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Double Point Manifolds of Immersions $M^7 \looparrowright \mathbb{R}^9$

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Abstract. In this note we study the double point manifolds of immersions of seven dimensional manifold into nine dimensional Euclidean space. The method is to evaluate the Stiefel-Whitney numbers of these manifolds.

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1. Introduction

Classifying the manifolds and maps is a very difficult problem. R. Thom has classified the manifolds up to cobordism. Two n-dimensional manifold M and N are called cobordant if there is another manifold W of dimension $(n+1)$ such that $\partial W = M \sqcup N$, where $\sqcup$ denotes the disjoint union. The un-oriented cobordism ring $\mathfrak{N}_*$ is isomorphic to polynomial algebra

$$\mathbb{Z}_2[V^n : n \neq 2^i - 1]$$

where V^{2k} can be chosen to be $2k$-dimensional real projective space $\mathbb{R}P^{2k}$ and V^{2k+1} can be chosen to odd dimensional Dold manifold. Since the manifolds are not classified for $n > 2$, then we will look the problem up to cobordism. Although this problem classifies the immersions up to multiple point manifolds, but it is also closely related to Hopf and Karwiare invariant one problems; in codimension one see [5]. Let $f : M^{n-k} \looparrowright \mathbb{R}^n$ be a self-transverse immersion of a compact closed smooth $(n-k)$-dimensional manifold in n-dimensional Euclidean space $(0 < k \leqslant n)$. A point of $\mathbb{R}^n$ is an *r-fold self-intersection point* of the immersion if it is the image under f of r distinct points of the manifold. The self-transversality of f implies that the set of r-fold self-intersection points is itself the image of an immersion

$$\theta_r(f) : \Delta_r(f) \looparrowright \mathbb{R}^n$$

of a compact manifold $\Delta_r(f)$, the r-fold self-intersection manifold, of dimension $n - rk$. If $n - kr < 0$, then $\Delta_r(f)$ is empty. If $n - kr = 0$, then $\Delta_r(f)$ is finite number of points. In this problem since $n = 9$ and $k = 2$, then $n - kr = 9 - 2r$ therefore, if $r = 2, 3, 4$ then $\Delta_r(f)$ have the positive dimension. But if $r = 3, 4$ then $\Delta_r(f)$ is of dimension 3 and 1 respectively. Since by Thom's theorem the cobordism groups of 1 and 3 dimensional manifolds are trivial, then $\Delta_r(f)$ are boundary. As a result it is enough to look the problem when $r = 2$, i.e. to detect $\Delta_2(f)$ for the immersions $M^7 \looparrowright \mathbb{R}^9$. We use the algebraic topology and in particular the correspondence between cobordism groups and homotopy groups of

Thom complexes to evaluate the Stiefel-Whitney numbers of $\Delta_2(f)$. In fact two manifolds are cobordant if and only if their tangent or normal Stiefel-Whitney numbers are equal, see for details [8]. In the next section we will explain how we can detect the double point manifolds by these numbers. The main result of this note is the following.

Theorem 1.1. *There is an immersion of a* 7*- dimensional boundary into nine dimensional Euclidean space whose double point manifold is cobordant to* 5*-dimensional Dold manifold* V^5.

2. The Double Point Manifolds

Let $\mathrm{Imm}(n-k,k)$ denote the group of bordism classes of immersions $M^{n-k} \looparrowright \mathbb{R}^n$ of compact closed smooth manifolds in Euclidean n-space. Details of cobordism in this setting have been given by R. Wells in [10]. By general position every immersion is regularly homotopic and so bordant to a self-transverse immersion and so each element of $\mathrm{Imm}(n-k,k)$ can be represented by a self-transverse immersion. In the same way bordism between self-transverse immersion can be taken to be self-transverse; it is clear that such a bordism will induce a bordism of the immersions of the double point self-intersection map

$$\theta_2 : \mathrm{Imm}(n-k,k) \to \mathrm{Imm}(n-2k,2k).$$

Let $MO(k)$ denote the Thom complex of the universal $O(k)$-bundle $\gamma^k : EO(k) \to BO(k)$. Using the Pontrjagin-Thom construction, Wells in [10] describes an isomorphism

