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On the Weighted Composition Operators

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Abstract. In this paper we consider the weighted composition operators on the weighted Hardy spaces. We investigate some relations between the function theoretic of the composition map and the weight function with the operator theoretic of the weighted composition operators.

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1. Introduction

Suppose that u is a holomorphic function on the open unit disk $\mathbb{D}$ and let φ be a holomorphic self-map of $\mathbb{D}$. A weighted composition operator $C_{u,\varphi}$ induced by the weight symbol u and composition symbol φ maps an analytic function f in a functional Banach space of analytic function into $(C_{u,\varphi}f)(z) = u(z)f(\varphi(z))$.

Note that $C_{u,\varphi}$ is the composition operator C_φ given by $C_\varphi(f) = fo\varphi$ when $u = 1$. A recent book of Cowen and MacCluer is a good reference for the theory of composition operators([1]). Weighted composition operators are recently the subject of attention due to the fact that of results have been obtained about composition operators in one and several complex variables.

Forelli showed that an isometry on $H^p, 1 < p < \infty$ and $p \neq 2$, is a weighted composition operator([4]). Throughout the paper $\{\beta(n)\}_{n=1}^{\infty}$ is a sequence of non-negative integers with $\beta(0) = 1$ and $1 < p < \infty$. we are mainly concerned about weighted Hardy spaces $H^p(\beta)$. In fact, $H^p(\beta), 1 < p < \infty$, is the space of formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ with

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

Note that, for $p = 2$, the classical Hardy spaces, Bergman spaces and the Dirichlet spaces are weighted Hardy spaces with $\beta(n) \equiv 1, \beta(n) \equiv (n+1)^{\frac{-1}{2}}$ and $\beta(n) \equiv (n+1)^{\frac{1}{2}}$, respectively . The space $H^2(\beta)$ becomes a Hilbert space with inner product

$$< f, g >= \sum_{n=0}^{\infty} a_n \bar{b}_n \beta(n)^2$$

where $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ are the elements of $H^2(\beta)$([1, 6]).

If $\liminf \beta(n)^{\frac{1}{n}} = 1$ or $\lim \frac{\beta(n+1)}{\beta(n)} = 1$,then $H^p(\beta)$ consists of functions analytic on $\mathbb{D}$. Generally the spaces $H^p(\beta)$ are reflexive Banach spaces with the norm $\|.\|_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ when $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{\frac{p}{q}} = \{\beta(n)^{\frac{p}{q}}\}_n$([5]).

A complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ , is bounded. A complex number λ is a bounded point evaluation on $H^p(\beta)$ if and only if $\left\{\frac{\lambda^n}{\beta(n)}\right\}_n \in l^q$([7]). If $\liminf \beta(n)^{1/n} = 1$ then for each λ in the open unit disk, the functional of evaluation atλ, e_λ , is a bounded linear functional on $H^p(\beta)$ and we have

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n z^n}{\beta(n)^p},$$

and

$$\|e_\lambda\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

Some sources on formal power series include ([2,3,7,8,9]).

2. Main Results

From now on suppose that $C_{u,\varphi}$ is a bounded weighted composition operator on $H^p(\beta), 1 < p < \infty$, $\beta(0) = 1$ and $\liminf \beta(n)^{1/n} = 1$. To avoid $C_{u,\varphi}$ being a multiplication operator , φ is taken to be different from identity. Clearly the space $H^p(\beta), 1 < p < \infty$, is contained in H^p and it is known as a small space when $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} < \infty$ and $1/p + 1/q = 1$ ([4]).

Lemma 1. *If* $C_{u,\varphi}$ *is bounded on* $H^p(\beta)$, $1 < p < \infty$ *, then*

$$|u(\omega)| \leqslant ||C_{u,\varphi}|| \frac{||e_\omega||}{||e_{\varphi(\omega)}||}$$

for any w in the open unit disk.

Proof. First of all note that for any f in $H^p(\beta)$,

$$< f, C_{u,\varphi}^* e_w > = < C_{u,\varphi} f, e_w > = < u.f \circ \varphi, e_w >$$

$$= u(w) f(\varphi(w)) = u(w) < f, e_{\varphi(w)} >,$$

so $C_{u,\varphi}^* e_w = u(w) e_{\varphi(w)}$. Note that $\|\frac{e_w}{\|e_w\|}\| = 1$ for any w in $\mathbb{D}$. By boundedness of $C_{u,\varphi}$,

$$\|C_{u,\varphi}^* \frac{e_w}{\|e_w\|}\| \leqslant \|C_{u,\varphi}^*\| \|\frac{e_w}{\|e_w\|}\| = \|C_{u,\varphi}\|.$$

Hence

$$\|u(\omega) e_{\varphi(\omega)}\| \leqslant \|C_{u,\varphi}\| \|e_\omega\|,$$

or

$$|u(\omega)| \leqslant \|C_{u,\varphi}\| \frac{\|e_\omega\|}{\|e_{\varphi(\omega)}\|}. \quad \square$$

Lemma 2. *Suppose that* $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} < \infty, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$ *and* $C_{u,\varphi}$ *is bounded on* $H^p(\beta)$ *. Then* $\|C_{u,\varphi}\|$ *is bounded below by* $\frac{\|u\|_\infty}{(\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q})^{1/q}}$.

