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Using Least Square Method to Find the Approximate Solution of an Overdetermined System of Linear Equations

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Abstract. In this paper an algorithm is introduced to find the approximate solution of an inconsistent linear system. The used norm in this approach is smooth and strictly convex . The algorithm is iterative and produce a sequence that tends to best solution for the inconsistent system. Two numerical examples are given to illustrate the procedure.

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1. Introduction

Consider the overdetermined system of linear equations:

$$Ax = b,$$

where $A = [a_{ij}]_{m\times n}$ is a real matrix, $b \in \mathbb{R}^m$, $m, n \geqslant 1$. The basic problem to be examined here can be posed as follows:

find $x \in \mathbb{R}^n$, to Minimize $\|r(x)\|$, where

$$r(x) = b - Ax. \tag{1}$$

Now we can change the optimization problem (1) to the following problem:

$$\text{Minimize } \|b - z\|, \ z \in K, \tag{2}$$

where K is a subspace of $\mathbb{R}^m$, and $b \in \mathbb{R}^m - K$.

2. Preliminaries and Backgrounds

The inner product in $\mathbb{R}^m$, denoted by $< . >$, is defined as:

$$< u, v >= \sum_{i=1}^{m} u_i v_i, \quad u = (u_1, \dots, u_m), \quad v = (v_1, \dots, v_m).$$

Let $E_{m\times m}$ be the orthogonal projection of $\mathbb{R}^m$on the subspace $K = ker(A^T)$ where A^T is the transposition matrix of A. So for any $u \in \mathbb{R}^m$, the vector $Eu \in K$ is the best approximation of u in Euclidian norm in subspace K, and if $u \in K$, them $Eu = u$. So it is clear that $K^\perp = Im(A)$, and if $s = Eb$, then the system $Ax = b$ is inconsistent, thus $s \neq 0$.(It is easy to show that $Eu = 0$ if and only if $u \in K^\perp$).

Thus the problem (2) can be rewritten as:

$$\text{Minimize } \|b - z\|, \tag{1}$$

subject to

$$z \in K^{\perp}, \tag{2}$$

where the defined norm is smooth and strictly convex.

We would like mention that $\|.\|$ is smooth if and only if from any point in $u \in \mathbb{R}^m$, with unite norm, only one hyperplane can pass, that support the sphere

$$B = \{x \in \mathbb{R}^m,\ \|x\| \leqslant 1\},$$

and $\|.\|$ is strictly convex if and only if there is not any segment in the set

$$S = \{x \in \mathbb{R}^m,\ \|x\| = 1\}.$$

Now we are going to define the dual problem (2). First define the dual norm $\|.\|'$, as follows

$$\|y\|' = \mathrm{Max}\{< x, y >;\ \|x\| = 1\ and\ x \in \mathbb{R}^m\}.$$

If $y \neq 0$, is a vector in $\mathbb{R}^m$, the vectors y' and y^* are respectively dual-$\|.\|$ and dual-$\|.\|'$ of y and defined by:

$$\|y'\| = 1,\ \ < y', y >= \|y\|',$$

$$\|y^*\| = 1,\ \ < y^*, y >= \|y\|.$$

If $\|.\|$ is strictly convex, then for every nonzero $y \in \mathbb{R}^m$, dual-$\|.\|$ is unique and the mapping $\nu \to \nu'$ is continuous, and if $\|.\|$ is smooth, then dual-$\|.\|'$ in the case of existence is unique and $\nu \to \nu^*$ is continuous.

Theorem 2.1. *$\|.\|$ is smooth if and only if $\|.\|'$ is strictly convex.*

Proof. See [1].□

Now from [5], where $1 < p < \infty$, for L_p norm,

$$\|u\|_p = (\sum_{i=1}^{m}(u_i)^p)^{\frac{1}{p}},$$

we have

$$\|.\|'_p = \|.\|_q, \quad p + q = pq,$$

and if $\nu \neq 0$, the ν' and ν^*, with ν'_i and ν^*_i, as their elements, defined by

$$\nu'_i = (\frac{|\nu_i|}{\|\nu\|_q})^{q-1}\text{Sign}\nu_i, \quad i = 1, \dots m,$$

$$\nu^*_i = (\frac{|\nu_i|}{\|\nu\|_q})^{p-1}\text{Sign}\nu_i, \quad i = 1, \dots m.$$

Now the problem (P') (dual of (P)) is as follows:

$$\text{Maximize} < b, y >,$$

subject to

$$y \in K, \quad \|y\|' = 1. \qquad (P')$$

(P) and (P'), respectively, are equivalent with the following problems:

$$\text{Minimize}\|s - z\|,$$

subject to

$$z \in K^{\perp}, \text{ where } s = Eb, \qquad (P)$$

and

$$\text{Maximize} < s, y >,$$

subject to

$$y \in K, \quad \|y\|' = 1. \qquad (P')$$

From now on, we use two latest problem (P) and (P'), and since $\mathbb{R}^m$ and subspaces K and $K^{\perp}$ are finite dimension and the norm is strictly convex, so (P) and (P'), have unique solution.

