Journal of Mathematical Extension
Vol. 10, No. 2, (2016), 75-86
ISSN: 1735-8299
URL: http://www.ijmex.com

A Note on An Engel Condition with Generalized Derivations in Rings

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Abstract. Let R be a prime ring with characteristic different from two, I be a nonzero ideal of R, and F be a generalized derivation associated with a nonzero derivation d of R. In the present paper we investigate the commutativity of R satisfying the relation $F([x,y]_k)^n = ([x,y]_k)^l$ for all $x, y \in I$, where l, n, k are fixed positive integers. Moreover, let R be a semiprime ring, $A = O(R)$ be an orthogonal completion of R, and $B = B(C)$ be the Boolean ring of C. Suppose $F([x,y]_k)^n = ([x,y]_k)^l$ for all $x, y \in R$, then there exists a central idempotent element e of B such that d vanishes identically on eA and the ring $(1-e)A$ is commutative.

AMS Subject Classification: 16N60; 16U80; 16W25
Keywords and Phrases: Prime and semiprime rings, generalized derivation, generalized polynomial identity (GPI), ideal

1. Introduction

Let R be an associative ring with center $Z(R)$. For each $x, y \in R$, define $[x,y]_k$ inductively by $[x,y]_1 = xy - yx$ and $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. The ring R is said to satisfy an Engel condition if there exists a positive integer k such that $[x,y]_k = 0$ for all $x, y \in R$. Note that an Engel condition is a polynomial

Received: July 2015; Accepted: November 2015
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$[x,y]_k = \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m}$ in non-commutative indeterminates x, y and $[x+z,y]_k = [x,y]_k + [z,y]_k$. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y+yd(x)$ holds, for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds, for all $x \in R$. An additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds, for all $x, y \in R$.

The Engel type identity with derivation first appeared in the well-known paper of Posner [17] which states that a prime ring admitting a nonzero derivation d must be commutative if $[d(x), x] \in Z(R)$ holds, for all $x \in R$. Since then, several authors have studied this kind of Engel type identities with derivations acting on an appropriate subset of prime and semiprime rings (see [6, 8, 19] for a partial bibliography). In 1992, Daif and Bell [4, Theorem 3] proved that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d of R such that $d([x,y]) = [x,y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In addition, if R is a prime ring, then R is commutative. In 2003, Quadri et al. [18] extended the result of Daif and Bell and proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x,y]) = [x,y]$ for all $x, y \in I$, then R is commutative. Very recently, Huang and Davvaz [9] generalized the result of Quadri et al. and proved that if R is a prime ring and F is a generalized derivation associated with a nonzero derivation d of R such that $F([x,y])^m = [x,y]^n$ for all $x, y \in R$, where m, n are fixed positive integers, then R is commutative.

On the other hand, in 1994 Giambruno et al. [7] established that a ring must be commutative if it satisfies $([x,y]_k)^n = [x,y]_k$. Inspired by the above mention results it is natural to investigate what we can say about the commutativity of ring satisfying the relation $F([x,y]_k)^n = ([x,y]_k)^l$, where F is a generalized derivation associated with a nonzero derivation d of R and l, n, k are fixed positive integers.
If we take $k = 1$, then we obtain the following:

Corollary 1.1. ([9, Theorem A]) *Let R is a prime ring and n, l are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x,y])^n = ([x,y])^l$ for all $x, y \in R$, then R is commutative.*

2. Generalized Derivation in Prime Ring

Throughout this section, we take R is a prime ring, I is a nonzero ideal, U is the Utumi quotient ring, C is the extended centroid and Q is the symmetric

Martindale quotient ring. For a complete and detailed description of the theory of generalized polynomial identities involving derivations, we refer to [1].

We denote by $Der(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \ldots d_m$ with each $d_i \in Der(U)$. Then a differential polynomial is a generalized polynomial with coefficients in U of the form $\Phi(\Delta_j x_i)$ involving non-commuting indeterminates x_i on which the derivation words Δ_j act as unary operations. The differential polynomial $\Phi(\Delta_j x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i. Let D_{int} be the C-subspace of $Der(U)$ consisting of all inner derivations on U and d be a nonzero derivation on R. By [11, Theorem 2], we have the following result (see also [13, Theorem 1].

