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Numerical Solution of The Linear Fredholm Integral Equations of the Second Kind

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Abstract. The theory of integral equation is one of the major topics of applied mathematics. The main purpose of this paper is to introduce a numerical method based on the interpolation for approximating the solution of the second kind linear Fredholm integral equation. In this case, the divided differences method is applied. At last, two numerical examples are presented to show the accuracy of the proposed method.

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1. Introduction

In recent years, the use of the Fredholm integral equation has increased in many physical applications, e.g. potential theory and Dirichlet problems, electrostatic, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat problems. Some valid methods for solving Fredholm integral equation have been developed such as quadrature methods, single-term Walsh series method ([7]), Lagrange interpolation ([6]) mixed interpolation collocation methods ([2]), Adomian's decomposition method ([9,3]),

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radial basis function networks ([4]). Furthermore, Mechanical algorithm methods ([8]) and approximate solution ([4]) for solving Fredholm integral equation had been developed. In this work, a new method is proposed to estimate the solution of a Fredholm integral equation of the second kind by using the divided differences method. The Fredholm functional integral equation of the second kind is defined by,

$$F(x) = f(x) + \lambda \int_a^b k(x,t)F(t)dt, \qquad a \leqslant x \leqslant b \tag{1}$$

where $\lambda > 0$ and $k(x,t)$ is an arbitrary kernel function over the square $a \leqslant x, t \leqslant b$. The rest of this essay is organized as follows. In Section 2, we present preliminaries. The proposed method is drawn in Section 3. The numerical examples and a comparison between the proposed method and the Adomian's decomposition method are given in Section 4. In Section 5, conclusion is discussed.

2. Preliminaries

Definition 2.1. *The interpolating polynomial $F_n(x)$, interpolating at the $n+1$ distinct points $x_0, x_1, ...x_n$ can be written as,*

$$F_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + ... + a_n(x-x_0)...(x-x_{n-1}), \tag{2}$$

and by Substituting, successive $x_0, x_1, ..., x_n$, we have,

$$a_0 = f[x_0], a_1 = f[x_0, x_1], ..., a_n = f[x_0, x_1, ..., x_n],$$

where

$$f[x_0] = f(x_0), f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, ..., f[x_0, ..., x_n]$$

$$= \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}.$$

Definition 2.2. *The $n \times n$ linear system*

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & y_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & y_2, \\ \vdots & & & & & & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & y_n, \end{cases} \tag{3}$$

where the given matrix of coefficients $A = (a_{ij}), 1 \leqslant i, j \leqslant n$, the unknown vector $X = (x_1, ..., x_n)^T$ and the right hand side vector $Y = (y_1, ..., y_n)^T$ all of them are real.

3. The Proposed Method

In this paper, we give a numerical method to compute numerical solutions of Fredholm integral equations by using divided differences interpolation. For this purpose, we approximate $F(x)$ and $\int_a^b k(x,t)F(t)dt$ with $F_n(x)$ and $\int_a^b k(x,t)F_n(t)dt$, respectively, as follow:

$$F_n(x) = F(x_0) + \sum_{j=0}^{n-1} F[x_0, ..., x_{j+1}](x - x_0)...(x - x_j), \tag{4}$$

and

$$\int_a^b k(x,t)F_n(t)dt = (b-a)k(x,x_0)F[x_0] + \sum_{j=0}^{n-1} k(x, x_{j+1})F[x_0, ...x_{j+1}]h_j, \tag{5}$$

where

$$h_j = \int_a^b (x - x_0)...(x - x_j)dx, \tag{6}$$

By Substituting (5) and (6) in (1), we get the following system of equations.

$$F[x_0] + \sum_{j=0}^{n-1} F[x_0, ..., x_{j+1}](x-x_0)...(x-x_j) = f(x_j) + \lambda((b-a)k(x, x_0)F[x_0]$$

$$+\sum_{j=0}^{n-1} k(x, x_{j+1})F[x_0, ...x_{j+1}]h_j). \qquad (7)$$

By replacing x with x_i for $i = 0, ..., n$, the (n+1)(n+1) equation system is obtained where by solving these system equation, $F[x_0], ..., F[x_0...x_n]$ are found. We also obtain $F_n(x)$ which is the interpolation polynomial for $F(x)$. Then the iterative procedure

$$F_{n,0}(x) = F_n(x),$$

$$F_{n,k+1}(x) = f(x) + \lambda \int_a^b k(x, t)F_{n,k}(t)dt$$

converges to the unique solution of Eq(1) ([10]).
In the following, the distance between approximate and exact solution are as follow:

$$D(F(x), F_{n,k}(x)) = |F(x) - F_{n,k}(x)|.$$

where $F_{n,k}(x)$ approximated from $F(x)$.

