Journal of Mathematical Extension
Vol. 2, N0. 1, 2 (2007-2008), 81–92

On the Intuitionistic Fuzzy Metric Spaces

S. Jahedi
Shiraz university of Technology

E. Azhdari
Shiraz university of Technology

Abstract. Introducing an appropriate concept of fuzzy metric spaces has been investigated by many authors from different point of views. In this paper by using the idea of intuitionistic fuzzy metric spaces we will introduce an intuitionistic fuzzy metric on the space of nonempty compact subsets of a given intuitionistic fuzzy metric space.

AMS Subject Classification : Primary 54B20; Secondary 54A40; 54E35.
Keywords and Phrases: Fuzzy metric, Continuous t-norm, Continuous t-conorm, Topology, Compact set.

1. Introduction

Zadeh in [8] defined a fuzzy set A in X as a function with domain X and values in [0,1]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek ([4]). Many authors have introduced the concept of fuzzy metric spaces in different ways ([2,3,4]). George and Veeramani [2] Modified the notion of fuzzy metric spaces with help of continuous t-norm. They also have obtained a Hausdorff topology for fuzzy metric spaces. In 1983,Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Here we review the definition of intuitionistic fuzzy metric spaces ([6]). Then we give a notion for the Hausdorff intuitionistic fuzzy metric of a given intuitionistic fuzzy metric space on the set of its non-empty compact subsets. In fact it it is a generalization of the Hausdorff fuzzy metric on compact sets ([7]).

2. Preliminaries

First of all we recall some concepts and results that will be required in the sequel.

Definition 2.1. ([6]) *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t-norm if* $*$ *is satisfying the following conditions:*

(1) $*$ *is commutative and associative,*
(2) $*$ *is continuous,*
(3) $a * 1 = a$ *for all* $a \in [0,1]$,
(4) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$, *and* $a, b, c, d \in [0,1]$.

Definition 2.2. ([6]) *A binary operation* $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm if* $\diamondsuit$ *is satisfying the following conditions:*

(1) $\diamondsuit$ *is commutative and associative,*
(2) $\diamondsuit$ *is continuous,*
(3) $a \diamondsuit 0 = a$ *for all* $a \in [0,1]$,
(4) $a \diamondsuit b \leqslant c \diamondsuit d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *and* $a, b, c, d \in [0,1]$.

Definition 2.3. ([6]) *A 5-tupple* $(X, M, N, *, \diamondsuit)$ *is said to be an intuitionistic fuzzy metric space if* X *is an arbitrary set,* $*$ *is continuous t-*

norm, $\Diamond$ *is a continuous t-conorm and M,N are fuzzy sets on* $X^2 \times (0,\infty)$ *satisfying the following conditions for all* $x, y, z \in X$, $s, t > 0$,

(1) $M(x,y,t) + N(x,y,t) \leqslant 1$,
(2) $M(x,y,t) > 0$,
(3) $M(x,y,t) = 1$ *if and only if* $x = y$,
(4) $M(x,y,t) = M(y,x,t)$,
(5) $M(x,y,t) * M(y,z,s) \leqslant M(x,z,t+s)$,
(6) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous,*
(7) $N(x,y,t) > 0$,
(8) $N(x,y,t) = 0$ *if and only if* $x = y$,
(9) $N(x,y,t) = N(y,x,t)$,
(10) $N(x,y,t) \Diamond N(y,z,s) \geqslant N(x,z,t+s)$,
(11) $N(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

(M,N) is called intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Following [2], ordered triple $(X, M, *)$ is a fuzzy metric space such that X is a set, $*$ is a continuous t-norm and M is a fuzzy set of $X^2 \times (0,\infty)$ satisfying the conditions (2)-(6) of definition 2.3.

Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space by defining $x \Diamond y = 1 - ((1-x) * (1-y))$ for all $x, y \in X$ ([5]).

Example 2.4. *Typical examples of continuous* $t-norm$ *and con– tinuous t-conorm are as follows:*

$a * b = ab$, $a * b = min(a,b)$, $a \Diamond b = min(a+b,1)$, $a \Diamond b = max(a,b)$.

