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The Laplace Transform Method for Linear Ordinary Differential Equations of Fractional Order

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Abstract. In this paper we use the Laplace transform to solve some ordinary linear differential equations of the fractional order.

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1. Introduction

We are all familiar with the idea of derivatives, $\frac{df}{dx}, \frac{d^2f}{dx^2}, \cdots$. But what would be the meaning of notation $\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$, and in general $D^\alpha f(x), \alpha \in \mathbf{R}$?

We want to introduce the fractional calculus in gentle manner. We explore the idea of a fractional derivative by first looking at examples of familiar nth order derivatives like $D^n e^{ax} = a^n e^{an}$ and then replacing the natural number n by other numbers like $\frac{1}{2}$.

We are familiar with the expressions for derivatives of e^{ax}, $D^1e^{ax} = ae^{ax}, D^2e^{ax} = a^2e^{ax}, \cdots, D^ne^{ax} = a^ne^{ax}, n \in N$. Could we replace n by $\frac{1}{2}$ and write $D^{\frac{1}{2}}e^{ax} = a^{\frac{1}{2}}e^{ax}$, and in general, can we let n be an irrational number like $\sqrt{2}$ or a complex number like $1+i$?

Is it true that

$$D^\alpha e^{ax} = a^\alpha e^{ax} \qquad (1-1)$$

for any value of α? If α is a negative integer we will get

$$D^{-1}e^{ax} = \int e^{ax}dx \text{ and } D^{-2}(e^{ax}) = \int\int e^{ax}dx\,dx,$$

as it is reasonable to interpret. When α is a negative integer $-n$, D^α is interpreted as the nth iterated integrals, in general D^α represents a derivative if α is a positive real number and an integral if α is a negative real number. Note that we have not yet given a definition for a fractional derivative of a general functions.

Liouville used this approach to fractional differentiation in his paper [2] and [3].

For trigonometric functions sine and cosine we are familiar with the derivatives of the sine function:

$$D^0 \sin x = \sin x, D \sin x = \cos x, D^2 \sin x = -\sin x, \cdots$$

This presents no obvious pattern form which to find $D^\alpha \sin x$. However, graphing the functions discloses a pattern. Each time we differentiate, the graph of $sinx$ is shifted $\pi/2$ to the left. Thus differentiating $sinx$, n times results in the graph of $\sin x$ being shifted $\frac{n\pi}{2}$ to the left. So $D^n \sin x = \sin(x + \frac{n\pi}{2})$, if we replace the positive integer n with an arbitrary number α we have the following expression for the general derivative of sine

$$D^\alpha \sin x = \sin(x + \frac{\alpha\pi}{2}).$$

Similarly for cosine we have

$$D^\alpha \cos x = \cos(x + \frac{\alpha\pi}{2})$$

Now we look at the derivative of powers of x. We have

$$D^0 x^p = x^p,\ D^1 x^p = p x^{p-1},\ \cdots,\ D^n x^p = p(p-1)\cdots(p-n+1)x^{p-n}$$

and

$$D^n x^p = \frac{p!}{(p-n)!} x^{p-n}.$$

If we replace n by arbitrary number α, we may use gamma function that introduced by Euler in the $18th$ century to generalize the notion of $z!$ to non-integer values of z,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Therefore, we have

$$D^\alpha x^p = \frac{\Gamma(p+1) x^{p-\alpha}}{\Gamma(p-\alpha+1)}$$

for any α. So we can extend the idea of a fractional derivative to a large number of functions. Let f be any function that can be expanded in a Taylor series in powers of x as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and assuming we can differentiate term by term. So

$$D^\alpha f(x) = \sum_{n=0}^{\infty} a_n D^\alpha x^n = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \qquad (1-2)$$

that is candidate for the definition of the fractional derivative for the wide variety of functions that can be expanded in a Taylor's series in power of x. But, we will soon see that it leads to contradictions.

We wrote the fractional derivative of e^x as

$$D^\alpha e^x = e^x \qquad (1-3)$$

Let us compare this with (1-1). Since $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, by (1-2) we have

$$D^\alpha e^x = \sum_{n=0}^{\infty} \frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}. \qquad (1-4)$$

But (1-3) and (1-4) do not match unless α is an integer number, if α is not an integer number, we have two entirely different functions. We have discovered a contradiction that historically has caused great problems. It appears as though our expression (1-1) for the fractional derivative of the exponential is inconsistent with our formula (1-2) for the fractional derivative of a power. We would write

$$D^{-1}f(u) = \int_0^x f(t)dt$$

$$D^{-2}f(x) = \int_0^x \int_0^{t_2} f(t)dt_1 dt_2.$$

If we interchange the order of integration, then we get

$$\begin{aligned} D^{-2}f(x) &= \int_0^x \int_{t_1}^x f(t_1)dt_2 dt_1 \\ &= \int_0^x f(t_1)(x-t_1)dt_1. \end{aligned}$$

Using the same procedure, we can show that

$$D^{-3}f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2 dt$$

and in general

$$D^{-n}f(x) = \frac{1}{(n+1)!}\int_0^x f(t)(x-t)^{n-1}dt.$$

Now as we have done previously, let us replace the $-n$ by arbitrary α and to get

$$D^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha+1}}\, dt. \qquad (1-5)$$

This is a general expression for fractional derivatives that has the potential of being used as a definition. But there is a problem if $\alpha > -1$ the integral is improper and this improper integral diverges for every $\alpha \geqslant 0$. If $-1 < \alpha < 0$, the improper integral converges and so if α is negative, there is no problem.

