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Domination Polynomial of Generalized Book Graphs

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Abstract. Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x) = \sum_{i=0}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Let n be any positive integer and B_n be the n-book graphs, formed by joining n copies of the cycle graph C_4 with a common edge. In this paper, we study the domination polynomials of some generalized book graphs. In particular we examine the domination roots of these families, and find the limiting curve for the roots.

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1. Introduction

Let $G = (V,E)$ be a simple graph. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) | \{u,v\} \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$

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is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. Let $\mathcal{D}(G,i)$ be the family of dominating sets of graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The domination polynomial $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G (see [1, 8]). This graph polynomial was introduced in the paper [5] that appeared in 2014 but numerous other papers on the polynomial appeared earlier. For some very recent developments on the polynomial see for example [4, 6].

A root of $D(G,x)$ is called a domination root of G. The set of distinct roots of $D(G,x)$ is denoted by $Z(D(G,x))$. Calculating the domination polynomial of a graph G is difficult in general, as the smallest power of a non-zero term is the domination number $\gamma(G)$ of the graph, and determining whether $\gamma(G) \leqslant k$ is known to be NP-complete [9]. But for certain classes of graphs, we can find a closed form expression for the domination polynomial.

In the next section, we consider generalized book graphs and compute their domination polynomials. Also we explore the nature and location of roots of domination polynomial of generalized book graphs in Section 3.

2. Domination Polynomial of Generalized Book Graphs

A book graph B_n, is defined as follows $V(B_n) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \leqslant i \leqslant n\}$ and $E(B_n) = \{u_1u_2\} \cup \{u_1v_i,\ u_2w_i,\ v_iw_i : 1 \leqslant i \leqslant n\}$. We consider the generalized book graph $B_{n,m}$ with vertex and edge sets by $V(B_{n,m}) = \{u_i : 1 \leqslant i \leqslant m-2\} \cup \{v_i, w_i : 1 \leqslant i \leqslant n\}$ and $E(B_{n,m}) = \{u_iu_{i+1} : 1 \leqslant i \leqslant m-3\} \cup \{u_iw_j : 1 \leqslant j \leqslant n,\ i = m-2\} \cup \{u_1v_i : 1 \leqslant i \leqslant n\} \cup \{v_iw_i : 1 \leqslant i \leqslant n\}$ (see Figure 1).

We need some preliminaries to study the domination polynomial of some generalized book graphs. The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see [12]). The following theorem is useful for finding the recurrence relations for the domination polynomials of graphs.

Figure 1. Graphs B_n and $B_{n,5}$, respectively.

Theorem 2.1. ([2, 10]) *Let G be a graph. For any vertex u in G we have*

$$D(G,x) = xD(G/u,x) + D(G-u,x) + xD(G-N[u],x) - (1+x)p_u(G,x),$$

where $p_u(G,x)$ is the polynomial counting the dominating sets of $G-u$ which do not contain any vertex of $N(u)$ in G.

The following theorem gives formula for the domination polynomial of B_n.

Theorem 2.2. ([3]) *For every $n \in \mathbb{N}$,*

$$D(B_n,x) = (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n.$$

It is interesting that for each natural number $n \geqslant 3$, the graph B_n/v (vertex v is in the common edge of B_n) is not $\mathcal{D}$-unique, that is, there is another non-isomorphic graph with the same domination polynomial. Because B_n/v and the friendship graph F_n have the same domination polynomial ([3]).

Here we compute domination polynomial of the book graphs $B_{n,5}$. We need some preliminaries. For two graphs $G = (V,E)$ and $H = (W,F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with

$|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H. The following theorem gives the domination polynomial of graphs of the form $H \circ K_1$, which is needed to obtain our result.

Theorem 2.3. ([1]) *Let G be a graph. Then $D(G, x) = (x^2 + 2x)^n$ if and only if $G = H \circ K_1$ for some graph H of order n.*

Given any two graphs G and H we define the *Cartesian product*, denoted $G\Box H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$. This product is well known to be commutative.

Theorem 2.4. ([11]) *The domination polynomial for $K_n\Box K_2$ is*

$$D(K_n\Box K_2, x) = ((1 + x)^n - 1)^2 + 2x^n.$$

The following theorem gives formula for the domination polynomial of $B_{n,5}$.

