Journal of Mathematical Extension
Vol. 11, No. 1, (2017), 1-13
ISSN: 1735-8299
URL: http://www.ijmex.com

Fourier-Dunkl Dini Lipschitz Functions in the Space $L^p_{\alpha,n}$

R. DAHER
University Hassan II

S. EL OUADIH*
University Hassan II

Abstract. In this paper, we consider the generalized Fourier-Dunkl transform associated with the Dunkl operator on $\mathbb{R}$ and we give condition of quite different kind for a function to have a transform belonging to certain $L_p-classes$.

AMS Subject Classification: 42B37
Keywords and Phrases: Differential-difference operator, generalized Fourier-Dunkl transform, generalized translation operator

1. Introduction

Theorems 5.1 and 5.2 in Younis [5] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have the following theorem.

Theorem 1.1. ([5]) *Let $f \in L^2(\mathbb{R})$. Then the following are equivalents*
(a) $\|f(x+h) - f(x)\| = o\left((\log \frac{1}{h})^{-1}\right), \quad as \quad h \to 0,$
(b) $\displaystyle\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^2 d\lambda = o\left((\log r)^{-1}\right), \quad as \quad r \to \infty,$

Received: August 2015; Accepted: January 2016
*Corresponding author

where $\widehat{f}$ stands for the Fourier transform of f.

Theorem 1.2. ([5]) *Let $f \in L^2(\mathbb{R})$. Then the following are equivalents*
(a) $\|f(x+h)-f(x)\| = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)$, *as* $h \to 0, 0<\delta<1, \gamma \geqslant 0$

(b) $\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$, *as* $r \to \infty$,

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we consider a first-order singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the Dunkl operator Λ_α. We prove an analog of Theorems 1.1 and 1.2 in the generalized Fourier-Dunkl transform associated to Λ in $L^p_{\alpha,n} := L^p(\mathbb{R}, |x|^{2\alpha+2n(2-p)+1}dx)$. For this purpose, we use a generalized translation operator.
In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [1] and [6]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.
Consider the first-order singular differential-difference operator Λ defined on $\mathbb{R}$ by

$$\Lambda f(x) = f'(x) + \left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x} - 2n\frac{f(-x)}{x}.$$

For $n=0$, we define the differential-difference operator Λ_α by

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha+1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics.

Define $L^p_{\alpha,n}$, $1 \leqslant p \leqslant \infty$, as the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha,n} < \infty$, where

$$\|f\|_{p,\alpha,n} = \left(\int_{\mathbb{R}} |f(x)|^p x^{2\alpha+2n(2-p)+1}\right)^{1/p}, \quad \text{if} \quad p<\infty,$$

and $\|f\|_{\infty,\alpha,n} = \|f\|_\infty = ess\sup_{x\in\mathbb{R}} |f(x)|$.
If $p = 2$, then we have $L^2_{\alpha,n} = L^2(\mathbb{R}, |x|^{2\alpha+1})$.
The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), z \in \mathbb{C}, \tag{1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\Gamma(m+\alpha+1)}, z \in \mathbb{C}, \tag{2}$$

is the normalized spherical Bessel function of index α. It is well-known that the functions e_α are the solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, u(0) = 1.$$

From (2) we see that

$$\lim_{z\to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0. \tag{3}$$

Hence, there exists $c > 0$ and $\eta > 0$ satisfying

$$|z| \leqslant \eta \Rightarrow |j_\alpha(z) - 1| \geqslant c|z|^2.$$

Lemma 1.3. *For $x \in \mathbb{R}$ the following inequalities are fulfilled*

(i) $|j_\alpha(x)| \leqslant 1$,

(ii) $|1 - j_\alpha(x)| \leqslant x^2/2$,

(iii) $|1 - j_\alpha(x)| \geqslant c$ *with* $|x| \geqslant 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof. Similarly as the proof of Lemma 2.9 in [2]. □

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x).$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).

Proposition 1.4. *(i) φ_λ satisfies the differential equation*

$$\Lambda\varphi_\lambda = i\lambda\varphi_\lambda.$$

(ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leqslant |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform that we call it the integral transform is defined by

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 1.5. *(i) For every $f \in L^p_{\alpha,n}$,*

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda\mathcal{F}_\Lambda(f)(\lambda).$$

(ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

By Plancherel equality and Marcinkiewics interpolation Theorem (see [7]) we get for $f \in L^p_{\alpha,n}$ with $1 \leqslant p \leqslant 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_\Lambda(f)\|_{q,\alpha+2n} \leqslant K\|f\|_{p,\alpha,n}, \tag{4}$$

where K is a positive constant.
The generalized translation operators τ^x, $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned}\tau^x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1}\frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt\\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1}\frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt,\end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)}(1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 1.6. *Let f be in $L^p_{\alpha,n}$, $1 \leqslant p \leqslant \infty$. Then for all $x \in \mathbb{R}$, the function $\tau^x f$ belongs to $L^p_{\alpha,n}$, and*

$$\|\tau^x f\|_{p,\alpha,n} \leqslant 2x^{2n}\|f\|_{p,\alpha,n}.$$

Furthermore,

$$\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n} e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \tag{5}$$

2. Dini-Lipschitz Condition

Definition 2.1. *Let $f \in L^p_{\alpha,n}$, $1 \leqslant p \leqslant \infty$, and define*

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leqslant C\frac{h^{\eta+2n}}{(\ln\frac{1}{h})^\gamma}, \quad \eta > 0, \gamma \geqslant 0,$$

i.e.,

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\ln\frac{1}{h})^\gamma}\right),$$

for all x in $\mathbb{R}$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a Dini-Lipschitz of order η, or f belongs to $Lip(\eta, \gamma, p)$.

Definition 2.2. *If however*

$$\frac{\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n}}{\frac{h^{\eta+2n}}{(\ln \frac{1}{h})^\gamma}} \to 0, \quad as \quad h \to 0, \gamma \geqslant 0,$$

then f is said to be belong to the little Dini-Lipschitz class $lip(\eta, \gamma, p)$.

Remark 2.3. *Let $\eta > 1$. If $f \in Lip(\eta, \gamma, p)$, then $f \in lip(1, \gamma, p)$.*

Proof. For $x \in \mathbb{R}$, h small and $f \in Lip(\eta, \gamma, p)$, we have

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leqslant C \frac{h^{\eta+2n}}{(\ln \frac{1}{h})^\gamma}.$$

Then

$$(\log \frac{1}{h})^\gamma \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leqslant C h^{\eta+2n}.$$

Therefore

$$\frac{(\log \frac{1}{h})^\gamma}{h^{1+2n}} \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leqslant C h^{\eta-1},$$

which tends to zero with $h \to 0$. Thus

$$\frac{(\log \frac{1}{h})^\gamma}{h^{1+2n}} \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \to 0, \quad h \to 0.$$

Then $f \in lip(1, \gamma, p)$. □

Remark 2.4. *If $\eta < \nu$, then $Lip(\eta, 0, p) \supset Lip(\nu, 0, p)$ and $lip(\eta, 0, p) \supset lip(\nu, 0, p)$.*

Proof. We have $0 \leqslant h \leqslant 1$ and $\eta < \nu$, then $h^\nu \leqslant h^\eta$.
Then the proof of theorem is immediate. □

3. New Results on Dini-Lipschitz Class

Theorem 3.1. *Let $\eta > 2$ and $1 \leqslant p \leqslant 2$. If f belongs to the Dini-Lipschitz class, i.e.,*

$$f \in Lip(\eta, \gamma, p), \quad \eta > 2, \gamma \geqslant 0, 1 \leqslant p \leqslant 2.$$

Then f is null almost everywhere on $\mathbb{R}$.

Proof. Assume that $f \in Lip(\eta, \gamma, p)$. Then we have

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leqslant C \frac{h^{\eta+2n}}{(\ln \frac{1}{h})^{\gamma}}, \quad \gamma \geqslant 0.$$

By using the formulas (1), (2), and (5) we have the generalized Fourier-Dunkl transform of $\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)$ is $2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\Lambda f(\lambda)$.
By formula (4), we get

$$2^q h^{2qn} \int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leqslant K^q C^q \frac{h^{q\eta+2qn}}{(\ln \frac{1}{h})^{q\gamma}}.$$

Therefore

$$\int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leqslant \frac{K^q C^q}{2^q} \frac{h^{q\eta}}{(\ln \frac{1}{h})^{q\gamma}}.$$

Then

$$\frac{\int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)}{h^{2q}} \leqslant \frac{K^q C^q}{2^q} \frac{h^{q\eta-2q}}{(\ln \frac{1}{h})^{q\gamma}},$$

Since $\eta > 2$ we have

$$\lim_{h\to 0} \frac{h^{q\eta-2q}}{(\ln \frac{1}{h})^{q\gamma}} = 0.$$

Thus

$$\lim_{h\to 0} \int_{\mathbb{R}} \left(\frac{|1 - j_{\alpha+2n}(\lambda h)|}{\lambda^2 h^2} \right)^q \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

And also from the formula (3) and Fatou theorem, we obtain

$$\int_{\mathbb{R}} \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

Hence $\lambda^2 \mathcal{F}_\Lambda f(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and so $f(x)$ is the null function. $\square$

Theorem 3.2. *Let $f \in L^p_{\alpha,n}, 1 \leqslant p \leqslant 2$. If f belongs to $lip(2,0,p)$. i.e.,*

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} = O(h^{2+2n}), \quad as \quad h \to 0.$$

Then f is null almost everywhere on $\mathbb{R}$.