Definition 1.1. *The fractional derivative of $f(t)$ of order $\mu > 0$ (if it exists) can be defined in terms of the fractional integral $D^{-\alpha}f(t)$ as*

$$D^{\mu}f(t) = D^{m}(D^{-(m-\mu)}f(t))$$

where m is an integer $\geqslant \lceil\mu\rceil$ and $\lceil\mu\rceil$ is the ceiling function, which gives the smallest integer $\geqslant \mu$.

Example 1.2. *The fractional derivative of the function t^{λ} is given by*

$$\begin{aligned} D^{\mu}t^{\lambda} &= D^{m}[D^{-(m-\mu)}t^{\lambda}] \\ &= D^{m}[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m-\mu+1)}t^{\lambda+m-\mu}] \\ &= \frac{\Gamma(\lambda+1)(\lambda-\mu+m-1)\cdots(\lambda-\mu+1)}{\Gamma(1+m+\lambda-\mu)}t^{\lambda-\mu} \\ &= \frac{\Gamma(\lambda+1)(1+\lambda-\mu)}{\Gamma(1+m+\lambda-\mu)}t^{\lambda-\mu} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu} \end{aligned}$$

for $\lambda > -1$ and $\mu > 0$.

The fractional derivative of the constant function $f(t) = c$ is given by

$$D^{\mu}c = c\lim_{\lambda\to 0}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu} = \frac{ct^{-\mu}}{\Gamma(1-\mu)}.$$

If the reader wishes to continue this study, we recommended the paper by Miller [5] and the books by Miller and Ross [6], also the book by Oldham and Spanier [8].

Other references of historical interest are ([4, 7, 9, 10, 11]).

2. Ordinary linear fractional differential equations

In this section we want to solve some ordinary fractional differential equations by the Laplace transform.

Definition 2.1. *A two parameter function of the Mittage-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0. \qquad (2-1)$$

We know that

$$L[t^k e^{\pm \alpha t}] = \frac{k!}{(s \mp \alpha)^{k+1}}\ , \ \ Re(s) > |\alpha|.$$

Therefore

$$\int_0^\infty e^{-st} t^k e^{\pm \alpha t} dt = \frac{k!}{(s \mp \alpha)^{k+1}}. \qquad (2-2)$$

substitution (2-1) in the integral below leads to

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(z t^\alpha) dt = \frac{1}{1-z}, \quad |z| < 1 \qquad (2-3)$$

and we obtain from (2-3) the Laplace transform of the function

$$t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm z t^\alpha).$$

Hence we get

$$\int_0^\infty e^{-st} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a t^\alpha) dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \qquad (2-4)$$

The particular case of (2-4) for $\alpha = \beta = \frac{1}{2}$

$$\int_0^\infty e^{-st} t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a\sqrt{t}) dt = \frac{k!}{(\sqrt{s} \mp a)^{k+1}}, \quad Re(s) > a^2$$

Example 2.2. *Suppose the differential equation*

$$D_t^{\frac{1}{2}} f(t) + a\, f(t) = 0, \quad D_t^{-\frac{1}{2}} f(t)|_{t=0} = c. \qquad (2-5)$$

Applying the Laplace transform of (2-5) we obtain

$$L[f] = F(s) = \frac{c}{s^{\frac{1}{2}} + a}, \quad c = D_t^{-\frac{1}{2}} f(t)|_{t=0}$$

and the inverse transform gives the solution of (2-5) as

$$f(t) = c\, t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t}).\square$$

Example 2.3 *Consider the equation*

$$D_t^p f(t) + D_t^q f(t) = h(t), \qquad (2-6)$$

where $0 < q < p < 1$.

The Laplace transform of (2-6) leads to

$$(s^p + s^q) F(s) = c + H(s)$$

where $H(s) = L[h(t)]$ and

$$c = (D_t^{q-1} f(t) + D_t^{p-1} f(t))|_{t=0}$$

so

$$F(s) = \frac{c + H(s)}{s^p + s^q} = \frac{c + H(s)}{s^q(s^{p-q} + 1)}$$
$$= (C + H(s))\frac{s^{-q}}{s^{p-q} + 1}.$$

The inverse transform for $\alpha = p - q$ and $\beta = p$ gives the solution:

$$f(t) = CG(t) + \int_0^t G(t - v)h(v)dv$$

$$C = (D_t^{q-1}f(t) + D_t^{p-1}f(t))|_{t=0}$$

$$G(f) = t^{p-1}E_{p-q,p}(-t^{p-q}).$$

The case $0 < q < p < n$ can be solved similarly.□

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