Remark 2.5. *In intuitionistic fuzzy metric space* X, $M(x,y,\cdot)$ *is non-decreasing and* $N(x,y,\cdot)$ *is non-increasing for all* $x, y \in X$.

Park J.H. [6] proved that every intuitionistic fuzzy metric (M,N)

on X is Hausdorff and generates a first countable and Hausdroff topology $\tau_{(M,N)}$ on X which has a base the family of open sets of the form $\{B(x,r,t) \ : x \in X \ , r \in (0,1), t > 0\}$.

A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$ for each $t > 0$.

3. Main Results

Denote $K(X)$ the set of nonempty closed and bounded subsets of $(X,\tau_{(M,N)})$. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. For all $A, B \in K(X)$ and $t > 0$ define functions H_M and H_N on $(K(X))^2 \times (0, \infty)$ by

$$H_M(A,B,t) = \min\left\{ \inf_{a\in A} \ M(a,B,t) \, , \inf_{b\in B} \ M(A,b,t)\right\}$$
$$H_N(A,B,t) = \max\left\{ \sup_{a\in A} \ N(a,B,t) \, , \sup_{b\in B} \ N(A,b,t)\right\}.$$

Rodrigues-Lopez and Romaguera [7] have proved that the function H_M is a metric on $K(X)$ where $(X, M, *)$ is a fuzzy metric space. Here we want to prove the following theorem.

Theorem3.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then $(K(X), H_M, H_N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.*

To prove the theorem we shall prove some lemmas and propositions.

Definition 3.2. *Let A be a nonempty subset of an intuitionistic fuzzy metric $(X, M, N, *, \diamondsuit)$. For $x \in X$, $t > 0$, let*

$$M(x, A, t) = \sup_{a \in A} M(x, a, t)$$
$$N(x, A, t) = \inf_{a \in A} N(x, a, t).$$

Proposition 3.3. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then M and N are continuous functions on $X^2 \times (0, \infty)$.*

Proof. Let $\{(x_n, y_n, t_n)\}$ be a sequence in $X^2 \times (0, \infty)$ that converges to (x_0, y_0, t_0). So there is a subsequence $\{(x_{n_k}, y_{n_k}, t_{n_k})\}$ such that $\{M(x_{n_k}, y_{n_k}, t_{n_k})\}$ and $\{N(x_{n_k}, y_{n_k}, t_{n_k})\}$ converge to some points. The sequence $\{t_{n_k}\}$ converges to $t_0 > 0$, then for $0 < \varepsilon < \frac{t_0}{2}$ there is $n_0 \in \mathbb{N}$ such that $|t_0 - t_{n_k}| < \varepsilon$ for all $n \geqslant n_0$. Since M is non-decreasing and N is non-increasing then

$$M(x_{n_k}, y_{n_k}, t_{n_k}) \geqslant M(x_{n_k}, x_0, \frac{\varepsilon}{2}) * M(x_0, y_0, t_0 - 2\varepsilon) * M(y_0, y_{n_k}, \frac{\varepsilon}{2})$$

and

$$N(x_{n_k}, y_{n_k}, t_{n_k}) \leqslant N(x_{n_k}, x_0, \frac{\varepsilon}{2}) \diamondsuit N(x_0, y_0, t_0 - 2\varepsilon) \diamondsuit N(y_0, y_{n_k}, \frac{\varepsilon}{2}).$$

Also

$$M(x_0, y_0, t_0 + 2\varepsilon) \geqslant M(x_0, x_{n_k}, \frac{\varepsilon}{2}) * M(x_{n_k}, y_{n_k}, t_{n_k}) * M(y_{n_k}, y_0, \frac{\varepsilon}{2})$$

and

$$N(x_0, y_0, t_0 + 2\varepsilon) \leqslant N(x_0, x_{n_k}, \frac{\varepsilon}{2}) \diamondsuit N(x_{n_k}, y_{n_k}, t_{n_k}) \diamondsuit N(y_{n_k}, y_0, \frac{\varepsilon}{2}),$$

for all $n \geqslant n_0$. Therefore,

(1) $\lim_k M(x_{n_k}, y_{n_k}, t_{n_k}) \geqslant 1 * M(x_0, y_0, t_0 - 2\varepsilon) * 1$

