contrary, we have $P \in D(I)$ and consequently $P \in \mathrm{Max}(R)$ which is a contradiction.
(b) $\Rightarrow$ (a). Suppose that $P \in D(I)$, by Theorem 2.2, we must show that $P \in \mathrm{Max}(R)$. Let $a \notin P$; since $P \in D(I)$, there exists an $i \in I \setminus P$. Clearly, $ai \in I \setminus P$ and by assumption, there exists $r \in R$ such that $ai = (ai)^2 r$. Hence, $ai(1 - air) = \circ \in P$ and so $1 - air \in P$ which imlies that $P + aR = R$. $\square$.

Corollary 3.6. *Every* P*-ideal in a reduced ring is a z°-ideal.*

The following proposition and theorem show the connection between P-ideals, PMP-ideals and pure ideals.

Proposition 3.7. *Let R be a reduced ring. Then*
(a) *$a \in R$ is regular if and only if aR is a pure ideal.*
(b) *I is a* P*-ideal if and only if every ideal contained in I is a pure ideal.*

Proof. (a $\Rightarrow$). Suppose that $x = ar \in I = aR$. By our hypothesis, there exists $s \in R$ such that $a = a^2 s$ and so $x = ar = a^2 sr \in aI$.
(a $\Leftarrow$). Since $I = aR$ is pure and $a \in I$, it follows that $a = a(ar) = a^2 r$ for some $r \in R$.
(b). By part (a), it is easy. $\square$

Theorem 3.8. *Let R be a reduced ring and I be an ideal of R. Then I is a* P*-ideal if and only if it is a pure* PMP*-ideal.*

Proof. ($\Rightarrow$). It is clear.
($\Leftarrow$). By Theorem 3.5, it is enough to show that I is a regular ideal. To see

this, let $a \in I$, by Proposition 3.4, it is enough to show that whenever $a \in M \in \text{Max}(R)$ then $a \in m(M)$. For this, let $a \in M$. If $M \in D(I)$, then by Theorem 2.2, we have $M \in \text{Min}(R)$ and so by part (f) of Theorem 3.3, M is a pure ideal. Hence, $a \in M = m(M)$. If $M \notin D(I)$, then $I \subseteq M$ and so by the purity of I, we have $a \in I = m(I) \subseteq m(M)$. $\square$

It is clear that a reduced ring R is regular if and only if every ideal of R is a P-ideal. In the next theorem we give a similar assertion for PMP-ideals.

Theorem 3.9. *Every proper ideal in a reduced ring R is a* PMP*-ideal if and only if R is regular or a local ring with* $\dim(R) = 1$.

Proof. ($\Rightarrow$). Assume that R is not regular. Hence, there exist $M_0 \in \text{Max}(R)$ and $P \in \text{Spec}(R)$ such that $P \subsetneq M_0$. It is enough to show that $\text{Max}(R) = \{M_0\}$ and $P \in \text{Min}(R)$. Let $M \in \text{Max}(R)$, since M is a PMP-ideal, by part (b) of Theorem 2.2, we have $M \subseteq M_0$ and hence $M = M_0$. This implies that $\text{Max}(R) = \{M_0\}$. Now, suppose that $Q \in \text{Spec}(R)$ and $Q \subseteq P$. Since M_0 is a PMP-ideal, by part (b) of Theorem 2.2, we conclude that $P = Q$. This implies that $P \in \text{Min}(R)$.

($\Leftarrow$). It is clear. $\square$

The following result shows that the existence of a maximal P-ideal or a pure maximal PMP-ideal in a reduced ring R implies that R is a regular ring.

Theorem 3.10. *Let R be a reduced ring. Then the following statements are equivalent:*

(a) *R is a regular ring.*

(b) *There exists an ideal $M \in \text{Max}(R)$ which is a* **P***-ideal.*

(c) *There exists a pure ideal $M \in \text{Max}(R)$ which is a* PMP*-ideal.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

($c \Rightarrow a$). Suppose that $M \in \text{Max}(R)$ is a pure PMP-ideal and $M \neq N \in \text{Max}(R)$. Clearly, $N \in D(M)$ and so $N \in \text{Min}(R)$. $\square$

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