Theorem 2.5. *For every $n \in \mathbb{N}$,*

$$D(B_{n,5}, x) \quad = x^2(x + 1)^{2n+1} - 2x^{n+1} + (x^2 + 2x)^n(2x^2 + 3x).$$

Proof. Consider graph $B_{n,5}$ in Figure 1. By Theorems 2.1 and 2.2 we have:

$$\begin{aligned}
D(B_{n,5}, x) &= xD(B_{n,5}/v, x) + D(B_{n,5} - v, x) + xD(B_{n,5} - N[v], x) \\
&\quad -(1 + x)p_v(B_{n,5}, x) \\
&= xD(B_n, x) + D(B_{n,5} - v, x) + x(D(\cup_{i=1}^{n} K_2, x)) \\
&\quad -(1 + x)[(x^2 + 2x)^n - 2x^n] \\
&= x[(x^2 + 2x)^n(2x + 1) + x^2(x + 1)^{2n} - 2x^n] + D(B_{n,5} - v, x) \\
&\quad +x(x^2 + 2x)^n - (1 + x)[(x^2 + 2x)^n - 2x^n] \\
&= x^3(x + 1)^{2n} + 2x^n + D(B_{n,5} - v, x) \\
&\quad +(x^2 + 2x)^n(2x^2 + x - 1). \qquad (1)
\end{aligned}$$

graph $B_{n,5} - v = G$ (see Figure 2). We have

$$\begin{aligned} D(B_{n,5} - v, x) &= xD(G/u, x) + D(G - u, x) \\ &\quad + xD(G - N[u], x) - (1 + x)p_u(G, x) \\ &= xD(G/u, x) + D(G - u, x) + xD(K_{1,n}, x) \\ &\quad -(1 + x)(x^n(1 + x)) \\ &= xD(G/u, x) + D(G - u, x) \\ &\quad + x(x^n + x(1 + x)^n) - x^n(1 + x)^2, \end{aligned} \tag{2}$$

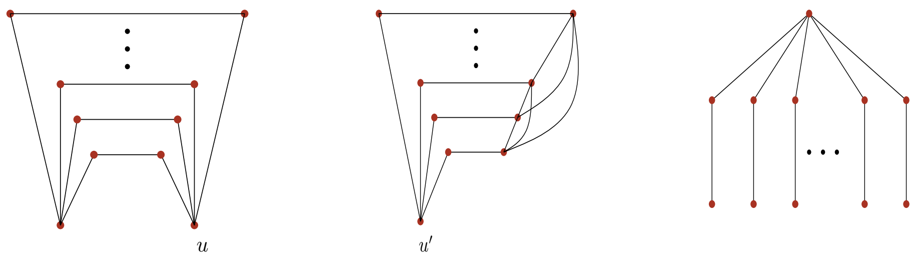


Figure 2. Graphs $B_{n,5} - v$, $(B_{n,5} - v)/u$ and $B_{n,5} - v - u$, respectively.

where $D(K_{1,n}, x) = x^n + x(1 + x)^n$. We use Theorems 2.1 and 2.3 to obtain the domination polynomial of the graph $B_{n,5} - v - u = G$ (see Figure 2). We have

$$\begin{aligned} D(B_{n,5} - v - u, x) &= xD(G/w, x) + D(G - w, x) + xD(G - N[w], x) \\ &\quad -(1 + x)p_w(G, x) \\ &= xD(K_n \circ K_1, x) + D(\cup_{i=1}^{n} P_2, x) \\ &\quad + x(x^n) - (1 + x)x^n \\ &= (2x + x^2)^n(x + 1) - x^n. \end{aligned} \tag{3}$$

Using Theorems 2.1, 2.3 and 2.4, we obtain the domination polynomial of the graph $(B_{n,5} - v)/u = G$ (see Figure 2). We have

$$\begin{aligned} D((B_{n,5} - v)/u, x) &= xD(G/u', x) + D(G - u', x) \\ &\quad + xD(G - N[u'], x) - (1 + x)p_{u'}(G, x). \end{aligned}$$

Since $G/u' = K_n \square K_2$, $G - u' = K_n \circ K_1$ and $G - N[u'] = K_n$, we have

$$\begin{aligned} D((B_{n,5}-v)/u,x) &= xD(K_n\square K_2,x) + D(K_n \circ K_1,x) \\ &\quad + xD(K_n,x) - (1+x)x^n \\ &= x((1+x)^n-1)^2 + 2x^n) + (2x+x^2)^n \\ &\quad + x((1+x)^n-1) - (1+x)x^n \\ &= x((1+x)^n-1)(1+x)^n \\ &\quad + (2x+x^2)^n + x^n(x-1). \end{aligned} \tag{4}$$

