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The Entropy of Fuzzy Dynamical Systems with Countable Partitions

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Abstract. In this paper, the entropy of fuzzy countable partitions is proposed. In addition, the notion of fuzzy conditional entropy, fuzzy relative entropy and entropy on a fuzzy dynamical system based on fuzzy countable partitions are introduced and some properties of these entropies are analyzed. Using the concept of fuzzy generator, it is proved that the entropy of a dynamical system is done through a fuzzy generator. Furthermore, one of the proven results in this paper is isomorphism invariant property of entropy with fuzzy countable partitions on a fuzzy dynamical system.

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1. Introduction

The notion of entropy was first introduced by Clausius in his works on thermodynamics. Later this concept was generalized to other areas [7, 11, 14]. Entropy is a tool which measures the amount of uncertainly in random events as well as the complex behavior of the orbits in a dynamical system. In addition, entropy makes classifying dynamical systems possible. Kolmogorov defined the important entropy that distinguishes two dynamical systems [8]. In this entropy,

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partitions and their joins have essential roles. The fuzzy dynamical system and its entropy were introduced by Markechow [9] and then were extended by other researchers [1, 2, 3, 4, 12]. The main idea of fuzzy entropy was to replace the partitions by fuzzy partitions. Entropy with infinite partitions is another entropy which has been proposed by some scholars in recent years [5, 6]. Hence the notion of fuzzy entropy with fuzzy countable partitions is introduced in Section 2. In Section 3, the fuzzy conditional entropy and fuzzy relative entropy based on a fuzzy countable partition are proposed and some of their properties are investigated. In the final section of this note, the entropy of a fuzzy dynamical system is put forward and then it is proved that this entropy is isomorphism invariant. Moreover, one of the most important results obtained in this section is a theorem that states $\tilde{h}(T) = h(T, A)$, whenever A is a fuzzy generator.

2. Fuzzy Entropy with Countable Partition

This section starts with the basic concept of fuzzy probability space and it continues with the investigation of properties of this space. Then, the important notions of fuzzy countable partition, refinement of partition, join refinement of two fuzzy countable partitions and entropy of a fuzzy countable partition are introduced.

Definition 2.1. *Let X be a non-empty set. $M \subseteq [0,1]^X$ (fuzzy subsets of X) is a $\sigma-$algebra, if:*

a) $1 \in M$,

b) if $f \in M$, *then* $f' = 1 - f \in M$,

c) if $f_n \in M$ *for* $n \in \mathbb{N}$, *then* $\bigvee_{i=1}^{\infty} f_n := \sup f_n \in M$.

The operations " $\vee$ ", " $\wedge$ " and partial ordering relation " $\leqslant$ " are defined in the following way:
For every $f, g \in M$, $f \vee g = sup\{f, g\}$, $f \wedge g = inf\{f, g\}$ and $f \leqslant g$ iff $f(x) \leqslant g(x)$ for all $x \in X$.

Lemma 2.2. *Let f, g and h be fuzzy sets. Then:*

a) $(f')' = f$, *for every* $f \in M$,

b) if $f \leqslant g$, *then* $g' \leqslant f'$,

c) $(f \wedge g)' = f' \vee g'$ *and* $(f \vee g)' = f' \wedge g'$,

d) $f \wedge g = g \wedge f$ *and* $f \vee g = g \vee f$,

e) $f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$ *and* $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$.

Proof. The proof can be found in [15]. □

Definition 2.3. *Two elements $f, g \in M$ are said to be orthogonal and write $f \perp g$ iff $f \leqslant 1 - g$.*

Definition 2.4. *Fuzzy probability space is a triplet (X, m, M), where X is a nonempty set, M is a $\sigma-$algebra and $m : M \to [0, 1]$ is a function satisfying:*

a) for every $f \in M$, $m(f \vee f') = 1$,

b) if $\{f_i\}_{i=1}^{\infty}$ is a sequence of pairwise orthogonal elements from M, then $m(\overset{\infty}{\underset{i=1}{\vee}} f_i) = \sum\limits_{i=1}^{\infty} m(f_i)$

Theorem 2.5. *$m : M \to [0, 1]$ has the following properties:*

a) $m(f') = 1 - m(f)$ for every $f \in M$,

b) if $f \leqslant g$, then $m(f) \leqslant m(g)$,

c) if $f \leqslant 1 - g$, then $m(f \wedge g) = 0$,

d) $m(g) = 1$ iff for all $f \in M$, $m(f \wedge g) = m(f)$,

e) $m(f \vee g) + m(f \wedge g) = m(f) + m(g)$,

f) $m(f \wedge f') = 0$.