Proof . By Lemma 1,

$$|u(\omega)| \leqslant \|C_{u,\varphi}\| . \frac{\|e_\omega\|}{\|e_{\varphi(\omega)}\|}$$

$$= ||C_{u,\varphi}|| \frac{(\sum_{n=0}^{\infty} \frac{|\omega|^{nq}}{\beta(n)^q})^{\frac{1}{q}}}{(\sum_{n=0}^{\infty} \frac{|\varphi(\omega)|^{nq}}{\beta(n)^q})^{\frac{1}{q}}}$$

$$\leqslant \frac{(\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q})^{\frac{1}{q}}}{(\sum \frac{|\varphi(\omega)|^{nq}}{\beta(n)^q})^{\frac{1}{q}}}.$$

Since $\|e_{\varphi(w)}\| \geqslant 1$, then

$$|u(w)| \leqslant \|C_{u,\varphi}\|(\sum \frac{1}{\beta(n)^q})^{1/q}.$$

This completes the proof. $\square$

Now we give the sufficient conditions to determine when the adjoint of a weighted composition operator on weighted Hardy space $H^p(\beta)$ is a composition operator.

Proposition 3. *Let* $C_{u,\varphi}$ *be a weighted composition operator on* $H^p(\beta), 1 < p < \infty$ *. If* $C^*_{u,\varphi} = C_\psi$ *is a composition operator then* $u = \bar{e}_w$ *with* $w = \psi(0)$ *and* φ *has Denjoy-Wolf point at 0.*

Proof. If $C^*_{u,\varphi} = C_\psi$, then $C^*_{u,\varphi} e_\lambda = C_\psi e_\lambda$. It follows that

$$u(\lambda) e_{\varphi(\lambda)}(z) = e_\lambda(\psi(z))$$

where λ and z are in the open unit disk. So

$$u(\lambda) \sum_{n=0}^{\infty} \frac{\overline{\varphi(\lambda)}^n z^n}{\beta(n)^p} = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n \psi^n(z)}{\beta(n)^p}. \tag{1}$$

Put $z = 0$ and $\psi(0) = w$ in (1), then we have

$$u(\lambda) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n w^n}{\beta(n)^p} = \overline{e_w(\lambda)}.$$

Put $u = \bar{e}_w$ in (1), then

$$\sum_{n=0}^{\infty} \frac{w^n \bar{\lambda}^n}{\beta(n)^p} \cdot \sum_{n=0}^{\infty} \frac{\left(\overline{\varphi(\lambda)}\right)^n z^n}{\beta(n)^p} = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n \psi^n(z)}{\beta(n)^p},$$

for any λ and z in the open unit disk and $w = \psi(0)$.

If $\lambda = 0$, then
$$\sum_{n=0}^{\infty} \frac{\overline{\varphi(0)}^n z^n}{\beta(n)^p} = \frac{1}{\beta(0)^p} = 1,$$
for each z in $\mathbb{D}$. Therefore $\varphi(0) = 0$. □

Theorem 4. *Let $C_{u,\varphi}$ be a compact weighted composition operator on $H^p(\beta), 1 < p < \infty$. Suppose that $C^*_{u,\varphi}$ is a composition operator. Then the spectrum of $C_{u,\varphi}$ is a subset of*

$$\Big\{0, 1,\ \varphi'(0),\ (\varphi'(0))^2,\ (\varphi'(0))^3,\ \cdots\Big\}.$$

Proof. Suppose that λ is an eigenvalue of $C_{u,\varphi}$. We will show that $\lambda = 0, 1$ or has the form $(\varphi'(0))^n$ for some n . Note that $C_{u,\varphi}$ is a compact operator on infinite dimensional space $H^p(\beta)$. Then zero is in the spectrum. Now if $\lambda \neq 0$, then for some nonzero $f \in H^p(\beta)$,

$$(C_{u,\varphi} f)(z) = \lambda f(z). \tag{1}$$

If $f(0) \neq 0$, then

$$u(0) f(\varphi(0)) = \lambda f(0).$$

But by proposition 3 , $\varphi(0) = 0$ and $u(0) = 1$. then $\lambda = 1$. Now suppose that zero is a root of f with multiplicity of $n > 0$.

Hence by differentiating from (1) , n times, we have

$$\sum_{i=0}^{n-1} c_i(z)(f^{(i)}(\varphi(z))) + u(z) f^{(n)}(\varphi(z))(\varphi'(z))^n = \lambda f^{(n)}(z) \tag{2}$$

where C_i's are functions of the different products of φ and derivatives of u. Put $z = 0$ in (2). Since $\varphi(0) = 0$ and $u(0) = 1$ then $f^{(n)}(0)(\varphi'(0))^n = \lambda f^{(n)}(0)$ and this completes the proof. □

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