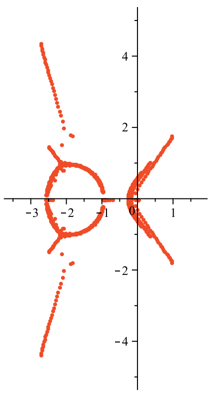
By equations (2), (3) and (4), we have:

$$D(B_{n,5}-v,x) = (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n(1+x).$$

Consequently, by this equation and equation (1), we have:

$$\begin{aligned} D(B_{n,5},x) &= x^3(x+1)^{2n} + 2x^n + (x^2+2x)^n(2x^2+x-1) \\ &\quad + (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n(1+x) \\ &= x^2(x+1)^{2n+1} - 2x^{n+1} + (x^2+2x)^n(2x^2+3x). \quad \square \end{aligned}$$

Figure 3 shows the domination roots of book graphs $B_{n,5}$ and B_n for $n \leqslant 30$, respectively.



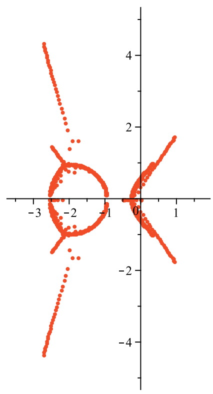


Figure 3. Domination roots of graphs $B_{n,5}$ and B_n, for $1 \leqslant n \leqslant 30$, respectively.

3. Limits of Domination Roots of Book Graphs B_n and $B_{n,5}$.

In this section, we consider the complex domination roots of book graphs. The plot in Figure 3 suggest that the roots tend to lie on a curve. In order to find the limiting curve, we will need a definition and a well known result.

Definition 3.1. *If $f_n(x)$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{R}(f_n(x))$, where $\mathbb{R}(f_n(x))$ is the union of the roots of the $f_n(x)$.*

The following restatement of the Beraha-Kahane-Weiss theorem can be found in [7].

Theorem 3.2. *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + ... + \alpha_k(x)\lambda_k(x)^n \tag{5}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either

(i) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(ii) for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.

The following Theorem gives the limits of the domination roots of book graphs B_n.

Theorem 3.3. *The limit of domination roots of book graphs are $x = -\frac{1}{2}$ and $x = 0$ together with the part of the circle $|x+2| = 1$ with real part at least $-\dfrac{3}{2} - \dfrac{\sqrt{2}}{2}$, the portions of the hyperbola $(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}$, $\Re(x) \notin [\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2}]$, plus the portion of the curve $|x+1|^2 = |x|$ with real part at most $-\dfrac{3}{2} - \dfrac{\sqrt{2}}{2}$.*

Proof. By Theorem 2.2, the domination polynomial of B_n is,

$$\begin{aligned} D(B_n, x) &= (2x+1)(x^2+2x)^n + x^2(x+1)^{2n} - 2x^n \\ &= \alpha_1(x)\lambda_1^n(x) + \alpha_2(x)\lambda_2^n(x) + \alpha_3(x)\lambda_3^n(x), \end{aligned}$$

where $\alpha_1(x) = 2x+1$, $\lambda_1(x) = x^2+2x$, $\alpha_2(x) = x^2$, $\lambda_2(x) = (x+1)^2$, and $\alpha_3(x) = -2$, $\lambda_3(x) = x$. Clearly α_1, α_2 and α_3 are not identically zero. Also, no $\lambda_i = \omega\lambda_j$ for $i \neq j$ and a complex number ω of modulus 1. Therefore, the initial conditions of Theorem 3.2 are satisfied.

Now, applying part (i) of Theorem 3.2, we consider four different cases: (i) $|\lambda_1| = |\lambda_2| = |\lambda_3|$, (ii) $|\lambda_1| = |\lambda_2| > |\lambda_3|$, (iii) $\lambda_1| = |\lambda_3| > |\lambda_2|$ and (iv) $|\lambda_2| = |\lambda_3| > |\lambda_1|$.