If $\Phi(x_1, \ldots, x_n, {}^d x_1, \ldots, {}^d x_n)$ is a differential identity on R, then one of the following assertions holds:

(i) either $d \in D_{int}$;

(ii) or, R satisfies the generalized polynomial identity

$$\Phi(x_1, \cdots, x_n, y_1, \cdots, y_n).$$

Before starting our result, we state the following theorem which is very crucial for developing the proof of our main result.

Theorem 2.1. ([14, Theorem 3]) *Every generalized derivation F on a dense right ideal of R can be uniquely extended to a generalized derivation of U and assumes of the form $F(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

Lemma 2.2. *Let R be a prime ring with characteristics different from two, n, k be the fixed positive integers and $b \in Q$ with $b \notin C$ such that $([b, x]_{k+1})^n = 0$ for all $x \in R$. Then R satisfies a nonzero generalized polynomial identity (GPI).*

Proof. By both [1, Theorem 6.4.1] and [3, Theorem 2], we have

$$([b, x]_{k+1})^n = 0 \text{ for all } x \in Q.$$

That is, the element $([b, X]_{k+1})^n$ in the free product $T = Q *_C C\{X\}$ is a generalized polynomial identity on R. As $b \notin C$, we can easily see that the term $(bX^{k+1})^n$ appears nontrivially in the expansion of $([b, X]_{k+1})^n$. So $([b, X]_{k+1})^n$ is a nonzero element in $T = Q *_C C\{X\}$. Therefore, R satisfies a nonzero generalized polynomial identity. □

Now, we prove our main result of this section.

Theorem 2.3. *Let R be a prime ring with characteristics different from two and I be a nonzero ideal of R. If R admits a nonzero generalized derivation F associated with a nonzero derivation d such that $F([x,y]_k)^n = ([x,y]_k)^l$ for all $x, y \in I$, where l, n, k are fixed positive integers, then R is commutative.*

Proof. Since R is a prime ring and $F([x,y]_k)^n = ([x,y]_k)^l$ for all $x, y \in I$. By Theorem 2.1, for some $a \in U$ and a derivation d on U such that I satisfies the differential identity

$$(a[x,y]_k + d([x,y]_k))^n = ([x,y]_k)^l,$$

which can be written as

$$\begin{aligned}\Big(a(\sum_{m=0}^{k}(-1)^m & \binom{k}{m} y^m x y^{k-m}) \\ + & \sum_{m=0}^{k}(-1)^m\binom{k}{m}(\sum_{i+j=m-1} y^i d(y) y^j) x y^{k-m} \\ + & \sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m d(x) y^{k-m} \\ + & \sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m x(\sum_{r+s=k-m-1} y^r d(y) y^s)\Big)^n \\ - & (\sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m x y^{k-m})^l = 0. \qquad (1)\end{aligned}$$

Firstly we assume that d is an outer derivation on Q. By Kharchencko's Theorem [11], I satisfies the generalized polynomial identity

$$\begin{aligned}&\Big(a(\sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m x y^{k-m}) + \sum_{m=0}^{k}(-1)^m\binom{k}{m}(\sum_{i+j=m-1} y^i z y^j) x y^{k-m} \\ &+ \sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m w y^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m x(\sum_{r+s=k-m-1} y^r z y^s)\Big)^n \\ &- (\sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m x y^{k-m})^l = 0.\end{aligned}$$

In particular $x = z = 0$, we have

$$(\sum_{m=0}^{k}(-1)^m\binom{k}{m} y^m w y^{k-m})^n = 0 \ \text{ for all } y, w \in I.$$

By Chuang [3, Theorem 2], this polynomial identity is also satisfied by Q and hence R as well, i.e., $(\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m w y^{k-m})^n = 0$ for all $y, w \in R$. Substituting y with $[b, w]$, where b is a noncentral element of R in the above identity, we have $([b, w]_{k+1})^n = 0$ for all $w \in R$. It follows from both [1, Theorem 6.4.1] and [3, Theorem 2] that $([b, w]_{k+1})^n = 0$ for all $w \in Q$.