$$\phi : \mathrm{Imm}(n-k,k) \cong \pi_n^S MO(k).$$

But the stable homotopy group $\pi_n^S MO(k)$ is known to be isomorphic to homotopy group $\pi_n QMO(k)$, where the QX stands for the direct limit $\Omega^\infty \Sigma^\infty X = \lim \Omega^n \Sigma^n X$, and Σ denotes the reduced suspension functor and Ω denotes the loop space functor. We consider the $\mathbb{Z}_2$-homology Hurewicz homomorphism

$$h : \pi_n^S MO(k) \cong \pi_n QMO(k) \longrightarrow H_n QMO(k) = H_n(QMO(k); \mathbb{Z}_2).$$

The main result of [1] describes how, for a self-transverse immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ corresponding to $\alpha \in \pi_n^S MO(k)$, the Hurewicz image $h(\alpha) \in H_n QMO(k)$ determines the normal Stiefel-Whitney numbers of the self-intersection manifold $\Delta_2(f)$. The case $r = 2$, may be outlined as follows:

The 2-adic construction on X denoted by D_2X is defined as follows:

$$D_2X = W\Sigma_2 \ltimes_{\Sigma_2} X \wedge X = (W\Sigma_2 \times_{\Sigma_2} X \wedge X)/(W\Sigma_2 \times_{\Sigma_2} \{*\}).$$

Here Σ_2 denotes the permutation group on tow elements and $W\Sigma_2$ is a contractible space with a free Σ_2-action. The group Σ_2 acts on the smash product $X \wedge X$ by permuting the factors. There is a natural map

$h^2 : QX \to QD_2X$ known as a stable James-Hopf map which induces stable Hopf invariant $h^2_* : \pi^S_n X \to \pi^S_n D_2X$ see [3]. If the self-transverse immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ corresponds to the element $\alpha \in \pi^S_n MO(k)$, then the immersion of the double point self-intersection manifold $\theta_2(f) : \Delta_2(f) \looparrowright \mathbb{R}^n$ corresponds to the element $h^2_*(\alpha) \in \pi^S_n D_2MO(k)$ given by the stable Hopf invariant (see [6] and [9]). The map

$$\xi_* : \pi^S_n D_2MO(k) \to \pi^S_n MO(2k)$$

induced by the map of Thom complexes $\xi : D_2MO(k) \to MO(2k)$ make the following commutative diagram.

$$\begin{array}{ccccc}
\pi^S_{n+k}MO(k) & \xrightarrow{h^2_*} & \pi^S_{n+k}D_2MO(k) & \xrightarrow{\xi_*} & \pi^S_{n+k}MO(2k) \\
\downarrow h & & \downarrow h^S & & \downarrow h^S \\
H_{n+k}QMO(k) & \xrightarrow{h^2_*} & H_{n+k}D_2MO(k) & \xrightarrow{\xi_*} & H_{n+k}MO(2k)
\end{array}$$

Diagram (1)

In this diagram the second and third vertical maps are stable Hurewicz homomorphism defined by using the fact that the Hurewicz homomorphisms commute with suspension. Notice that the normal Stiefel-Whitney numbers(and so bordism class) of the multiple point self-intersection manifold $\Delta_r(f)$ of an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ corresponding to $\alpha \in \pi^S_n MO(k)$ are determined by (and determine) the Hurewicz image

$h^S(\beta)$ of the element $\beta = \xi_* h_*^r(\alpha) \in \pi_n^S MO(rk)$ corresponding to the immersion $\theta_r(f)$.

To read of these numbers we need to find the spherical elements of $H_nQMO(k)$, so we need to have a good description of this group. Recall that $H^*MO(k) \cong w_k\mathbb{Z}_2[w_1, w_2, \cdots w_k]$, where w_i denotes the Stiefel-Whitney classes. In homology we will work with another bases rather than dual bases, for the calculation is easier than dual one.