Proof. The proof of parts a, d, and e can be found in [10]. We just prove parts b, c and f.

b) $f \leqslant g$, so $f \wedge f' \leqslant f \leqslant g = 1 - g'$ and also $f \leqslant f \vee g' = 1 - (f' \wedge g)'$ and these imply $(f, f' \wedge g)$ and $(f \wedge f', g')$ are orthogonal. By definition 2.4, $m(f \vee (f' \wedge g)) = m(f) + m(f' \wedge g)$ and $m(g' \vee (f \wedge f')) = m(g') + m(f \wedge f')$. Now we have
$m(f) \leqslant m(f) + m(f' \wedge g) = m(f \vee (f' \wedge g)) =$
$m((f \vee g) \wedge (f' \vee f)) = m(g \wedge (f' \vee f)) = 1 - m(g' \vee (f' \wedge f)) = 1 - m(g') - m(f' \wedge f) = m(g)$.

c) $f \leqslant 1 - g$ so $f \perp g$ and $m(f \vee g) = m(f) + m(g)$. By part e, $m(f \wedge g) = 0$.
f) By part c, the proof is obvious. □

Definition 2.6. *A countable sequence $A = \{f_i\}_{i=1}^{\infty}$ of elements of M is a fuzzy countable partition of X, if :*

a) for $i \neq j$, $f_i \leqslant 1 - f_j$,

b) $m(\overset{\infty}{\underset{i=1}{\vee}} f_i) = 1$.

Definition 2.7. *Let $B = \{g_j\}_{j=1}^{\infty}$ and $A = \{f_i\}_{i=1}^{\infty}$ be fuzzy countable parti-*

tions. Partition B is a refinement of partition A and we write $A \prec B$, if for any g_j there exists f_i such that $g_j \leqslant f_i$. We say that $A \sqcup B = \{f_i \wedge g_j | f_i \in A, g_j \in B\}$ is a join refinement of A and B if $A \sqcup B$ is a fuzzy countable partition and $\sup_{i\,j} m(f_i \wedge g_j) \geqslant \sup_i m(f_i) \sup_j m(g_j)$.

Definition 2.8. *Let $A = \{f_i\}_{i=1}^{\infty}$ be a fuzzy countable partition of X.we define the entropy of A by:*

$$H(A) := -log \sup_{i \in \mathbb{N}} m(f_i).$$

Example 2.9. $A = \{0, 1\}$ is a partition and $H(A) = 0$.

Lemma 2.10. *Let $f \triangle g = (f \wedge g') \vee (f' \wedge g)$. If $m(f \triangle g) = 0$, then $m(f) = m(g)$.*

Proof. Since $m(f \triangle g) = 0$, part b of Theorem 2.5 implies that $m(f \wedge g') = m(f' \wedge g) = 0$. $m(f) = m(f \wedge (g \vee g')) = m(f \wedge g) + m(f \wedge g') - m((f \wedge g) \wedge (f \wedge g'))$. $m(f \wedge g') = m((f \wedge g) \wedge (f \wedge g')) = 0$ so $m(f) = m(f \wedge g)$. With the same argument, we can prove that $m(g) = m(f \wedge g)$ thus $m(f) = m(g)$. □

Definition 2.11. *Let $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$ be countable partitions. A and B are independent if $m(f_i \wedge g_j) = m(f_i)m(g_j)$ for any $i, j \in N$. $A \overset{\circ}{\subseteq} B$ if for any f_i there exists g_j such that $m(f_i \triangle g_j) = 0$ and $A \overset{\circ\circ}{\subseteq} B$ if for any f_i there exists g_j such that $m(f_i \wedge g_j') = m(f_i' \wedge g_j) = 0$.*

Example 2.12. $A = \{f_i\}_{i=1}^{\infty}$ and $C = \{0, 1\}$ are independent.

