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Some Properties of Autosoluble Groups

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Abstract. In this paper we introduce a new concept of autosoluble groups, which is in a way a generalized version of the notion of soluble groups. Using the autocommutators, a new series will be constructed, which is some how a generalization of the derived series of a given group G. We then determine the structure of such groups, when the generalized series are terminated.

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1. Introduction

Let G be a group and $Aut(G)$ the full automorphisms group of G, then for $\alpha \in Aut(G)$ and $g \in G$, $[g,\alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$ is the *autocommutator* of g and α. Clearly, if $\alpha = \varphi_x$ $(x \in G)$ is an inner automorphism then $[g,\varphi_x] = g^{-1}g^{\varphi_x} = g^{-1}x^{-1}gx$, which is the ordinary commutator of the element g and x of G. We may define the autocommutator of higher weight inductively as follows:

$$[g,\alpha_1,\alpha_2,\ldots,\alpha_n] = [[g,\alpha_1,\alpha_2,\ldots,\alpha_{n-1}],\alpha_n],$$

for all $\alpha_1,\alpha_2,\ldots,\alpha_n \in Aut(G)$, $g \in G$ and $n \geqslant 1$.

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The subgroup

$$K(G) = [G, Aut(G)] = \langle [g, \alpha] \mid g \in G, \alpha \in Aut(G) \rangle$$

is called the *autocommutator subgroup* of G (see [3]). Assume $K_0(G) = G$ and $K_1(G) = K(G)$, then for $n \geqslant 1$ we may define

$$\begin{aligned} K_n(G) &= [K_{n-1}(G), Aut(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G) \rangle, \end{aligned}$$

which is the natural generalization of $\gamma_{n+1}(G)$, the $(n+1)^{st}$-term of the lower central series of G. Clearly, $K_n(G) = \gamma_{n+1}(G)$, when all the automorphisms α_i's are taken to be the inner automorphisms of G. One can easily see that $\gamma_{n+1}(G) \leqslant K_n(G)$, $n \geqslant 1$ and $K_n(G)$ is a characteristic subgroup of G. Hence, we obtain the following descending series of G.

$$G \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq \ldots \supseteq K_n(G) \supseteq \ldots \quad . \tag{1}$$

We may also define

$$K^{(2)}(G) = K(K(G)) = [K(G), Aut(K(G))]$$

and inductively,

$$K^{(n)}(G) = K(K^{(n-1)}(G)) \quad , \quad n \geqslant 2,$$

which is called the n^{th}-*autocommutator subgroup* of G. Clearly, if we consider the inner automorphisms of G, we obtain the n^{th}-*derived subgroup*, $G^{(n)}$ of G and hence $G^{(n)}$ is contained in $K^{(n)}(G)$.
The *absolute centre* of G is defined as follows:

$$L(G) = \{x \in G \mid [x, \alpha] = 1 \ , \ \forall \alpha \in Aut(G)\},$$

which is contained in $Z(G)$, the centre of G. Now, assume $L_1(G) = L(G)$ and the n^{th}-absolute centre of G is defined in the following way

$$\frac{L_n(G)}{L_{n-1}(G)} = L\left(\frac{G}{L_{n-1}(G)}\right) \quad , \quad \text{for } n \geqslant 2.$$

Clearly, if we consider the canonical homomorphism $\varphi : G \longrightarrow \frac{G}{L_{n-1}(G)}$, one may define $L_n(G) = \varphi^{-1}(L(\frac{G}{L_{n-1}}))$. Now we call G to be an *autonilpotent group*, whenever $L_n(G) = G$, for some $n \geqslant 1$. One can easily see that $L_n(G) \leqslant Z_n(G)$ and so every autonilpotent group is nilpotent. Also in [6, Theorem 2.13], we have proved that any finite abelian group is autonipotent if and only if is a cyclic 2-group. It can be verified that for any natural number n,

$$G^{(n)} \leqslant \gamma_{n+1}(G) \leqslant K_n(G) \leqslant K^{(n)}(G).$$

One observes that, if $L_n(G) = G$ then $K_n(G) = 1$. By the above discussion, we may define the following