In case C is infinite, we have $([b, w]_{k+1})^n = 0$ for all $w \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both C and $Q \otimes_C \overline{C}$ are centrally closed [5, Theorem 2.5 and Theorem 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $([b, w]_{k+1})^n = 0$ for all $w \in R$. By Lemma 2.2, R is a nontrivial generalized polynomial identity (GPI). By Martindale's Theorem [15], R is a primitive ring and so is isomorphic to a dense subring of linear transformations on a vector space $\mathcal{V}$ over C.

Suppose that $\mathcal{V}$ is infinite dimensional over C. For any $v \in \mathcal{V}$, we claim that v and vb are C-dependent. On contrary suppose that v and vb are C-independent. We choose $v_1, v_2, \cdots, v_k$ such that $v, vb, v_1, \cdots, v_k$ are C-dependent. By the density of R on $\mathcal{V}$, there exists $x_0 \in R$ such that

$$vx_0 = 0,\ vbx_0 = v_1,\ v_i x_0 = v_{i+1},\ v_k x_0 = v,\ \text{where } i = 1, 2, \cdots, k-1.$$

We see that

$$v[b, x_0]_{k+1} = vbx_0^{k+1} = v_1 x_0^k = v_2 x_0^{k-1} = \cdots = v_k x_0 = v,$$

and so $0 = v([b, x_0]_{k+1})^n = v \neq 0$, a contradiction. Our next goal is to show that there exists $\alpha \in C$ such that $bv = v\alpha$, for any $v \in \mathcal{V}$. Now choose $v, w \in \mathcal{V}$ such that they are linearly C-independent. By the previous argument there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that $bv = v\alpha_v,\ bw = w\alpha_w,\ b(v + w) = (v + w)\alpha_{v+w}$. Moreover $v\alpha_v + w\alpha = (v + w)\alpha_{v+w}$. Hence $v(\alpha_v - \alpha_{v+w}) + w(\alpha_w - \alpha_{v+w}) = 0$, and because v, w are linearly C-independent, we have $\alpha_v = \alpha_w = \alpha_{v+w}$, that is, α does not depend on the choice of v. Now for $r \in R, v \in \mathcal{V}$, we have $bv = v\alpha$, $r(bv) = r(v\alpha)$ and also $b(rv) = (rv)\alpha$. Thus $0 = [b, r]v$, for any $v \in \mathcal{V}$, that is $[b, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[b, r] = 0$, for all $r \in R$, i.e., $b \in C$, a contradiction.

So $\mathcal{V}$ must be of finite dimensional, i.e., $R \cong M_t(\mathbb{F})$ for some $t > 1$. Now we assume that $t = 2$, i.e., $M_2(\mathbb{F})$ satisfies $([b, w]_{k+1})^n = 0$. Let e_{ij} be the usual unit matrix with 1 in (i, j)-entry and zero elsewhere. Take $b = \sum_{i,j=1}^{2} b_{ij}e_{ij}$ with $b_{ij} \in \mathbb{F}$ and by choosing $w = e_{11}$, we see that $[b, e_{11}]_{k+1} = (-1)^{k+1}b_{12}e_{12} + b_{21}e_{21}$. Thus we have