Theorem 2.13. *Let $A = \{f_i\}_{i=1}^{\infty}, B = \{g_j\}_{j=1}^{\infty}, C = \{h_k\}_{k=1}^{\infty}$ and D be fuzzy countable partitions of X, then:*

a) $A \prec A \sqcup B$,

b) if $A \prec B$, then $A \sqcup C \prec B \sqcup C$,

c) if $A \prec B$ and $C \prec D$, then $A \sqcup C \prec B \sqcup D$,

d) if $A \overset{\circ\circ}{\subseteq} B$, then $A \sqcup C \overset{\circ\circ}{\subseteq} B \sqcup C$,

e) if $A \overset{\circ\circ}{\subseteq} B$ and $C \overset{\circ\circ}{\subseteq} D$, then $A \sqcup C \overset{\circ\circ}{\subseteq} B \sqcup D$,

f) if $A \overset{\circ\circ}{\subseteq} B$, then $A \overset{\circ}{\subseteq} B$,

g) $A \overset{\circ\circ}{\subseteq} A \sqcup A$.

Proof.

a) $f_i \wedge g_j \leqslant f_i$.

b) Let $g_j \wedge h_k \in B \sqcup C$ be given. There exists $f_i \epsilon A$ such that $g_j \leqslant f_i$ and this implies $g_j \wedge h_k \leqslant f_i \wedge h_k$.

c) $A \sqcup C \prec B \sqcup C \prec B \sqcup D$.

d) For every f_i there exists g_j such that $m(f_i \wedge g'_j) = m(f'_i \wedge g_j) = 0$. $m(f_i \wedge h_k \wedge (g_j \wedge h_k)') = m(f_i \wedge h_k \wedge (g'_j \vee h'_k)) = m(f_i \wedge h_k \wedge g'_j) + m(f_i \wedge h_k \wedge h'_k) - m(f_i \wedge h_k \wedge g'_j \wedge h'_k) = m(f_i \wedge h_k \wedge g'_j) \leqslant m(f_i \wedge g'_j) = 0$. With the same argument can be proved $m(g_j \wedge h_k \wedge (f_i \wedge h_k)') = 0$.

e) $A \sqcup C \overset{\circ\circ}{\subseteq} B \sqcup C \overset{\circ\circ}{\subseteq} B \sqcup D$.

f) The proof is trivial.

g) For every f_i, $f_i = f_i \wedge f_i$ and $m(f_i \wedge f'_i) = 0$. □

Theorem 2.14. *Let $A = \{f_i\}_{i=1}^{\infty}, B = \{g_j\}_{j=1}^{\infty}$ and $C = \{h_k\}_{k=1}^{\infty}$ be fuzzy countable partitions and P be the set of all fuzzy countable partitions of X. The entropy $H : P \to [0, \infty)$ has the following properties:*

a) $H(A) \geqslant 0$, for every A,

b) if $A \prec B$, then $H(A) \leqslant H(B)$,

c) if $A \prec B$, then $H(A \sqcup C) \leqslant H(B \sqcup C)$,

d) if $A \overset{\circ}{\subseteq} B$, then $H(A) \geqslant H(B)$,

e) $max\{H(A), H(B)\} \leqslant H(A \sqcup B) \leqslant H(A) + H(B)$,

f) if A and B are independent, then $H(A \sqcup B) = H(A) + H(B)$,

g) $H(A) = H(A \sqcup A)$.

Proof.
a) Since for every $f_i \in A, 0 \leqslant m(f_i) \leqslant 1$ thus $H(A) = -log \sup_{ieN} m(f_i) \geqslant 0$.

b) For every $g_j \epsilon B$ there exists $f_i \in A$ such that $g_j \leqslant f_i$ and this implies $\sup_j m(g_j) \leqslant \sup_i m(f_i)$.

c) By Theorem 2.13 and part b is trivial.

d) For every f_i there exists g_j such that $m(f_i \triangle g_j) = 0$ so $m(f_i) = m(g_j)$ and this implies that $\sup_i m(f_i) \leqslant \sup_j m(g_j)$.

e) By definition $\sup_{i,j} m(f_i \wedge g_j) \geqslant \sup_i m(f_i) \sup_j m(g_j)$.

g) The proof would obtain from part b and d of Theorem 2.14. □

3. Relative and Conditional Fuzzy Entropy with Countable Partition

In this section based on fuzzy countable partitions, two important entropies, named relative and conditional fuzzy entropies, are defined and some interesting theorems analyzing the properties of these two entropies are proved.

Definition 3.1. *Let $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$ be two fuzzy countable partitions. we define conditional entropy as follows:*
$H(A|B) := -log \sup_{i,j}(\frac{m(f_i \wedge g_j)}{m(g_j)}), m(g_j) \neq 0.$

Theorem 3.2. *For every fuzzy countable partitions $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$:*

a) $H(A|B) \geqslant 0$,

b) $H(A|B) + H(B) \leqslant H(A \sqcup B)$,

c) $H(A|B) \leqslant H(A)$. $H(A|B) = H(A)$ if A and B are independent.

