Journal of Mathematical Extension
Vol. 2, N0. 1, 2 (2007-2008), 103–111

Hypercyclicity On Some Function Spaces

B. Yousefi

Payame-Noor University

"Dedicated to Mola Ali"

Abstract. In this paper we characterize the hypercyclicity of the composition operator acting on some function spaces of analytic functions.

AMS Subject Classification: Primary 47B37; Secondary 47A25.
Keywords and Phrases: Hilbert spaces of analytic functions,Reproducing kernels,Hypercyclic vector Composition operator, Denjoy-Wolff Theorem, Hypercyclicity Criterion, Hilbert spaces of formal power series.

1. Introduction

Let H be a Hilbert space of analytic functions on the open unit disk U such that for each $z \in U$, the evaluation function $e_\lambda : H \to C$ defined by $e_\lambda(f) = f(\lambda)$ is bounded on H. By the Riesz Representation Theorem there is a vector $k_z \in H$ such that $f(z) =< f, k_z >$ for every $z \in U$. Furthermore, we suppose that ϕ is a holomorphic self map of U. The adjoint of a composition operator has not been yet well characterized on any spaces of holomorphic functions. Nevertheless its action on reproducing kernels is determined. In fact $C_\phi^*(k_z) = k_{\phi(z)}$ for every $z \in U$.

Remember that a bounded linear operator T on a Banach space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit

$$Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$$

is dense in X and in this case we refer to x as a hypercyclic vector for T. Relating on these subjects we refer to([1–6]).

Let U be the open unit disk of the plane, then we denote by $H(U)$ the space of all complex-valued functions holomorphic on U.

By $A(U)$ we will denote the disc algebra on the open unit disc U, which contains the functions that are analytic on U and are continuous on $\bar{U}$.

2. Main Results

In the main theorems of this paper we investigate the hypercyclicity of the composition operators on the Hilbert space H and some other function spaces.

Proposition 1. *If* ϕ *is an analytic self-map of* U *that fixes a point of* U*, then* C_ϕ *is not hyprcyclic on* H.

Proof. Suppose $\alpha \in U$ is a fixed point for ϕ. Then $< f, e_\alpha >= f(\alpha)$ for all f in H. Fix $f \in H$, to be regarded as a hypercyclic vector candidate.

If g belongs to the closure of $Orb(C_\phi, f)$, then for some subsequence $n_k \to +\infty$ we have $C_{\phi_{n_k}} f \to g$. Thus we have

$$\begin{aligned} g(\alpha) &= < g, e_\alpha > \\ &= \lim_k < C_{\phi_{n_k}} f, e_\alpha > \\ &= \lim_k < f, C^*_{\phi_{n_k}} e_\alpha > \\ &= \lim_k < f, e_{\phi_{n_k}(\alpha)} > \\ &= \lim_k f(\phi_{n_k}(\alpha)) \\ &= f(\alpha). \end{aligned}$$

This implies that no orbit can be dense in H and so C_ϕ is not hypercyclic on H. This completes the proof. □

Proposition 2. *If* $H \subset A(U)$ *and* ϕ *is an analytic self-map of* U *that fixes a point of* $\bar{U}$*, then* C_ϕ *is not hypercyclic on* H.

Proof. By the same proof applies to the above proposition, we can complete the proof. □

Theorem 3. *Let H^2 be continuously contained in H and suppose that the set of polynomials is dense in H. If ϕ is an analytic self-map of U and C_ϕ is hypercyclic on H, then λC_ϕ is hypercyclic on H whenever $|\lambda| = 1$.*

Proof. Let $T = C_\phi$ and X be the set of polynomials vanishing in the boundary fixed point w. Since convergence in the Hardy space H^2 implies convergence in H, with both spaces containing the polynomials as a dense subset, so X is a dense subset of H. Note that for every $f \in X$, we have $T^n(f) = f \circ \phi_n$ tending to zero, as $n \to \infty$. Now fix the open subsets U' and V' and the open neighborhood W of zero in H. Since T is hypercyclic and the sequence $\{T^n\}$ converges point wise to zero on the dense subset X, so there exists some positive integer n such that $T^n(U') \cap W \neq \emptyset$ and $T^n(W) \cap V' \neq \emptyset$. But for each positive integer n, $\lambda^n W = W$. This implies that

$$(\lambda T)^n(U') \cap W \neq \emptyset$$

and

$$(\lambda T)^n(W) \cap V' \neq \emptyset.$$

Now clearly λT is hypercyclic.□

Remark 4. *The above results are also true for a Banach space X*

instead of the Hilbert space H with the same assumptions.

Note that $H(U)$ can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(U)$ converges to $f \in H(U)$ if and only if $f_n \to f$ uniformly on every compact subsets of U.