$0 = ([b, e_{11}]_{k+1})^{2n} = (-1)^{(k+1)n}(b_{12}b_{21})^n e_{11} + (-1)^{(k+1)n}(b_{12}b_{21})^n e_{22}$ which gives $b_{12}b_{21} = 0$ and so either $b_{12} = 0$ or $b_{21} = 0$. Now we assume that $b_{21} = 0$. Let χ be any automorphism of R such that $\chi(x) = (1+e_{21})x(1-e_{21})$. Therefore $\chi(b) = (b_{11} - b_{12})e_{11} + b_{12}e_{12} + (b_{11} - b_{12} - b_{22})e_{21} + (b_{12} + b_{22})e_{22}$. Since $([\chi(b), w]_{k+1})^n = 0$ for all $x \in R$, then it can be easily seen that $b_{12}(b_{11} - b_{12} - b_{22}) = 0$. Hence either $b_{12} = 0$ or $(b_{11} - b_{12} - b_{22}) = 0$. Suppose that $(b_{11} - b_{12} - b_{22}) = 0$. If k is even, then by easy computation we see that $0 = ([b, e_{11} + e_{21}]_{k+1})^{2n} = (2b_{12}^2)^n e_{11} + (2b_{12}^2)^n e_{22}$. It implies that $(2b_{12}^2)^n = 0$ and so $b_{12} = 0$. If k is odd, then we have $0 = ([b, e_{11} + e_{21}]_{k+1})^{2n} = (-2b_{12}^2)^n e_{11} + (-2b_{12}^2)^n e_{22}$, which implies that $(-2b_{12}^2)^n = 0$ and so $b_{12} = 0$. Thus in all, b is a diagonal matrix. As above we know that $\chi(b) = \sum_{i=1}^2 b_{ii}e_{ii} + (b_{11} - b_{22})e_{21}$ is a diagonal matrix. Therefore, $b_{11} = b_{22}$, and so, b is central in R, a contradiction.

Now we consider the case when $t > 2$. Let $b = \sum\limits_{i,j=1}^{t}$ with $b_{ij} \in \mathbb{F}$. Write $b = \begin{pmatrix} b_{11} & \mathcal{A} \\ \mathcal{B} & \mathcal{C} \end{pmatrix}$ where $\mathcal{A} = (b_{12}, \cdots, b_{1t})$ $\mathcal{B} = (b_{21}, \cdots, b_{t1})^T$ and $\mathcal{C} = (b_{ij})$ with $2 \leqslant i, j \leqslant t$. Note that $[b, e_{11}]_{k+1} = \begin{pmatrix} 0 & (-1)^{k+1}\mathcal{A} \\ \mathcal{B} & 0 \end{pmatrix}$. By given hypothesis, one can have

$$([b, e_{11}]_{k+1})^{2n} = \begin{pmatrix} (-1)^{n(k+1)}(\mathcal{A}\mathcal{B})^n & 0 \\ 0 & (-1)^{n(k+1)}(\mathcal{B}\mathcal{A})^n \end{pmatrix}.$$

In particular $(-1)^{n(k+1)}(\mathcal{A}\mathcal{B})^n = 0$ and so $\mathcal{A}\mathcal{B} = 0$.

Let χ_{ij} be an inner automorphism of R given by $\chi_{ij}(x) = (1 + e_{ij})x(1 - e_{ij})$ for $x \in R$. Write $1+e_{21} = \begin{pmatrix} 1 & 0 \\ \mathcal{E}_2 & \mathcal{I}_{t-1} \end{pmatrix}$ where $\mathcal{E}_2 = (1, 0, \cdots, 0)^T$ and $\mathcal{I}_{t-1}$ is the $(n-1)$-identity matrix. Thus $\chi_{21}(b) = \begin{pmatrix} b_{11} - b_{12} & \mathcal{A} \\ b_{11}\mathcal{E}_2 - b_{12}\mathcal{E}_2 + \mathcal{B} - \mathcal{C}\mathcal{E}_2 & \mathcal{E}_2\mathcal{A} + \mathcal{C} \end{pmatrix}$. By easy calculation, it follows that $b_{11}b_{12} - b_{12}^2 - \mathcal{A}\mathcal{C}\mathcal{E}_2 = 0$. Suppose first that k is even. We can easily see that $[b, e_{11} + e_{21}]_{k+1} = \begin{pmatrix} b_{12} & -\mathcal{A} \\ \mathcal{J}_1 & -\mathcal{E}_2\mathcal{A} \end{pmatrix}$ where $\mathcal{J}_1 = \mathcal{B} + \mathcal{C}\mathcal{E}_2 - \mathcal{E}_2 b_{11}$. Therefore $([b, e_{11} + e_{21}]_{k+1})^2 = \begin{pmatrix} b_{12}^2 - \mathcal{A}\mathcal{J}_1 & 0 \\ * & -\mathcal{J}_1\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A} \end{pmatrix}$. Making use of both $\mathcal{A}\mathcal{B} = 0$ and $b_{11}b_{12} - b_{12}^2 - \mathcal{A}\mathcal{C}\mathcal{E}_2 = 0$, we get $\mathcal{A}\mathcal{J}_1 = -b_{12}^2$. Thus $([b, e_{11} + e_{21}]_{k+1})^2 = \begin{pmatrix} 2b_{12}^2 & 0 \\ * & -\mathcal{J}_1\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A} \end{pmatrix}$. Therefore by assumption, we have