Proof. a) $m(f_i \wedge g_j) \leqslant m(g_j)$.

b) $\sup_{i,j}(\frac{m(f_i \wedge g_j)}{m(g_j)}) \geqslant \frac{\sup_{i,j} m(f_i \wedge g_j)}{\sup_j m(g_j)}$.

c) $\sup_{i,j}(\frac{m(f_i \wedge g_j)}{m(g_j)}) \geqslant \frac{\sup_{i,j} m(f_i \wedge g_j)}{\sup_j m(g_j)} \geqslant \frac{\sup_i m(f_i)\ \sup_j m(g_j)}{\sup_j m(g_j)} = \sup_i m(f_i)$. If A and B are independent, then $\frac{m(f_i \wedge g_j)}{m(g_j)} = m(f_i)$. □

Theorem 3.3. *For every fuzzy countable partitions $A = \{f_i\}_{i=1}^{\infty}$, $B = \{g_j\}_{j=1}^{\infty}$ and $C = \{h_k\}_{k=1}^{\infty}$:*

a) $H(A \sqcup B|C) \geqslant H(B|A \sqcup C) + H(A|C)$,

b) $H(A \sqcup C) \geqslant H(A) + H(C|A)$,

c) if $A \prec B$, then $H(A|C) \leqslant H(B|C)$.

Proof.
a)

$$\frac{m((f_i \wedge g_j) \wedge h_k)}{m(h_k)} = \frac{m((f_i \wedge g_j) \wedge h_k)}{m(h_k)}\ \frac{m(f_i \wedge h_k)}{m(f_i \wedge h_k)}$$

and this implies

$$\sup_{i,j,k} \frac{m((f_i \wedge g_j) \wedge h_k)}{m(h_k)} \leqslant \sup_{i,j,k} \frac{s((f_i \wedge g_j) \wedge h_k)}{m(f_i \wedge h_k)}\ \sup_{i,k} \frac{m(f_i \wedge h_k)}{m(h_k)}.$$

b) $m(f_i \wedge h_j) = \frac{m(f_i \wedge h_j)}{m(f_i)} m(f_i)$, so

$$\sup_{i,j} m(f_i \wedge h_j) \leqslant \sup_{i,j} \frac{m(f_i \wedge h_j)}{m(f_i)} \sup_i \ m(f_i).$$

c) For every g_j there exists f_i such that $g_j \leqslant f_i$ and this implies

$$\frac{m(f_i \wedge h_k)}{m(h_k)} \geqslant \frac{m(g_j \wedge h_k)}{m(h_k)}. \quad \square$$

Definition 3.4. *Let $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$ be two fuzzy countable partitions. We define the relative entropy as follows:*

$$H(A \parallel B) := log \ \sup_{i,j}(\frac{m(f_i)}{m(g_j)}), m(g_j) \neq 0.$$

Theorem 3.5. *Let $A = \{f_i\}_{i=1}^{\infty}, B = \{g_j\}_{j=1}^{\infty}, C = \{h_k\}_{k=1}^{\infty}, D$ and E be fuzzy countable partitions, then:*

a) if $A \prec B$ or $B \overset{\circ}{\subseteq} A$, then $H(A \parallel B) \geqslant 0$,

b) if $A \prec B$, then $H(A \parallel C) \geqslant H(B \parallel C)$,

c) if $A \prec B$, then $H(A \sqcup D \parallel C) \geqslant H(B \sqcup D \parallel C)$,

d) if $A \prec B$ and $C \prec D$, then $H(A \sqcup C \parallel E) \geqslant H(B \sqcup D \parallel E)$,

e) if $B \overset{\circ}{\subseteq} A$, then $H(B \parallel C) \leqslant H(A \parallel C)$,

f) $H(A \| B) \geqslant H(A)$,

g) $H(A \sqcup B \| C) \leqslant H(A \sqcup B \| B) + H(B \| C)$.

Proof.

a) $\sup_{i,j}(\frac{m(f_i)}{m(g_j)}) \geqslant 1$.

b) $\sup_{i,k}(\frac{m(f_i)}{m(h_k)}) \geqslant \sup_{j,k}(\frac{m(g_j)}{m(h_k)})$.