Homology of $MO(k)$ and $QMO(k)$: Let $e_i \in H_iBO(1) \cong \mathbb{Z}_2$ be the non-zero element (for $i \geqslant 0$). By a counting argument we can show that

$$\{e_{i_1}e_{i_2}...e_{i_k} \mid 0 \leqslant i_1 \leqslant i_2 \leqslant ... \leqslant i_k\}$$

is a basis for $H_*BO(k)$. Since the Thom complex $MO(k)$ is homotopy equivalent to the quotient space $BO(k)/BO(k-1)$. It follows that

$$\{e_{i_1}e_{i_2}...e_{i_k} \mid 1 \leqslant i_1 \leqslant i_2 \leqslant ... \leqslant i_k\}$$

is a basis for $\tilde{H}_*MO(k)$.

Dyer and Lashof (see [4] and [7]) make use of the Kudo-Araki operations $Q^i : H_mQX \to H_{m+i}QX$ to describe the homology of QX. These operations are trivial for $i < m$ and equal to the Pontrjagin square for $i = m$. If I denotes the sequence $(i_1, i_2, ..., i_r)$ then we write $Q^Ix = Q^{i_1}Q^{i_2}...Q^{i_r}x$. The sequence I is admissible if $i_j \leqslant i_{j+1}$ for $1 \leqslant j < r$ and its excess is given by $e(I) = i_1 - i_2 - ... - i_r$. With this notation we can give the description of H_*QX as a polynomial algebra: if $\{x_\lambda \mid \lambda \in \Lambda\}$ is homogeneous basis for $\tilde{H}_*X \subseteq H_*QX$ where X is path-connected space then

$$H_*QX = \mathbb{Z}_2[Q^I x_\lambda \mid \lambda \in \Lambda, I \text{admissible of excess} e(I) > \text{dim} x_\lambda].$$

We may define a height function ht on the monomial generators of H_*QX by $ht(x_\lambda) = 1$, $ht(Q^i u) = 2ht(u)$ and $ht(u \cdot v) = ht(u) + ht(v)$ (where $u \cdot v$ represents the pontrjagin product). The detection of spherical elements of a homology group is another difficult problem, still open in general, but in the case of $H_n QMO(k)$ and because of the following lemma it is sometimes possible to find these elements by a long calculations.

lemma 2.1. (1) *If an homology class* $u \in H_n X$ *is spherical then it is primitive with respect to the cup coproduct, that is*

$$\psi(u) = u \otimes 1 + 1 \otimes u,$$

where $\psi : H_n X \to H_n(X \times X) \cong \Sigma_i H_i X \otimes H_{n-i} X$ *is the map induced by the diagonal map.*

(2) *If an homology class* $u \in H_n X$ *is spherical (or stably spherical,i.e. in the image of* $h^S : \pi_n^S X \to H_n X$*) then it is annihilated by the reduced Steenrod algebra, i.e.*

$$Sq_*^i(u) = 0$$

for all $i > 0$*, where* $Sq_*^i : H_n X \to H_{n-i} X$ *is the vector space dual of the usual Steenrod square cohomology operation* $Sq^i : H^{n-i} X \to H^n X$*.*

Proof. For a proof see [2]. □

Note that if $Sq^i_* u = 0$ for all $i > 0$ in the sense of Lemma 2.1., we say that u is $\mathcal{A}_2$-annihilated.

3. Homology Group of $H_9 QMO(2)$

The homology group $H_9 QMO(2)$ is generated by the following elements.

$$e_1e_8, \quad e_2e_7, \quad e_3e_6, \quad e_4e_5, \quad e_1^2 \cdot e_1e_6, \quad e_1^2 \cdot e_2e_5, \quad e_1^2 \cdot e_3e_4, \quad e_1e_2 \cdot e_1e_5$$

$$e_1e_2 \cdot e_2e_4, \quad e_1e_2 \cdot e_3^2, \quad e_1e_3 \cdot e_1e_4, \quad e_1e_3 \cdot e_2e_3, \quad e_2^2 \cdot e_1e_4, \quad e_2^2 \cdot e_2e_3, \quad Q^6e_1e_2$$

$$Q^5e_1e_3, \quad Q^5e_2^2, \quad Q^7e_1^2, \quad e_1^2 \cdot e_1^2 \cdot e_1e_4, \quad e_1^2 \cdot e_1^2 \cdot e_2e_3, \quad e_1^2 \cdot e_1e_2 \cdot e_1e_3, \quad e_1^2 \cdot e_1e_2 \cdot e_2^2$$

$$e_1^2 \cdot e_1^2 \cdot Q^3e_1^2, \quad e_1e_3 \cdot Q^3e_1^2, \quad e_2^2 \cdot Q^3e_1^2, \quad e_1^2 \cdot Q^4e_1e_2$$

$$e_1e_2 \cdot Q^4e_1^2, \quad e_1e_2 \cdot e_1e_2 \cdot e_1e_2, \quad e_1^2 \cdot e_1^2 \cdot e_1^2 \cdot e_1e_2$$