$$0 = ([b, e_{11} + e_{21}]_{k+1})^{2n} = \begin{pmatrix} (2b_{12}^2)^n & 0 \\ * & (-\mathcal{J}_1\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A})^n \end{pmatrix}.$$

In particular, $(2b_{12}^2)^n = 0$, and so $b_{12} = 0$. Next suppose that k is odd. By computation we have $[b, e_{11} + e_{21}]_{k+1} = \begin{pmatrix} -b_{12} & \mathcal{A} \\ \mathcal{J}_2 & \mathcal{E}_2\mathcal{A} \end{pmatrix}$ where $\mathcal{J}_2 = \mathcal{B} + \mathcal{C}\mathcal{E}_2 - (b_{11} + 2b_{12})\mathcal{E}_2 b_{11}$. Thus

$$([b, e_{11} + e_{21}]_{k+1})^2 = \begin{pmatrix} b_{12}^2 + \mathcal{A}\mathcal{J}_2 & 0 \\ * & \mathcal{J}_2\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A} \end{pmatrix}.$$

Applying both $\mathcal{A}\mathcal{B} = 0$ and $b_{11}b_{12} - b_{12}^2 - \mathcal{A}\mathcal{C}\mathcal{E}_2 = 0$, we get $\mathcal{A}\mathcal{J}_2 = -3b_{12}^2$. Thus

$$([b, e_{11} + e_{21}]_{k+1})^2 = \begin{pmatrix} -2b_{12}^2 & 0 \\ * & \mathcal{J}_2\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A} \end{pmatrix},$$

and so

$$0 = ([b, e_{11} + e_{21}]_{k+1})^{2n} = \begin{pmatrix} (-2b_{12}^2)^n & 0 \\ * & (\mathcal{J}_2\mathcal{A} + b_{12}\mathcal{E}_2\mathcal{A})^n \end{pmatrix}.$$

In particular, $(-2b_{12}^2)^n = 0$, and so $b_{12} = 0$.

Now we claim that b is a diagonal matrix. Since $([\chi_{j2}(b), x]_{k+1})^n = 0$ for all $x \in R$, where $j > 2$, as what has been shown, we get that $-b_{1j} = \chi_{j1}(b)_{12} = 0$. So $b_{1j} = 0$ for $j > 1$. For $1 < j < s \leqslant t$, we get from $([\chi_{j2}(b), x]_{k+1})^n = 0$ for all $x \in R$, that $b_{js} = \chi_{1j}(b)_{1s} = 0$. This shows that b must be lower triangular. Since the transpose of b satisfies the same condition, b is indeed diagonal. We have shown that $b = \sum_{i=1}^{n} b_{ii}e_{ii}$ with $b_{ii} \in \mathbb{F}$. For $1 \leqslant i \neq j \leqslant t$, as above we get that $\chi_{ij}(b)$ is a diagonal matrix. On the other hand, $\chi(b) = b + (b_{jj} - b_{ii})e_{ij}$, we infer that $b_{jj} = b_{ii}$, and so b is central in R, a contradiction.