The proofs of parts c and d are obvious because of part b and Theorem 2.13.

e) For every $g_j \epsilon B$ there exists $f_i \in A$ such that $m(g_j) = m(f_i)$, so $\sup_{i,k}(\frac{m(f_i)}{m(h_k)}) \geqslant \sup_{j,k}(\frac{m(g_j)}{m(h_k)})$.

f) $\frac{m(f_i)}{m(g_j)} \geqslant m(f_i)$.

g) $\sup_{i,j,k}(\frac{m(f_i \wedge g_j)}{m(h_k)}) \leqslant \sup_{i,j}(\frac{m(f_i \wedge g_j)}{m(g_j)}) \ \sup_{j,k}(\frac{m(g_j)}{m(h_k)})$. $\square$

4. Fuzzy Entropy of Dynamical System

This section presents a definition for the entropy of a fuzzy dynamical system. It continues proving that this entropy is isomorphism invariant. Moreover, the notion of a fuzzy generator is introduced and it is proved that the entropy of a fuzzy dynamical system is equal with the entropy of a fuzzy generator partition.

Definition 4.1. *A mapping $\varphi : M \to M$ is an m-preserving σ-homomorphism if:*

i) $\varphi(1) = 1$,

ii) for every $f \epsilon M$, $\varphi(f') = (\varphi(f))'$,

iii) for every sequence $\{f_i\}_{i=1}^{\infty} \subset M$, $\varphi(\bigvee_{i=1}^{\infty} f_i) = \bigvee_{i=1}^{\infty} \varphi(f_i)$,

iv) for every $f, g \epsilon M$, $\varphi(f \wedge g) = \varphi(f) \wedge \varphi(g)$,

v) $m(\varphi(f)) = m(f)$.

Definition 4.2. *Quadruple (X, M, m, φ) is a fuzzy dynamical system if (X, M, m) is a fuzzy probability space and φ is a m-preserving σ-homomorphism.*
We define $\varphi^n = \varphi \circ \varphi^{n-1}$, where φ^0 is the identity map on M.

Theorem 4.3. *Let $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$ be fuzzy countable partitions, then:*

a) if $f \leqslant g$, *then* $\varphi(f) \leqslant \varphi(g)$,

b) if $A \prec B$, *then* $\varphi(A) \prec \varphi(B)$,

c) if $A \overset{\circ}{\subseteq} B$, *then* $\varphi(A) \overset{\circ}{\subseteq} \varphi(B)$.

Proof.

a) $f \vee g = g$ and $\varphi(g) = \varphi(f \vee g) = \varphi(f) \vee \varphi(g)$, so $\varphi(f) \leqslant \varphi(g)$.

b) By definition of refinement and part a, the proof is obvious.

c)$\varphi(f) \bigtriangleup \varphi(g) = \varphi(f \bigtriangleup g)$. □

Theorem 4.4. *Let $A = \{f_i\}_{i=1}^{\infty}$ and $B = \{g_j\}_{j=1}^{\infty}$ be fuzzy countable partitions, then:*

a) $\{\varphi(f_i)\}_{i=1}^{\infty} \subset M$ *is a fuzzy countable partition of X,*

b) $H(\varphi^n(A)) = H(\varphi(A)) = H(A)$,

c) $H(\varphi^n(A)|\varphi^n(B)) = H(A|B)$,

d) $H(\varphi(A)\|\varphi(B)) = H(A\|B)$, *e)* $H(\varphi(A) \sqcup \varphi(B)) = H(A \sqcup B)$.

Proof.

a) By Definition 4.1, $m\left(\bigvee_{i=1}^{\infty} \varphi(f_i)\right) = m\left(\varphi(\bigvee_{i=1}^{\infty} f_i)\right) = m\left(\bigvee_{i=1}^{\infty} f_i\right) = 1$. $f_i \leqslant 1 - f_j = f_j'$, Definition 4.1 and Theorem 4.3 imply

$$\varphi(f_i) \leqslant \varphi(f_j') = (\varphi(f_j))' = 1 - \varphi(f_j).$$

b) $m(\varphi(f_i)) = m(f_i)$.

c) $\frac{m(\varphi(f_i) \wedge \varphi(g_j))}{m(\varphi(g_j))} = \frac{m(\varphi(f_i \wedge g_j))}{m(\varphi(g_j))} = \frac{m(f_i \wedge g_j)}{m(g_j)}$.

d) $\frac{m(\varphi(f_i))}{m(\varphi(g_j))} = \frac{m(f_i)}{m(g_j)}$.