4. Examples of Linear Fredholm Integral Equations

Example 1. Consider the following Fredholm integral equation

$$F(x) = 2\sin(\frac{x}{2}) + 0.1 \int_0^{2\pi} \sin(t)\sin(\frac{x}{2})F(t)dt,$$

The exact solution in this case is given by:

$$F(x) = 2\sin(\frac{x}{2}).$$

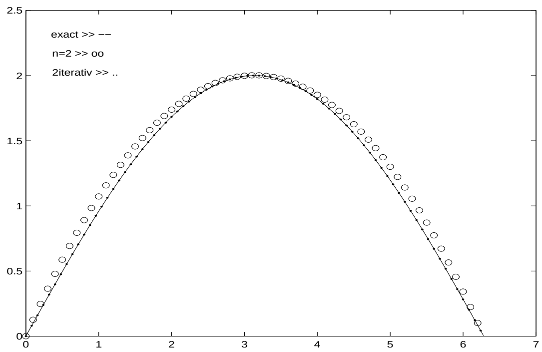

Figure 1. Comparison between approximate solutions with n=2 by 0, 2 iteration and the exact solution

Example 2. ([1]) Consider the following Fredholm integral equation .

$$F(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xtF(t)dt, \quad 0 \leqslant x \leqslant 1$$

with the exact solution :

$$F(x) = e^{3x}.$$

where the results are shown in figure 2 and Table 1 and 2.

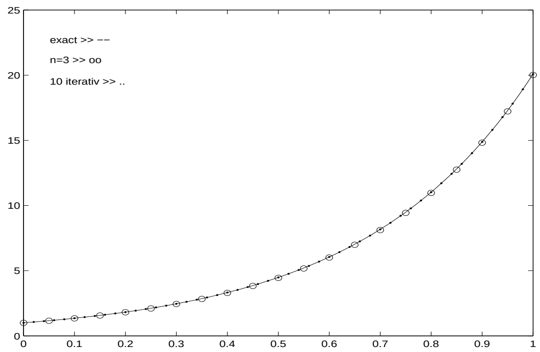

Figure 2. comparison between approximate solutions with n=3 by 0 ,10 iteration and the exact solution

Table 1. The distance of the exact solution and approximate solution of the proposed method for n=3 by 0,5,10,15 and 20 iterations.

x	$D(F, F_{3,0iter})$	$D(F, F_{3,5iter})$	$D(F, F_{3,10iter})$	$D(F, F_{3,15iter})$	$D(F, F_{3,20iter})$
0.0	0.000	0.000	0.000	0.000	0.000
0.1	4.444×10^{-1}	8.65×10^{-5}	3.560×10^{-7}	1.600×10^{-9}	2.000×10^{-10}
0.2	5.594×10^{-1}	1.730×10^{-4}	7.120×10^{-7}	3.200×10^{-9}	4.000×10^{-10}
0.3	4.664×10^{-1}	2.595×10^{-4}	1.068×10^{-6}	4.800×10^{-9}	6.000×10^{-10}
0.4	2.714×10^{-1}	3.459×10^{-4}	1.424×10^{-6}	6.400×10^{-9}	8.000×10^{-10}
0.5	6.030×10^{-2}	4.324×10^{-4}	1.781×10^{-6}	8.000×10^{-9}	1.000×10^{-9}
0.6	1.081×10^{-1}	5.189×10^{-4}	2.137×10^{-6}	9.600×10^{-9}	1.200×10^{-9}
0.7	2.121×10^{-1}	6.054×10^{-4}	2.493×10^{-6}	1.120×10^{-8}	1.400×10^{-9}
0.8	2.795×10^{-1}	6.919×10^{-4}	2.849×10^{-6}	1.280×10^{-8}	1.600×10^{-9}
0.9	4.056×10^{-1}	7.784×10^{-4}	3.205×10^{-6}	1.440×10^{-8}	1.800×10^{-9}
1.0	7.760×10^{-1}	8.648×10^{-4}	3.561×10^{-6}	1.600×10^{-8}	2.000×10^{-9}

Table 2. The distance between the exact solution and approximate solution of the Adomian's decomposition method for n=3 by 0,5,10,15 and 20 iterations.