Secondly we assume that d is an inner derivation induced by an element $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$. Therefore from (1), we have

$$(a[x, y]_k + [q, [x, y]_k])^n = ([x, y]_k)^l \quad \text{for all } x, y \in I.$$

By Chuang [3, Theorem 2], I and Q satisfy the same generalized polynomial identities, thus we have

$$(a[x, y]_k + [q, [x, y]_k])^n = ([x, y]_k)^l \quad \text{for all } x, y \in Q.$$

In case the center C of Q is infinite, we have

$$(a[x, y]_k + [q, [x, y]_k])^n = ([x, y]_k)^l \text{ for all } x, y \in Q \otimes_C \overline{C},$$

where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [5, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$

according as C is finite or infinite. Thus we may assume that R is centrally closed over C ($i.e., RC = R$) which is either finite or algebraically closed and $(a[x,y]_k + [q,[x,y]_k])^n = ([x,y]_k)^l$ for all $x, y \in R$. By Martindale's Theorem [15, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring. Hence by Jacobson's Theorem [10, p.75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$, then the density of R on $\mathcal{V}$ implies that $R \cong M_t(\mathcal{D})$, where $t = dim_{\mathcal{D}}\mathcal{V}$.

Assume first that $dim_{\mathcal{D}}\mathcal{V} \geqslant 3$. First of all, we want to show that for any $v \in \mathcal{V}$, v and qv are linearly $\mathcal{D}$-dependent. If $v = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Now suppose that $v \neq 0$ and $\{v, qv\}$ is linearly $\mathcal{D}$-independent. Since $dim_{\mathcal{D}}\mathcal{V} \geqslant 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ is also linearly $\mathcal{D}$-independent. By the density of R there exist $x, y \in R$ such that

$$\begin{array}{lll} xv = v, & xqv = 0, & xw = v \\ yv = 0, & yqv = w, & yw = w. \end{array}$$

This implies that $(-1)^n v = (a[x,y]_k + [q,[x,y]_k])^n v - ([x,y]_k)^l v = 0$, a contradiction. So, we conclude that $\{v, qv\}$ are linearly $\mathcal{D}$-dependent, for all $v \in \mathcal{V}$. A standard argument shows that $q \in C$ and $d = 0$, which contradicts our hypothesis.

Therefore $dim_{\mathcal{D}}\mathcal{V}$ must be $\leqslant 2$. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [12, Lemma 2], it follows that there exists a suitable filed $\mathbb{F}$ such that $R \subseteq M_t(\mathbb{F})$, the ring of all $t \times t$ matrices over $\mathbb{F}$, and moreover, $M_t(\mathbb{F})$ satisfies the same generalized polynomial identity of R.

If we assume $t \geqslant 3$, then by the same argument as above, we get a contradiction. Obviously if $t = 1$, then R is commutative. Thus we may assume that $t = 2$, i.e., $R \subseteq M_2(\mathbb{F})$, where $M_2(\mathbb{F})$ satisfies $(a[x,y]_k + [q,[x,y]_k])^n = ([x,y]_k)^l$. Since by choosing $x = e_{12},\ y = e_{22}$ we have $(ae_{12} + qe_{12} - e_{12}q)^n = 0$. Right multiplying by e_{12}, we get $(-1)^n(e_{12}q)^n e_{12} = 0$. Now set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation, we find that $(-1)^n \begin{pmatrix} 0 & q_{21}^n \\ 0 & 0 \end{pmatrix} = 0$, which implies that $q_{21} = 0$. In the same manner, we can see that $q_{12} = 0$. Thus we conclude that q is a diagonal matrix in $M_2(\mathbb{F})$. Let $\chi \in Aut(M_2(\mathbb{F}))$. Since $(\chi(a)[\chi(x),\chi(y)]_k + [\chi(q),[\chi(x),\chi(y)]_k])^n = ([\chi(x),\chi(y)]_k)^l$, then $\chi(q)$ must be diagonal matrix in $M_2(\mathbb{F})$. In particular, let $\chi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\chi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(\mathbb{F})$, which leads to $d = 0$, a contradiction. This completes the proof of the theorem. $\square$

The following example shows that the primeness of R is necessary in the hypothesis.