e) $m(\varphi(f_i) \wedge \varphi(g_j)) = m(f_i \wedge g_j)$. □

Theorem 4.5. *Let* $\{a_i\}_{i=1}^{\infty}$ *be sequence of nonnegative numbers such that* $a_{r+s} \leqslant a_r + a_s$ *for each* $r, s = 1, 2, \dots$ *, then:* $\lim_{n\to\infty} \frac{1}{n} a_n$ *exists.*

Proof. The proof can be found in [13]. □

Theorem 4.6. *Let A be a fuzzy countable partition. Then:*
$h(\varphi, A) := \lim_{n\to\infty} \frac{1}{n} H(\bigsqcup_{i=0}^{n-1} \varphi^i A)$ *exists.*

Proof. We have $a_{r+s} = H(\bigsqcup_{i=0}^{r+s-1} \varphi^i A) = H(\left(\bigsqcup_{i=0}^{r-1} \varphi^i A\right) \sqcup \left(\bigsqcup_{i=r}^{r+s-1} \varphi^i A\right)) \leqslant H(\bigsqcup_{i=0}^{r-1} \varphi^i A) + H(\varphi^r(\bigsqcup_{i=0}^{s-1} \varphi^i A)) = H(\bigsqcup_{i=0}^{r-1} \varphi^i A) + H(\bigsqcup_{i=0}^{s-1} \varphi^i A) = a_r + a_s$. □

Definition 4.7. *The entropy of a fuzzy dynamical system is defined as follows:* $\tilde{h}(\varphi) := \sup\{h(\varphi, A) : A$ *is a fuzzy countable partition of* $X\}$.

Theorem 4.8. *Let* $A = \{f_i\}_{i=1}^{\infty}$ *and* $B = \{g_j\}_{j=1}^{\infty}$ *be fuzzy countable partitions of X, then:*

a) $h(\varphi, A) \leqslant H(A)$,

b) $h(\varphi, A \sqcup B) \leqslant h(\varphi, A) + h(\varphi, B)$,

c) $h(\varphi, \varphi(A)) = h(\varphi, A)$,

d) $h(\varphi, \bigsqcup_{j=0}^{r} \varphi^j A) = h(\varphi, A)$, $r \geqslant 1$,

e) if $A \prec B$, *then* $h(\varphi, A) \leqslant h(\varphi, B)$,

f) if $B \overset{\circ\circ}{\subseteq} A$, *then* $h(\varphi, A) \leqslant h(\varphi, B)$, *g)* $h(\varphi^k, \overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A) = kh(\varphi, A)$ *for* $k > 0$*,*

h) $\tilde{h}(\varphi^k) = k\tilde{h}(\varphi)$ *for* $k > 0$*.*

Proof.

a) $H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i A) \leqslant \sum_{i=0}^{n-1} H(\varphi^i A) = \sum_{i=0}^{n-1} H(A) = nH(A)$.

b) $H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i(A \sqcup B)) = H((\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i A) \sqcup (\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i B)) \leqslant H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i A) + H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i B)$,

c) $H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i(\varphi(A)) = H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi(\varphi^i A)) = H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i A)$.

d) $h(\varphi, \overset{r}{\underset{j=0}{\sqcup}} \varphi^j A) = \lim_{n\to\infty} \frac{1}{n} H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i \overset{r}{\underset{j=0}{\sqcup}} \varphi^j A) = \lim_{n\to\infty} \frac{1}{n} H(\overset{n+r-1}{\underset{s=0}{\sqcup}} \varphi^s A) = \lim_{p\to\infty} \frac{p}{p-r} \frac{1}{p} H(\overset{p-1}{\underset{s=0}{\sqcup}} \varphi^s A)$.

e) $A \prec B$ and Theorem 4.3 implies $\varphi^i(A) \prec \varphi^i(B)$. Considering Theorems 2.13 and 2.14, we have

$$H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i(A)) \leqslant H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i(B)).$$

f) The proof is the same as part e.

g) $h(\varphi^k, \overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A) = \lim_{n\to\infty} \frac{1}{n} H(\overset{n-1}{\underset{j=0}{\sqcup}} (\varphi^k)^j (\overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A)) =$

$\lim_{n\to\infty} \frac{k}{kn} H(\overset{nk-1}{\underset{s=0}{\sqcup}} \varphi^s A) = kh(\varphi, A)$.