Definition 1.1. *A group G is called autosoluble if $K^{(n)}(G) = \langle 1 \rangle$, for some natural number n.*
Clearly, the autosolubility of groups implies solubility and nilpotency, while their converses are not valid, in general. For counter examples, consider the cyclic group $\mathbb{Z}_p$ of odd prime order p then $K(\mathbb{Z}_p) = \mathbb{Z}_p$. Also, the symmetric group S_3 is soluble, which is not autosoluble.
For a given group G, we have the following descending chain of characteristic autocommutator subgroups

$$G \supseteq K(G) \supseteq K^{(2)}(G) \supseteq \ldots \supseteq K^{(n)}(G) \supseteq \ldots \quad .$$

So, one is interested to know under what conditions the above series terminates, i.e., the group G is autosoluble. This is the concept, which will be studied in the next section.

2. Properties of Autosoluble Groups

In this section, we give some properties of autosoluble groups. In fact, we show that an abelian group is autosoluble if and only if it is a cyclic group.
The following result of [3] is useful in our investigation.

Theorem 2.1. *If G and H are finite groups with $(|G|,|H|) = 1$, then*

$$Aut(G \times H) \cong Aut(G) \times Aut(H).$$

The proof of the following result may be verified easily.

Theorem 2.2. *Let H_1 and H_2 be characteristic subgroups of a given group $G = H_1 \times H_2$. Then*

$$K(H_1 \times H_2) = K(H_1) \times K(H_2).$$

Theorem 2.3. *Let G and H be autosoluble finite groups with coprime orders. Then $G \times H$ is also autosoluble.*

Proof. The proof follows using induction and the above results. □
The following lemmas are needed for proving our main theorem.

Lemma 2.4. *If H is a characteristic subgroup of index two of a given group G, then $K(G)$ is contained in H.*

Proof. Clearly $G = H \cup gH$ and since $\alpha(g) \notin H$, for all $\alpha \in Aut(G)$, the result follows by the definition of $K(G)$. □

Lemma 2.5. *Let G be a finite cyclic group, then $K^{(n)}(G) = G^{2^n}$.*

Proof. It is enough to prove the result for $n = 1$, then the claim follows inductively. So let $G = \langle x \mid x^m = 1 \rangle$ be the cyclic group of order m, then the map $\alpha : G \longrightarrow G$ given by $\alpha(x) = x^{-1}$ is an automorphism of G. Hence $x^2 = [x^{-1}, \alpha]^{-1} \in K(G)$, which implies that $G^2 \subseteq K(G)$. If m is odd, then it is easily seen that $K(G) \leqslant G = G^2$. The case m is even, implies that G^2 is a characteristic subgroup of index 2 in G and hence by Lemma 2.4, the autocommutator subgroup $K(G)$ is contained in G^2, which completes the proof. □

Lemma 2.6. *Let G be a finite abelian group of odd order, then $K^{(n)}(G) = G$, for all $n \in \mathbb{N}$.*
The following proposition determines a class of abelian groups, which are non-autosoluble, and its proof can be seen easily.

Proposition 2.7. *If* $G \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{n-times} \quad (n \geqslant 2)$*, then* $K(G) = G$.

Remark 2.8. *If*

$$G \cong \underbrace{\mathbb{Z}_{2^m} \times \ldots \times \mathbb{Z}_{2^m}}_{n-times} \times \mathbb{Z}_{2^{k_1}} \ldots \times \mathbb{Z}_{2^{k_r}} \quad (m > k_1 \geqslant \ldots \geqslant k_r \geqslant 0, n \geqslant 2),$$

then $K(G) = G$.

One notes that for a nontrivial group G, $K(G) = \langle 1 \rangle$ *if and only if* $G \cong \mathbb{Z}_2$. *Therefore no groups of odd order can be autosoluble, because they must contain a cyclic subgroup of order 2. In the other words, such groups have even orders.*
In the following, we determine the structure of abelian 2-groups which are autosoluble.