Lemma 5. *Let X be the set of polynomials vanishing in the boundary fixed point w, then X is a dense subset of $H(U)$.*

Proof. Suppose Λ is a continuous linear functional on $H(U)$ that vanishes on X. By the Hahn-Banach Theorem it is enough to show that $\Lambda \equiv 0$ on $H(U)$. For each $0 < r < 1$ and each $f \in H(U)$ let

$$\| f \|_r = sup\{|f(z)| : |z| \leqslant r\}.$$

Then $\| \, . \, \|_r$ is a norm on $H(U)$ and the open balls for each of these norms forms a basis for the topology of $H(U)$ and so there exists some $0 < r < 1$ such that the set

$$\{f \in H(U) : \| f \|_r \leqslant 1\}$$

is contained in $\Lambda^{-1}(U)$. Thus Λ is a bounded linear functional relative to the norm $\| \, . \, \|_r$. So by the Hahn-Banach theorem it extends to a bounded linear functional on $C(U_r)$, where

$$U_r = \{z \in C : |z| \leqslant r\}.$$

The Riesz Representation Theorem provides a finite Borel measure μ on U_r such that

$$\Lambda(f) = \int_{U_r} f d\mu$$

for every $f \in H(U)$. For each positive integer n consider the polynomials $z^{n+1} - wz^n$ which are zero in w. But our hypothesis, states that Λ vanishes on X, so

$$\int_{U_r} z^{n+1} d\mu = w \int_{U_r} z^n d\mu$$

whence

$$|\int_{U_r} z^{n+1} d\mu| = |\int_{U_r} z^n d\mu|.$$

On the other hand

$$\int_{U_r} z^n d\mu \to 0$$

as $n \to \infty$. This implies that

$$\int_{U_r} z^n d\mu$$

=0 for each positive integers, and therefore

$$\int_{U_r} f d\mu = 0$$

for every polynomial f, and so for every $f \in H(U)$. □

Proposition 6. *Suppose that ϕ has no fixed point in U, then λC_ϕ is hypercyclic on $H(U)$ for every complex number λ with $|\lambda| = 1$.*

Proof. Let $T = C_\phi$ and X be the set stated in the preceding lemma. Note that for every $f \in X$, $T^n(f) = f \circ \phi_n$ tending to zero, as $n \to \infty$.

Now by the same method used in the proof of Theorem 3, we can see that λT is hypercyclic. $\square$

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leqslant p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$||f||^p = ||f||_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

shall be used whether or not the series converges for any value of z. Let $H^p(\beta)$ denotes the space of such formal power series. These are reflexive Banach spaces with the norm $||.||_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_n$. The following propositions follows immediately from the above results.

Proposition 7. *Let* $(\beta(n+1)/\beta(n)) \to 1$ *as* $n \to \infty$. *If* ϕ *is an analytic self-map of* U *that fixes a point of* U, *then* C_ϕ *is not hyprcyclic on* $H^p(\beta)$.

Proposition 8. *Let* $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} < \infty$ *where* $\frac{1}{p} + \frac{1}{q} = 1$. *If* ϕ *is an analytic self-map of* U *that fixes a point of* $\bar{U}$, *then* C_ϕ *is not hypercyclic on* $H^p(\beta)$.

Proposition 9. *Let* $(\beta(n+1)/\beta(n)) \to 1$ *and* $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} = +\infty$. *If* φ

is a conformal automorphism of the unit disk with no fixed point in the interior, then C_ϕ *is hypercyclic on* $H^p(\beta)$*.*

Proof. Let $\alpha \in \partial U$ be the unique fixed point of φ that comes in the Denjoy-Wolff Theorem and denote the other fixed point by β. Then this too must lie on the unit circle ∂U. Let Y_α denotes the set of polynomials that vanish at α. Since $\phi_n \to \alpha$ uniformly on compact subsets of U, we get

$$C_{\phi_n} p = p o \phi_n \to 0$$

for all p in Y_α. Thus $C_{\phi_n} \to 0$ on Y_α which is dense in $H^p(\beta)$. Put $S = C_\phi^{-1} = C_{\phi^{-1}}$ and let Y_β be the set of all continuous functions on the closed unit disk that are analytic in U and vanish at β. Then S maps the dense set Y_β into itself and $S^n \to 0$ on Y_β. The hypothesis of the Hypercyclicity Criterion are therefore satisfied and so C_ϕ is hypercyclic on $H^p(\beta)$. $\square$

References

[1] P. S. Bourdon and J. H. Shapiro, Cyclic composition operator on H^2, *Proc. Symp. Pure Math.*, 51 (2) (1990), 43-53.

[2] P. S. Bourdon and J. H. Shapiro, Cyclic phenomena for composition operator, *Mem. Amer. Math. Soc.*, 596 (1997).

[3] G. Godefroy and J. H. Shapiro, Operators with dense invariant cyclic vector manifolds, *J. Func. Anal.*, 98 (1991), 229-269.

[4] J. H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag New York, (1993).

[5] B. Yousefi and H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, 135 (10) (2007), 3263-3271.

[6] B. Yousefi and H. Rezaei, On the supercyclicity and hpercyclicity of the operator algebra, *Acta Math. Sinica*, 24 (7) (2008), 1221-1232.

Bahmann Yousefi
Payame-Noor University, Iran
Shiraz, Moallem Square,
South Iman Street
Shiraz Payame-Noor University
P.O.Box:71345-1774
Shiraz, IRAN
Email: byousefi@shirazu.ac.ir