h) $k\tilde{h}(\varphi) = k \sup_A h(\varphi, A) = \sup_A h(\varphi^k, \overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A) \leqslant \sup_C h(\varphi^k, C) = \tilde{h}(\varphi^k)$. On the other hand since $A \prec \overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A$ by part e, $h(\varphi^k, A) \leqslant h(\varphi^k, \overset{k-1}{\underset{i=0}{\sqcup}} \varphi^i A) = kh(\varphi, A)$. □

Definition 4.9. *Two fuzzy dynamical systems* $(X_1, M_1, m_1, \varphi_1)$ *and* $(X_2, M_2, m_2, \varphi_2)$ *are said to be isomorphic, if there exists a bijective map* $\psi : M_1 \to M_2$ *such that for any* $f \epsilon M_1$ *and any sequence* $\{f_n\}_{n=1}^{\infty} \subset M_1$*:*

i) $\psi(\bigvee_{i=1}^{\infty} f_i) = \bigvee_{i=1}^{\infty} \psi(f_i)$ *and* $\psi(f') = 1 - \psi(f)$*,*

ii) $m_1(f) = m_2(\psi(f))$*,*

iii) $\psi(\varphi_1(f)) = \varphi_2(\psi(f))$*.*

Theorem 4.10. *If* $(X_1, M_1, m_1, \varphi_1)$ *and* $(X_2, M_2, m_2, \varphi_2)$ *are isomorphic, then:*

a) $A = \{f_i\}_{i=1}^{\infty}$ *is a countable partition of* X_1 *iff* $\psi(A) = \{\psi(f_i)\}_{i=1}^{\infty}$ *is a countable partition of* X_2*,* *b)* $H(A) = H(\psi(A))$,

c) $h(\varphi_1, A) = h(\varphi_2, \psi(A))$,

d) $\tilde{h}(\varphi_1) = \tilde{h}(\varphi_2)$, *i.e. the entropy of fuzzy dynamical system is an isomorphism invariant.*

Proof.

a) $m_2(\bigvee_{i=1}^{\infty} \psi(f_i)) = m_2(\psi(\bigvee_{i=1}^{\infty} f_i)) = m_1(\bigvee_{i=1}^{\infty} f_i) = 1$. Whenever $i \neq j$, then $\psi(f_i) \leqslant (\psi(f_j))'$ iff $\psi(f_i) = \psi(f_i) \wedge (\psi(f_j))' = \psi(f_i) \wedge (\psi(f_j')) = \psi(f_i \wedge f_j')$ iff $f_i = f_i \wedge f'_j$ iff $f_i \leqslant f'_j$, so $f_i \leqslant 1 - f_j$. We can do the same argument with ψ^{-1}.

b) $H(\psi(A)) = -log \sup_{i\in\mathbb{N}} m_2(\psi(f_i)) = -log \sup_{i\in\mathbb{N}} m_1(f_i)) = H(A)$.

c) $h(\varphi_2, \psi(A)) = \lim_{n\to\infty} \frac{1}{n} H(\bigsqcup_{i=0}^{n-1} (\varphi_2^i \psi(A))) = \lim_{n\to\infty} \frac{1}{n} H(\bigsqcup_{i=0}^{n-1} \psi(\varphi_1^i A))$
$= \lim_{n\to\infty} \frac{1}{n} H(\psi(\bigsqcup_{i=0}^{n-1} \varphi_1^i A)) = \lim_{n\to\infty} \frac{1}{n} H(\bigsqcup_{i=0}^{n-1} \varphi_1^i A) = h(\varphi_1, A)$.

d) By part c, for any fuzzy countable partition A of M_1 and B of M_2 we have $h(\varphi_1, A) = h(\varphi_2, \psi(A))$ and $h(\varphi_2, B) = h(\varphi_1, \psi^{-1}(B))$, so $\sup\{h(\varphi_1, A)$: A is a fuzzy countable partition of $X_1\} = \sup\{h(\varphi_2, A)$: A is a fuzzy countable partition of $X_2\}$.

Definition 4.11. *A fuzzy countable partition A of X is said to be a fuzzy generator of the fuzzy dynamical system (X, M, m, φ), if there exists an integer $r > 0$ such that:*

$$B \prec \bigsqcup_{i=0}^{r} \varphi^i(A),$$

for every fuzzy countable partition B of X.