(2) $\lim_k N(x_{n_k}, y_{n_k}, t_{n_k}) \leqslant 0 \diamondsuit N(x_0, y_0, t_0 - 2\varepsilon) \diamondsuit 0$

and

(3) $M(x_0, y_0, t_0 + 2\varepsilon) \geqslant 1 * \lim_k M(x_{n_k}, y_{n_k}, t_{n_k}) * 1 = \lim_k M(x_{n_k}, y_{n_k}, t_{n_k})$

(4) $N(x_0, y_0, t_0 + 2\varepsilon) \leqslant 0 \diamondsuit \lim_k N(x_{n_k}, y_{n_k}, t_{n_k}) \diamondsuit 0 = \lim_k N(x_{n_k}, y_{n_k}, t_{n_k})$

From ((1),(3)), ((2),(4)), continuity of $t \longmapsto M(x, y, t)$ and $t \longmapsto N(x, y, t)$, we have

$$M(x_0, y_0, t_0) = \lim_k M(x_k, y_{n_k}, t_{n_K})$$

and

$$N(x_0, y_0, t_0) = \lim_k N(x_{n_k}, y_{n_k}, t_{n_K}).$$

This completes the proof. □

Lemma 3.3. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then for each $x \in X$ and $A \in K(X)$, the functions $t \longmapsto N(x, A, t)$ and $t \longmapsto M(x, A, t)$ are continuous on $(0, \infty)$.*

Proof. Remember that $N(x, A, t) = \inf_{a \in A} N(x, a, t)$ and $M(x, A, t) = \sup_{a \in A} M(x, a, t)$. Also by definition 2.3, the functions $N(x, y, \cdot)$ and $M(x, y, \cdot)$ are continuous on $(0, \infty)$. So $t \longmapsto N(x, a, t)$ is upper semi continuous and $t \longmapsto M(x, y, t)$ is lower semicontinuous on $(0, \infty)$. On the other hand, for fix $t_0 > 0$, let $\{t_n\} \subseteq (0, \infty)$ be a sequence converges to t_0. By continuity of the functions $y \longmapsto M(x, y, t)$, $y \longmapsto N(x, y, t)$ and compactness of A, there exist $a_M, a_N \in A$ such that $\inf_{a \in A} N(x, a, t) =$

$N(x, a_N, t)$ and $\sup_{a \in A} M(x, a, t) = M(x, a_M, t)$. Hence for each $n \in \mathbb{N}$ there is a_{N_n} and a_{M_n} in A such that $N(x, A, t_n) = N(x, a_{N_n}, t_n)$ and $M(x, A, t_n) = M(x, a_{M_n}, t_n)$. So there exist subsequences $\{a_{N_{n_k}}\}$ and $\{a_{M_{n_k}}\}$ such that they converge to a_{N_0} and a_{M_0} in A, respectively.

Thus by proposition 3.3,

$$\lim_k N(x, a_{N_{n_k}}, t_{n_k}) = N(x, a_{N_0}, t)$$

and

$$\lim_k M(x, a_{M_{n_k}}, t_{n_k}) = M(x, a_{M_0}, t).$$

Hence

$$\lim_k N(x, A, t_{n_k}) = N(x, a_{N_0}, t) \geqslant N(x, A, t)$$
$$\lim_k M(x, A, t_{n_k}) = M(x, a_{M_0}, t) \leqslant M(x, A, t).$$

Therefore $t \longmapsto N(x, A, t)$ is lower semicontinuous and $t \longmapsto M(x, A, t)$ is upper semicontinuous. □