Proof. Assume that $f \in lip(2,0,p)$. Then we have

$$\frac{\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n}}{h^{2+2n}} \to 0, \quad as \quad h \to 0$$

By using the formulas (1), (2) and (5) we have the generalized Fourier-Dunkl transform of $\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)$ is $2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\Lambda f(\lambda)$.
By formula (4), we get

$$\frac{2^q h^{2qn} \displaystyle\int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)}{h^{2q+2nq}} \to 0, \quad as \quad h \to 0$$

Thus

$$\lim_{h \to 0} \int_{\mathbb{R}} \left(\frac{|1 - j_{\alpha+2n}(\lambda h)|}{\lambda^2 h^2} \right)^q \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

And also from the formula (3) and Fatou theorem, we obtain

$$\int_{\mathbb{R}} \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$

Hence $\lambda^2 \mathcal{F}_\Lambda f(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and so $f(x)$ is the null function. □

Now, we give another the main result of this paper analog of Theorem 1.2.

Theorem 3.3. *Let $f \in L^p_{\alpha,n}$. If $f(x)$ belongs to $Lip(\eta,\gamma,p)$. Then*

$$\int_{|\lambda| \geqslant r} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-q\eta}}{(\ln r)^{q\gamma}} \right), \quad as \quad r \to \infty,$$

where $1 \leqslant p \leqslant 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $f \in Lip(\eta,\gamma,p)$. Then we have

$$\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|_{p,\alpha,n}=O\left(\frac{h^{\eta+2n}}{(\ln\frac{1}{h})^{\gamma}}\right)\quad \text{as}\quad h\to 0.$$

From formulas (1), (2) and (5) we have the generalized Fourier-Dunkl transform of $\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)$ is $2h^{2n}(j_{\alpha+2n}(\lambda h)-1)\mathcal{F}_\Lambda f(\lambda)$.
By formula (4), we obtain

$$2^q h^{2qn}\int_{\mathbb{R}}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\leqslant K^q\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^q_{p,\alpha,n}.$$

If $|\lambda|\in[\frac{1}{h},\frac{2}{h}]$, then $|\lambda h|\geqslant 1$ and (*iii*) of Lemma 1.3 implies that

$$1\leqslant \frac{1}{c^q}|j_{\alpha+2n}(\lambda h)-1|^q.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}\leqslant|\lambda|\leqslant\frac{2}{h}}|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &\leqslant \frac{1}{c^q}\int_{\frac{1}{h}\leqslant|\lambda|\leqslant\frac{2}{h}}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\\ &\leqslant \frac{1}{c^q}\int_{-\infty}^{+\infty}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\\ &\leqslant \frac{h^{-2qn}K^q}{2^q c^q}\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^q_{p,\alpha,n}\\ &= O\left(\frac{h^{q\eta}}{(\ln\frac{1}{h})^{q\gamma}}\right).\end{aligned}$$

So we obtain

$$\int_{r\leqslant|\lambda|\leqslant 2r}|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\leqslant C\frac{r^{-q\eta}}{(\ln r)^{q\gamma}},\qquad r\to\infty.$$

where C is a positive constant. Now, we have

$$\begin{aligned}\int_{|\lambda|\geqslant r}|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r\leqslant|\lambda|\leqslant 2^{i+1}r}|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\\ &\leqslant C\left(\frac{r^{-q\eta}}{(\ln r)^{q\gamma}}+\frac{(2r)^{-q\eta}}{(\ln 2r)^{q\gamma}}+\frac{(4r)^{-q\eta}}{(\ln 4r)^{q\gamma}}+\cdots\right)\\ &\leqslant C\frac{r^{-q\eta}}{(\ln r)^{q\gamma}}\left(1+2^{-q\eta}+(2^{-q\eta})^2+(2^{-q\eta})^3+\cdots\right)\\ &\leqslant K_\eta\frac{r^{-q\eta}}{(\ln r)^{q\gamma}},\end{aligned}$$

where $K_\eta = C(1-2^{-q\eta})^{-1}$ since $2^{-q\eta} < 1$.
Consequently

$$\int_{|\lambda|\geqslant r} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-q\eta}}{(\ln r)^{q\gamma}}\right), \quad as \quad r\to\infty,$$

and this completes the proof. $\square$

Theorem 3.4. *Let $f \in L^2_{\alpha,n}$, $0<\eta<1$ and $\gamma \geqslant 0$. If*

$$\int_{|\lambda|\geqslant r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty,$$

then $f \in Lip(\eta,\gamma,2)$.