Lemma 2.9. *Let* $G = \mathbb{Z}_{2^n}$ *be the cyclic group of order* 2^n *and* H *be an abelian 2-group of exponent* 2^m *with* $m < n$. *Then*

$$K(G \times H) = G^2 \times H.$$

Proof. See [1]. □
The above lemma gives the following

Corollary 2.10. *Let* $n > m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r$ *be natural numbers, then*

$$K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \ldots \times \mathbb{Z}_{2^{m_r}}) = \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^{m_1}} \times \ldots \times \mathbb{Z}_{2^{m_r}}.$$

Now, we are able to prove our main theorem of this section.

Theorem 2.11. *The finite abelian group* G *is autosoluble if and only if* $G \cong \mathbb{Z}_{2^n}$, *for some natural number* n.

Proof. Assume G is a finite abelian autosoluble group, then the group G contains a sylow 2-subgroup. So G is the direct product of its sylow subgroups, i.e., $G \cong P_1 \times P_2 \times \ldots \times P_r$, with at least one sylow 2-subgroup.

Now, if $|G|$ has a prime divisor p $(p \neq 2)$, then the sylow p-subgroup P say, is of odd order and hence can not be autosoluble. Thus $K^{(m)}(P) \neq \langle 1 \rangle$, for all $m \in \mathbb{N}$. One notes that, since the orders of P_i's are mutually coprime, by Theorem 2.2, we have $K^{(m)}(G) = K^{(m)}(P_1) \times \ldots \times K^{(m)}(P_r)$, for all $m \in \mathbb{N}$. Now, as G is autosoluble we must have $K^{(s)}(P_i) = \langle 1 \rangle$, for some $s \in \mathbb{N}$ and all $1 \leqslant i \leqslant r$, which is a contradiction. Therefore $|G|$ does not have any prime divisors except 2 and so either G is cyclic or

$$G \cong \mathbb{Z}_{2^{m_1}} \times \ldots \times \mathbb{Z}_{2^{m_t}}, \qquad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_t \geqslant 0, t \geqslant 2.$$

If for some i, $m_i \neq 0$, then by repeated applications of Corollary 2.10 there exists $d \in \mathbb{N}$ such that

$$K^{(d)}(G) \cong \underbrace{\mathbb{Z}_{2^m} \times \ldots \times \mathbb{Z}_{2^m}}_{n-times} \times \mathbb{Z}_{2^{k_1}} \ldots \times \mathbb{Z}_{2^{k_r}} \quad (k_1, \ldots, k_r, m \in \mathbb{N}, n \geqslant 2).$$

Clearly, by Remark 2.8 the group G can not be autosoluble, which gives a contradiction and hence $G \cong \mathbb{Z}_{2^n}$, as required. □

Conversely, Lemma 2.5 gives the result.
By the discussion before the Definition 1.1, we have the following

Corollary 2.12. *Every abelian autonilpotent group is autosoluble.*
Finally, we give an example of a family of non-abelian autosoluble groups.

Example 2.13. The dihedral 2-groups are autosoluble. To see this let

$$G = \langle a, b \mid a^{2^{n-1}} = b^2 = 1 \ , \ bab = a^{-1} \rangle,$$

be the dihedral 2-group of order 2^n. Then clearly the group of automorphisms of G consists of the following set:

$$Aut(G) = \left\{ \varphi_{ij} \;\middle|\; \varphi_{ij} : \begin{matrix} a \longmapsto a^i \\ b \longmapsto a^j b \end{matrix} \ , \ i \ \text{ is odd}, \ 1 \leqslant i \leqslant n \ \text{ and } \ 0 \leqslant j < 2^{n-1} \right\}.$$

An easy calculation implies that $K(G) \cong \mathbb{Z}_{2^{n-1}}$ and since $\mathbb{Z}_{2^{n-1}}$ is autosoluble of *length* $n-1$, it implies that G is autosoluble of *length* n. We remark that the generalized quaternion groups are not autosoluble. This can be verified using the structure of such groups.

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