Proposition 3.5. *Let* $(X, M, N, *, \diamondsuit)$ *be an intuitionistic fuzzy metric space. Then for each* $A, B \in K(X)$ *the functions*

$$t \longmapsto \sup_{a \in A} N(a, B, t) \quad (1)$$

$$t \longmapsto \inf_{a \in A} M(a, B, t) \quad (2)$$

are continuous on $(0, \infty)$.

proof. By continuity of the functions $t \longmapsto N(a, B, t)$ and $t \longmapsto M(a, B, t)$ on $(0, \infty)$, the function $t \longmapsto \sup_{a \in A} N(a, B, t)$ is lower semicontinuous and $t \longmapsto \inf_{a \in A} M(a, B, t)$ is upper semicontinuous.

To prove that (1) is upper semicontinuous, let $t_0 > 0$ and let $\{t_n\}$ be a sequence of positive real numbers that converges to t_0. Then there is $a_{N_0} \in A$ such that

$$\sup_{a\in A} N(a, B, t_0) = N(a_{_N}, B, t_0) \tag{3}$$

In fact, if $\alpha = \sup\limits_{a\in A} N(a, B, t_0)$, then there is a sequence $\{a_n\}$ in A such that

$$\alpha < N(a_n, B, t_n) + \frac{1}{n} \,. \tag{4}$$

By compactness of A, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to some point $a_{N_0} \in A$ in $(X, N, \diamondsuit)$. By proposition 3.3,

$$\lim_k N(a_{n_k}, b, t_0) = N(a_{N_0}, b, t_0)$$

for an arbitrary $b \in B$. So by taking limit on k of relation (4), we have $\alpha \leqslant N(a_{N_0}, b, t_0)$ for all $b \in B$. This proves the equality in (1).
Similarly the relation

$$\inf_{a\in A} M(a, B, t_0) = M(a_{M_0}, B, t_0)$$

holds for some $a_{M_0} \in A$. On the other hand, there is $b_{N_0} \in B$ such that $N(a_{N_0}, b_{N_0}, t_0) = N(a_{N_0}, B, t_0)$. Thus by continuity of N on $X^2 \times (0, \infty)$, $\lim\limits_k N(a_{n_k}, b_{N_0}, t_{n_k}) = N(a_{N_0}, b_{N_0}, t_0)$.

So for $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that

$$N(a_{N_0}, b_{N_0}, t_0) > \varepsilon + N(a_{N_k}, b_{N_0}, t_{n_k})$$

for all $k \geqslant k_0$. Then for all $k \geqslant k_0$ we have

$$\sup_{a\in A} N(a,B,t_0) \geqslant N(a_{N_0}, b_{N_0}, t_0) \quad > \varepsilon + N(a_{n_k}, b_{N_0}, t_{n_k}) \\ = \varepsilon + \sup_{a\in A} N(a,B,t_{n_k})$$

This implies that the function $t \longmapsto \sup_{a\in A} N(a,B,t_{n_k})$ is upper semicontinuous on $(0,\infty)$. By the same method the function $t \longmapsto \inf_{a\in A} M(a,B,t)$ is lower semicontinuous and the proof is complete. $\square$

Now we are ready to prove the theorem.

Proof of Theorem 3.1.

Let $A,B,C \in K(X)$ and $s,t > 0$. There exist $a_{N_0} \in A$ and $b_{N_0} \in B$ such that

$$\sup_{a\in A} \ N(a,B,t) = N(a_{N_0}, B, t),$$

and

$$\sup_{b\in B} \ N(A,b,t) = N(A, b_{N_0}, t).$$

Thus $H_N(A,B,t) > 0$. Similarly $H_M(A,B,t) > 0$. If $A = B$ then $H_N(A,B,t) = 0$ and $H_M(A,B,t) = 1$. Also $H_N(A,B,t) = H_N(B,A,t)$ and $H_M(A,B,t) = H_M(B,A,t)$. Now we shall prove that H_M and H_N are satisfied to the conditions (5) and (10) of definition 2.3. First of all note that for $a \in A$, $N(a,B,t) = N(a,b_{N_0},t)$ for some $b_{N_0} \in B$. Now for each $c \in C$ we have

$$N(a,C,t+s) \leqslant N(a,c,t+s) \leqslant N(a,b_{N_0},t) \Diamond N(b_{N_0},c,s).$$