Theorem 2.2. *Let y be the solution of (P'). Then the linear system*

$$Ax = b - < b, y > y'$$

is consistent. In fact every solution of this system is a solution of (P), and the remainder is $< b, y >$.

Proof. See [7].□

The following theorems help us to illustrate our algorithm. One can see the proof of these Theorems in [2].

Theorem 2.3. *Let* $s = Eb \neq 0$ *and* $t = s + u$, *where* $u \in K^{\perp}$. *Then* t^* *is a minimizer of* (P) *if and only if* $t^* \in K$. *In this case* t^* *is a maximizer of* (P').

Lemma 2.4. *Let* $0 \neq s \in K$, $y \in K$, $\|y\|' = 1$, *and* $< s, y >> 0$. *Define* $t \in \mathbb{R}^m$ *as follows:*

$$t = s + < s, y > (I - E)y'.$$

Then $Et^* \neq 0$, *(i.e.* $t^* \in K^{\perp}$*).*

Theorem 2.5. *Let* $s = Eb \neq 0$, $t \in \mathbb{R}^m$, *be such that*

$$t = s + < s, y > (I - E)y', \quad < s, y >> 0,$$

$$where y = (\|Et^*\|')^{-1} Et^*.$$

Then t^* *is a minimizer of* (P) *and* y *is a maximizer of* (P') *and* $t^* = y$.

3. Algorithm

Now, as a consequence of the above theorems, we can present the basic algorithm to solve the problem (P). This algorithm consists of the following steps:

Step 1. Close $\epsilon > 0$, $y_1 \in K$, such that $\|y_1\|' = 1$, and $< s, y >> 0$. One suggestion for this, can be $y_1 = \|s\|^{-1} s$. Choose the counter $k = 1$.

Step 2. Choose $t_k = s + < s, y_k > (I - E)y_k'$.

Step 3. If $1 - (\|t_k\|' < s, y_k >) \leqslant \epsilon$, goto 10.

Step 4. If $\|(I - E)t_k\| \leqslant \epsilon$, then $y_k = t_k$, goto 10.

Step 5. Choose $r_k = (\|Et_k\|')^{-1} Et_k$.

Step 6. If $< s, r_k >< r_k', y_k > \geqslant < s, y_k >$, then $y_{k+1} = r_k$, $\alpha_k = 1$, goto 9.

Step 7. Choose α_k $(0 \leqslant \alpha_k \leqslant 1)$ such that

$$
\begin{aligned}
&< s, \alpha_k r_k + (1 - \alpha_k) y_k >< (\alpha_k r_k + (1 - \alpha_k) y_k)', r_k - y_k > \\
&= \|\alpha_k r_k + (1 - \alpha_k) y_k\|' < s, r_k - y_k > .
\end{aligned}
$$

Step 8. Choose $y_{k+1} = (\|\alpha_k r_k + (1 - \alpha_k) y_k\|')^{-1} (\alpha_k r_k + (1 - \alpha_k) y_k)$.

Step 9. Choose $k = k + 1$, goto 2.

Step 10. y_k is a solution of (P') and t_k is a solution of minimizer in (P).

Now from above algorithm, any solution of the following linear system :

$$Ax = b - t_k$$

is an approximate solution of the overdetermined system

$$Ax = b$$

in the least square approximation.

Theorem 3.1. *The sequence* $\{y_k\}$ *and* $\{t_k\}$ *from above algorithm, tend respectively to the maximizer and minimizer of problems* (P') *and* (P).

4. Numerical result

In this section an example is given to illustrate the algorithm. This example is chosen from [1, 3, 4, 5].

Example 4.1. *Consider the following overdetermined linear system of equation*

$$x_1 = 1.52$$

$$x_1 + x_2 = 1.025$$

$$x_1 + 2x_2 = 0.475$$

$$x_1 + 3x_2 = 0.01$$

$$x_1 + 4x_2 = -0.475$$

$$x_1 + 5x_2 = -1.005.$$

Table 1:

P	N	x_1	x_2	ρ
7	81	1.500685	-0.499662	0.02911148
6	46	1.501486	-0.499711	0.02990692
5	19	1.502293	-0.499813	0.03105973
4	9	1.503757	-0.500057	0.03287394
3	4	1.506860	-0.500710	0.03612070
2	0	1.514762	-0.502571	0.04321596
1.8	3	1.517371	-0.503201	0.04566427
1.6	7	1.519679	-0.503750	0.04879303
1.5	16	1.520005	-0.503800	0.05079019
1.4	11	1.520126	-0.503787	0.05327584
1.3	15	1.520215	-0.503744	0.05643632
1.2	23	1.520187	-0.503621	0.06054325
1.1	65	1.520037	-0.503390	0.06597922

In Table 1, one can see the numerical results. In this Table , (P) indicates the L_p norm. N is the number of iteration, and ρ is the norm of error in the (P) problem.

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