Theorem 4.12. *If the fuzzy countable partition A is a fuzzy generator, then:*

a) $h(\varphi, B) \leqslant h(\varphi, A)$, *for every fuzzy countable partition B of X,*

b) $\tilde{h}(\varphi) = h(\varphi, A)$.

Proof.

a) A is a generator, so for each fuzzy countable partition B of X, $B \prec \bigsqcup_{i=0}^{r} \varphi^i(A)$. Theorem 4.8 implies $h(\varphi, B) \leqslant h(\varphi, \bigsqcup_{i=0}^{r} \varphi^i(A)) = h(\varphi, A)$.

b) From part a, for every fuzzy countable partition B of X we have $h(\varphi, B) \leqslant h(\varphi, A)$. Taking the supremum with respect to all fuzzy countable partitions of X, we obtain $\tilde{h}(\varphi) = \sup_B h(\varphi, B) = h(\varphi, A)$. □

Lemma 4.13. *Let $\varphi : M \to M$ be an m-preserving σ-homomorphism, invertible and φ^{-1} be an m-preserving σ-homomorphism, then:*

$$\tilde{h}\,(\varphi) = \tilde{h}\,\left(\varphi^{-1}\right).$$

Proof. It suffices to show that $h\,(\varphi,\, A) = h\left(\varphi^{-1},\, A\right)$, for all fuzzy countable partition A. But

$$H\left(\overset{n-1}{\underset{i=1}{\vee}}\, \varphi^{i}\,(A)\right) = H\left(\varphi^{-(n-1)}\overset{n-1}{\underset{i=1}{\vee}}\, \varphi^{i}\,(A)\right) = H\left(\overset{n-1}{\underset{j=1}{\vee}}\, \varphi^{-j}\,(A)\right),$$

because φ^{-1} is an m-preserving σ-homomorphism and Theorem 4.4 part b implies $H\,(\varphi^{-n}\,(A)) = H\,(A)$. $\square$

Theorem 4.14. *Let $\varphi : M \to M$ be an m-preserving σ-homomorphism, then:*

a) if $\varphi = 1_M$, then $\tilde{h}(\varphi) = 0$,

b) if φ is invertible, φ^{-1} be an m-preserving σ-homomorphis and there exists $k \in \mathbb{Z} - \{0\}$ such that $\varphi^k = 1_M$, then $\tilde{h}(\varphi) = 0$.

Proof.

a) For each fuzzy countable partition A, we have $\varphi(A) = A$ and this implies $\lim\limits_{n\to\infty} \frac{1}{n} H(\overset{n-1}{\underset{i=0}{\sqcup}} \varphi^i A) = \lim\limits_{n\to\infty} \frac{1}{n} H(\overset{n-1}{\underset{i=0}{\sqcup}} A) = \lim\limits_{n\to\infty} \frac{1}{n} H(A) = 0$.

b) First, we demonstrate that $\tilde{h}\left(\varphi^k\right) = |k|\,\tilde{h}\,(\varphi)$ for $k \in \mathbb{Z}$. For $k > 0$, it is clear by Theorem 4.8. part(h). If $k = 0$, then $\tilde{h}\left(\varphi^0\right) = \tilde{h}\,(1_M) = 0 = 0\,\tilde{h}\,(\varphi)$, where 1_M is the identity map over M. Let $k < 0$. We have $\tilde{h}\left(\varphi^k\right) = \tilde{h}\left(\left(\varphi^{-1}\right)^{|k|}\right) = |k|\,\tilde{h}\left(\varphi^{-1}\right)$. By the previous Lemma, $\tilde{h}\,(\varphi) = \tilde{h}\,\left(\varphi^{-1}\right)$. Therefore, we get $\tilde{h}\left(\varphi^k\right) = |k|\,\tilde{h}\,(\varphi)$.

Now, let $k \neq 0$. We have $\tilde{h}\,(\varphi) = \frac{1}{|k|}\,\tilde{h}\left(\varphi^k\right) = \frac{1}{|k|}\,\tilde{h}\,(1_M) = 0$. $\square$

5. Concluding Remarks

In this work, some properties of fuzzy dynamical systems with regard to fuzzy countable partitions are investigated. Mostly results are similar to those ones obtained in the classical theory. This new introduced entropy of a fuzzy dynamical system is also an isomorphism invariant, which is a useful property. In this paper, the join is suggested using the fuzzy infimum operation, but it seems that most of the results stated in this paper remain valid if we consider product fuzzy operation. Another interesting open problem could be trying to find other properties of fuzzy generators.

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