Proof. Suppose that

$$\int_{|\lambda|\geqslant r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty.$$

and write

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|^2_{2,\alpha,n} \;=\; 4h^{4n}(I_1+I_2),$$

where

$$I_1 = \int_{|\lambda|<\frac{1}{h}} |j_{\alpha+2n}(\lambda h)-1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

and

$$I_2 = \int_{|\lambda|\geqslant\frac{1}{h}} |j_{\alpha+2n}(\lambda h)-1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Firstly, we use the formulas $|j_{\alpha+2n}(\lambda h)| \leqslant 1$ and

$$I_2 \leqslant 4\int_{|\lambda|\geqslant\frac{1}{h}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \left(\frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}\right), \quad as \quad h\to 0.$$

Set

$$\phi(x) = \int_x^{+\infty} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Integrating by parts we obtain

$$\begin{aligned}\int_0^x \lambda^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= \int_0^x -\lambda^2\phi'(\lambda)d\lambda = -x^2\phi(x) + 2\int_0^x \lambda\phi(\lambda)d\lambda \\ &\leqslant C_1\int_0^x \lambda\lambda^{-2\eta}(\log\lambda)^{-2\gamma}d\lambda = O(x^{2-2\eta}(\log x)^{-2\gamma}),\end{aligned}$$

where C_1 is a positive constant.
We use the formula (ii) of Lemma 1.2

$$\begin{aligned}\int_{-\infty}^{+\infty}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= O\left(h^2\int_{|\lambda|<\frac{1}{h}}\lambda^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right) \\ &+ \left(\frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\ &= O\left(h^2\frac{h^{2\eta-2}}{(\log\frac{1}{h})^{2\gamma}}\right)+O\left(\frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\ &= O\left(\frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}\right),\end{aligned}$$

and this ends the proof. □

By analogy with the proof of the Theorems 3.3 and 3.4, we can establish the following results.

Theorem 3.5. *Let* $f \in L^p_{\alpha,n}$. *If*

$$\|\tau^h f(x) + \tau^{-h}f(x) - 2h^{2n}f(x)\|_{p,\alpha,n} = o\left(h^{2n}(\ln\frac{1}{h})^{-1}\right), \quad as \quad h \to 0,$$

then

$$\int_{|\lambda|\geqslant r}|\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = o\left((\ln r)^{-q}\right), \quad as \quad r\to\infty,$$

where $1 \leqslant p \leqslant 2$ *and* q *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 3.6. *Let* $f \in L^2_{\alpha,n}$. *If*

$$\int_{|\lambda|\geqslant r}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = o\left((\ln r)^{-2}\right), \quad as \quad r\to\infty,$$

then

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n} = o\left(h^{2n}(\ln\frac{1}{h})^{-1}\right), \quad as \quad h \to 0.$$

Acknowledgements
The authors would like to thank the any nomos referees for their valuable comments and suggestions.

References

[1] S. A. Al Sadhan, R. F. Al Subaie, and M. A. Mourou, Harmonic analysis associated with a first-order singular differential-difference operator on the real line. *Current Advances in Mathematics Research,* 1 (2014), 23-34.

[2] E. S. Belkina and S. S. Platonov, Equivalence of K-Functionnals and Modulus of Smoothness Constructed by Generalized Dunkl Translations, *Izv. Vyssh. Uchebn. Zaved. Mat.,* 8 (2008), 3-15.

[3] C. F. Dunkl, Differential-difference operators associated to reflection groups. *Transactions of the American Mathematical Society,* 311 (1989), 167-183.

[4] C. F. Dunkl, Hankel transforms associated to finite reflection groups. *Contemporary Mathematics,* 138 (1992), 128-138.

[5] M. S. Younis, Fourier transforms of Dini-Lipschitz functions. Int., *J. Math. Math. Sci.,* 9 (2) (1986), 301-312. doi:10.1155/S0161171286000376.

[6] R. F. Al Subaie and M. A. Mourou, Inversion of two Dunkl type Intertwining Operators on R Using Generalized Wavelets, *Far East Journal of Applied Mathematics,* 88 (2014), 91-120.

[7] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Claredon, oxford, 1948, Komkniga. Moxow. 2005.

RADOUAN DAHER
Department of Mathematics
Faculty of Sciences Aïn Chock
Ph.D. Student of Mathematics
University Hassan II
Casablanca, Morocco
E-mail: rjdaher024@gmail.com

SALAH EL OUADIH
Department of Mathematics
Faculty of Sciences Aïn Chock
Ph.D. Student of Mathematics
University Hassan II
Casablanca, Morocco
E-mail: salahwadih@gmail.com