x	$D(F, F_{3,0iter})$	$D(F, F_{3,5iter})$	$D(F, F_{3,10iter})$	$D(F, F_{3,15iter})$	$D(F, F_{3,20iter})$
0.0	0.000	0.000	0.000	0.000	0.000
0.1	4.575×10^{-1}	1.9×10^{-3}	7.75×10^{-6}	3.21×10^{-8}	3.22×10^{-10}
0.2	9.149×10^{-1}	3.8×10^{-3}	1.549×10^{-5}	6.41×10^{-8}	6.43×10^{-10}
0.3	1.3724	5.6×10^{-3}	2.324×10^{-5}	9.62×10^{-8}	9.65×10^{-10}
0.4	1.8298	7.5×10^{-3}	3.099×10^{-5}	1.283×10^{-7}	1.287×10^{-9}
0.5	2.2873	9.4×10^{-3}	3.874×10^{-5}	1.604×10^{-7}	1.609×10^{-9}
0.6	2.7447	1.13×10^{-2}	4.648×10^{-5}	1.924×10^{-7}	1.930×10^{-9}
0.7	3.2022	1.32×10^{-2}	5.423×10^{-5}	2.245×10^{-7}	2.252×10^{-9}
0.8	3.6597	1.51×10^{-2}	6.198×10^{-5}	2.566×10^{-7}	2.574×10^{-9}
0.9	4.1171	1.69×10^{-2}	6.973×10^{-5}	2.886×10^{-7}	2.896×10^{-9}
1.0	4.5746	1.88×10^{-2}	7.747×10^{-5}	3.207×10^{-7}	3.217×10^{-9}

Example 3. ([4]) Consider the integral equation

$$F(x) + \frac{1}{3}\int_0^1 e^{2x-\frac{5t}{3}} F(t)dt = e^{2x+\frac{1}{3}}, \quad 0 \leqslant x \leqslant 1$$

with the exact solution $F_e(x) = e^{2x}$. The numerical results are shown in Table 3.

Table 3. The distance between the exact solution and the approximate solution of radial basic function (RBF) networks and the proposed methods for n=5 by 10 iterations.

x	$D(F, F_{RBF})$	$D(F, F_{5,10iter})$
0.0	5.40631×10^{-7}	6.06×10^{-6}
0.1	4.17207×10^{-7}	7.4×10^{-6}
0.2	1.62255×10^{-7}	9.04×10^{-6}
0.3	9.97279×10^{-8}	1.104×10^{-5}
0.4	5.33277×10^{-7}	1.349×10^{-5}
0.5	$2.5.12821 \times 10^{-7}$	1.647×10^{-5}
0.6	8.86581×10^{-8}	2.012×10^{-5}
0.7	3.82386×10^{-7}	2.458×10^{-5}
0.8	6.76977×10^{-7}	3.002×10^{-5}
0.9	3.36868×10^{-7}	3.666×10^{-5}
1.0	5.00635×10^{-7}	4.478×10^{-5}

However, in Table 3 the numerical results showed that the RBF mehtod in comparison with the proposed method has more accuracy, but in RBF method, non-linear minimization problem is resulted that must be solved. Solving this kind of problems is not easy.

Example 4. ([5]) We consider a Fredholm integral of the second kind given by

$$F(x) - \int_{-1}^{1} (xt + x^2t^2)F(t)dt = 1, \quad -1 \leqslant x \leqslant 1$$

having the exact solution as follows:

$$F(x) = 1 + \frac{10}{9}x^2.$$

The numerical results are shown in Table 4.

Table 4. The distance between the exact solution and the approximate solution of Bernstein polynomials (BP) and the proposed methods (PM) for n=3 by 35 iterations.

x	0.0	±0.2	±0.4	±0.6	±0.8
F(x)(exact)	1.00000	1.04444	1.17777	1.40000	1.71111
F(x)(BP)	0.99999	1.04444	1.17777	1.40000	1.71111
F(x)(PM)	1.00000	1.04444	1.17777	1.40000	1.71111

5. Conclusion

In this paper, the divided differences method is applied to solve the linear Fredholm integral equation of the second kind. In this method, the coefficients of divided differences are given by solving a system of equations. In the comparison, the proposed method is better than the Adomian's decomposition method to approximate the exact solution. The advantage of the proposed method over other methods is that the integral equation is solved by having support points of the solution of integral equation.

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