Example 2.4. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$, where S is any non-commutative ring. We define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$ associated with a nonzero derivation $d = [e_{11}, x]$. Then it is easy to see that F is a nonzero generalized derivation and I is a nonzero ideal of R which satisfies $F([x, y]_k)^n = ([x, y]_k)^l$ for $x, y \in I$. However, R is not commutative.

3. Generalized Derivation in Semiprime Ring

In this section, we assume that R is a semiprime ring with extended centroid C. We denote $A = O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of Q containing R. Also $B = B(C)$ and $spec(B)$ denotes Boolean ring of C and the set of all maximal ideal of B, respectively. It is well know that if $M \in spec(B)$ then $R_M = R/RM$ is prime [1, Theorem 3.2.7]. We use the notations Ω-Δ-ring, Horn formulas and Hereditary formulas. For more details see ([1], pages 37, 38, 43, 120). In order to prove our main result, we need the following two results which can be found in [1].

Lemma 3.1. ([1], Proposition 2.5.1) *Any derivation d of a semiprime ring R can be extended uniquely to a derivation of U (we shall let d also denote its extension to U).*

Lemma 3.2. ([1], Theorem 3.2.18) *Let R be an orthogonally complete Ω-Δ-ring with extended centroid C, $\Psi_i(x_1, x_2, \cdots, x_n)$ Horn formulas of signature of Ω-Δ, $i = 1, 2, \cdots$ and $\Phi(y_1, y_2, ..., y_m)$ a hereditary first-order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \cdots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \cdots, c_m) \in R^{(m)}$. Suppose that $R \models \Phi(c)$ and for every maximal ideal M of the Boolean ring $B = B(C)$, there exists a natural number $i = i(M) > 0$ such that*

$$R_M \models \Phi(\phi_M(\vec{c})) \Rightarrow \Psi_i(\phi_M(\vec{a})).$$

Then there exist a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, \cdots, e_k \in B$ such that $e_1 + e_2 + \cdots + e_k = 1$ and $e_iR \models \Psi_i(e_i\vec{a})$ for all $e_i \neq 0$.

Now, we prove our main result of this section.

Theorem 3.3. *Let R is a 2-torsion free semiprime ring and F is a nonzero generalized derivation associated with a nonzero derivation d of R such that $F([x,y]_k)^n = ([x,y]_k)^l$ for all $x, y \in R$, where l, n, k are fixed positive integers. Further, let $A = O(R)$ is the orthogonal completion of R and $B = BC$, where C is the extended centroid of R. Then there exists a central idempotent element $e \in B$ such that d vanishes identically on eA and the ring $(1-e)A$ is commutative.*

Proof. By the given hypothesis, we have R satisfies

$$F([x,y]_k)^n = ([x,y]_k)^l.$$

By Theorem 2.1, the generalized derivation F can be extended uniquely to a generalized derivation on U. Since U and R satisfy the same differential identities (see [13]), we have $(a[x,y]_k + [q,[x,y]_k])^n = ([x,y]_k)^l$ for all $x, y \in U$. According to ([1], Remark 3.1.16) $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore, A is an orthogonally complete Ω-Δ- ring where $\Omega = \{0, +, ,., d\}$. Consider the formulas

$\Phi = (\forall x)(\forall y) \parallel (a[x,y]_k + [q,[x,y]_k])^n - ([x,y]_k)^l = 0 \parallel$,

$\Psi_1 = (\forall x)(\forall y) \parallel xy = yx \parallel$,

$\Psi_2 = (\forall x) \parallel d(x) = 0 \parallel$.

One can easily verify that Φ is a hereditary first-order formula and $\neg\Phi$, Ψ_1, Ψ_2 are Horn formulas. Using Theorem 2,3, we can easily check that all the conditions of Lemma 3.2 are fulfilled. Hence there exist two orthogonal idempotent e_1 and e_2 such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_i A \models \Psi_i$, $i = 1, 2$. This completes the proof. $\square$

Acknowledgment
The authors wishes to thank the referee for his/her their valuable comments, suggestions.

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