By continuity of $\diamondsuit$ we have,

$$N(a, C, t+s) \leqslant N(a, b_{N_0}, t) \diamondsuit N(b_{N_0}, C, s).$$

Hence

$$\begin{aligned} \sup_{a \in A} N(a, C, t+s) &\leqslant \sup_{a \in A} N(a, B, t) \diamondsuit \sup_{a \in A} (b_{N_0}, C, s) \\ &\leqslant \sup_{a \in A} N(a, B, t) \diamondsuit \sup_{b \in B} (b, C, s). \end{aligned}$$

Similarly we have

$$\sup_{c \in C} N(A, c, t+s) \leqslant \sup_{b \in B} N(A, b, t) * \sup_{c \in C} N(B, c, s).$$

Therefore

$$H_N(A, C, t+s) \leqslant H_N(A, B, t) \diamondsuit H_N(B, C, s).$$

By the same method

$$H_M(A, C, t+s) \geqslant H_M(A, B, t) * H_M(B, C, s).$$

Clearly $H_M(A, B, t) + H_N(A, b, t) \leqslant 1$. If not, put $H_M(A, B, t) = M(a_0, b_0, t)$ and $H_N(A, B, t) = N(a_1, b_1, t)$ for some $a_0, a_1 \in A$ and $b_0, b_1 \in B$. Then

$$M(a_1, b_1, t) \geqslant M(a_0, b_0, t) > 1 - N(a_1, b_1, t),$$

or

$$M(a_1, b_1, t) + N(a_1, b_1, t) > 1 \ ,$$

a contraction. Finally by proposition 3.5, we deduce that the functions $t \longmapsto H_M\ (A, B, t)$ and $t \longmapsto H_N\ (A, B, t)$ are continuous on $(0, \infty)$. $\square$

The intuitionistic fuzzy metric $(H_M, H_N, *, \diamondsuit)$ will be called the Hausdorff intuitionistic fuzzy metric of $(M, N, *, \diamondsuit)$ on $K(X)$.

4 . Conclusion

By introducing the Hausdorff intuitionistic fuzzy metric on the set of nonempty compact subsets of X. It may provides the relation between the completeness, precompactness and so on, of $(X, M, N, *, \diamondsuit)$ with the space $(K(X), H_M, H_N, *, \diamondsuit)$.

References

[1] K. Atanassov*, Intuitionistic fuzzy sets,* In: sgurev V,editor. VII ITKR'S session; Sofia June, 1983 Central Sci. and Tech. Library, Bulg. Academy of sciences, 1984.

[2] A. George and P. Veeramani,On some results in fuzzy metric spaces, *fuzzy sets and systems* 64 (1994), 395-399.

[3] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy sets and systems* 12 (1984), 215-299.

[4] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces,*Kybernetika* ,11 (1975), 326-334.

[5] R. Lowen, *Fuzzy set theory,* Dorderecht, Kluwer Academic publishers, 1996.

[6] J.H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons and fractals* , 22 (2004), 1039-1046.

[7] J. Rodrigoues-Lopez and S. Romaguera,The Hausderff Fuzzy metric on compact sets, *Fuzzy sets and syatems,* 147 (2004), 273-283.

[8] L.A. Zadeh, Fuzzy sets, *Inform. and Control,* 8 (1965), 338-353.

Sedigheh Jahedi
Mathematics Department
College of Basic Sciences
Shiraz University of Technology
Modarres BLV. P.O.Box 71555 - 313,
Shiraz, Iran
E-mail: jahedi@ sutech.ac.ir

Ebrahim Azhdari
Mathematics Department
College of Basic Sciences
Shiraz University of Technology
Modarres BLV. P.O.Box 71555 - 313,
Shiraz, Iran
E-mail: ebi_azhdari7@yahoo.com