Case (i)**:** Assume that $|x^2+2x| = |(x+1)^2| = |x|$. Then $|x^2+2x| = |x|$ implies that x lies on the unit circle centered -2 ($|x-(-2)| = 1$) and $|x^2+2x| = |(x+1)^2|$ by setting $y = x+1$, that is, $|y^2-1| = |y^2|$. To find this curve, let $a = \Re(y)$ and $b = \Im(y)$. Then by substituting in $y = a+ib$ and squaring both sides, we have $(a^2-1-b^2)^2 + (2ab)^2 = (a^2-b^2)^2 + (2ab)^2$. This is equivalent to $a^2 - b^2 = \frac{1}{2}$, a hyperbola. Hence, we converting back to variable x, we have the following hyperbola

$$(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}.$$

Now suppose that $|(x+1)^2| = |x|$, this curve is semi-cardioid which has shown in Figure 4. Therefore, the two points of intersection, $\frac{-3-\sqrt{2}}{2} \pm \frac{\sqrt{1+2\sqrt{2}}}{2}i$, are limits of roots.

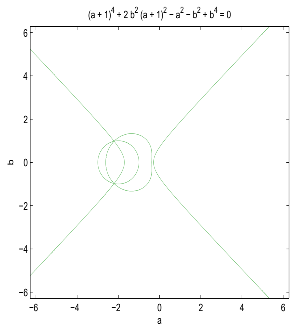


Figure 4. The curves in case (i) in the proof of Theorem 3.3.

Case (ii)**:** Assume that $|x^2+2x| = |(x+1)^2| > |x|$. Then $|x^2+2x| = |(x+1)^2|$ implies that x lies on the hyperbola $(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}$. And $|x^2+2x| > |x|$ implies that x lies outside the unit circle centered -2 ($|x-(-2)| = 1$), and $|(x+1)^2| > |x|$ implies that x lies outside the curve $|(x+1)^2| = |x|$. Therefore, the complex numbers x that satisfy

$$(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}, \quad \Re(x) \notin [\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2}],$$

are limits of roots.

Case (iii)**:** Assume that $|x^2+2x| = |x| > |(x+1)^2|$. Then $|x^2+2x| = |x|$ implies that x lies on the unit circle centered -2 ($|x-(-2)| = 1$) and $|x^2+2x| > |(x+1)^2|$ implies that x satisfy in the following inequality

$$(\Re(x)+1)^2 - (\Im(x))^2 < \frac{1}{2}.$$

The inequality $|x| > |(x+1)^2|$ implies that x lies inside the curve $|(x+1)^2| = |x|$. Therefore, the complex numbers x that satisfy $|x-(-2)| = 1$ with real part at least $\frac{-3-\sqrt{2}}{2}$ are limits of roots.

Case (iv)**:** Assume that $|(x+1)^2| = |x| > |x^2+2x|$. As we observed before, the equality $|(x+1)^2| = |x|$ is semi-cardioid which has shown in Figure 4. The inequality $|x| > |x^2+2x|$ implies that x lies inside the unit circle centered -2 ($|x-(-2)| = 1$), and $|(x+1)^2| > |x^2+2x|$ implies that x satisfy in the following inequality

$$(\Re(x)+1)^2 - (\Im(x))^2 > \frac{1}{2}.$$

Therefore, the complex numbers x that satisfy on the curve $|(x+1)^2| = |x|$ with real part at most $\frac{-3-\sqrt{2}}{2}$ are limits of roots.

Finally by Part (ii) of Theorem 3.2, since α_3 is never 0, and $\alpha_2 = 0$ iff $x = 0$, in this case $|\lambda_2(0)| = |1| > 0 = |\lambda_1(0)| = |\lambda_3(0)|$, and $\alpha_1 = 0$ iff $x = -\frac{1}{2}$, and also in this case $|\lambda_1(-\frac{1}{2})| = |-\frac{3}{4}| > \frac{1}{4} = |\lambda_2(-\frac{1}{2})|$ and $|\lambda_1(-\frac{1}{2})| = |-\frac{3}{4}| > \frac{1}{2} = |\lambda_3(-\frac{1}{2})|$, so we conclude $x = 0$ and $x = -\frac{1}{2}$ are limit of domination roots of book graphs.

The union of the curves and points above yield the desired result.

Along the same lines, we can show:

Theorem 3.4. *The limit of roots of the domination polynomial of the book graphs* $B_{n,5}$*, consist of the part of the circle* $|x+2| = 1$ *with real part at least* $-\dfrac{3}{2} - \dfrac{\sqrt{2}}{2}$*, the portions of the hyperbola* $(\Re(x)+1)^2 - (\Im(x))^2 = \frac{1}{2}$*,* $\Re(x) \notin [\frac{-3-\sqrt{2}}{2}, \frac{-2-\sqrt{2}}{2}]$*, plus the portion of the curve* $|x+1|^2 = |x|$ *with real part at most* $-\dfrac{3}{2} - \dfrac{\sqrt{2}}{2}$.

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