Now according to Lemma 2.1. first we are going to find the primitive $\mathcal{A}_2$-annihilated sub-module of $H_9 QMO(2)$. The following elements are primitive.

$$e_1e_8, \quad Q^7e_1^2, \quad Q^6e_1e_2, \quad Q^5e_1e_3$$

To see which linear combination of non-primitive elements are primitive we look the action of ψ, the cup co-product, on non-primitive elements. The calculation is too long, therefore we just write some action of ψ on typical elements.

$$\psi(e_2e_7) = e_2e_7 \otimes 1 + e_1e_6 \otimes e_1^2 + e_1e_5 \otimes e_1e_2 + e_1e_4 \otimes e_1e_3 + e_1e_3 \otimes e_1e_4 + e_1e_2 \otimes e_1e_5 + e_1^2 \otimes e_1e_6 + 1 \otimes e_2e_7$$

$$\psi(e_1^2 \cdot e_1e_6) = e_1^2 \cdot e_1e_6 \otimes 1 + e_1^2 \otimes e_1e_6 + e_1e_6 \otimes e_1^2 + 1 \otimes e_1^2 \cdot e_1e_6$$

$$\psi(e_1e_2 \cdot Q^4e_1^2) = e_1e_2 \cdot Q^4e_1^2 \otimes 1 + e_1e_2 \otimes Q^4e_1^2 + Q^4e_1^2 \otimes e_1e_2 + 1 \otimes e_1e_2 \cdot Q^4e_1^2$$

We add the above calculation in the following lemma.

lemma 3.1. *The primitive sub -module of $H_9QMO(2)$ is generated by the following elements.*

$$e_1e_8, \quad Q^7e_1^2, \quad Q^6e_1e_2, \quad Q^5e_1e_3$$

$$A = e_1^2 \cdot e_1e_6 + e_1e_2 \cdot e_1e_5 + e_1e_3 \cdot e_1e_4 + e_2e_7$$

$$B = e_1e_2 \cdot e_1e_2 \cdot e_1e_2 + e_1^2 \cdot e_1^2 \cdot e_1e_4 + e_1e_2 \cdot e_1e_5 + e_2^2 \cdot e_1e_4 + e_1e_3 \cdot e_2e_3 + e_1e_2 \cdot e_2e_4 + e_1^2 \cdot e_2e_5 + e_3e_6$$

$$C = e_1^2 \cdot e_1^2 \cdot e_1^2 \cdot e_1e_2 + e_1e_2 \cdot e_1e_2 \cdot e_1e_2 + e_1^2 \cdot e_1^2 \cdot e_2e_3 + e_2^2 \cdot e_2e_3 + e_1e_2 \cdot e_2e_4 + e_1e_3 \cdot e_1e_4 + e_1e_3 \cdot e_2e_3 + e_1e_2 \cdot e_3^2 + e_1^2 \cdot e_3e_4 + e_4e_5$$

Now we find the primitive $\mathcal{A}_2$-annihilated sub-module. So we look the action of Sq_*^i on the above elements.

$$Sq_*^1(Q^7e_1^2) = 0, \quad Sq_*^1(e_1e_8) = e_1e_7, \quad Sq_*^1(Q^6e_1e_2) = Q^5e_1e_2$$

$$Sq^1_*(Q^5e_1e_3) = 0, \quad Sq^1_*(A) = e_1e_3 \cdot e_1e_3 + e_1e_7$$

$$\begin{aligned} Sq^1_*(B) &= e_1^2 \cdot e_1e_2 \cdot e_1e_2 + e_1^2 \cdot e_1^2e_1e_3 + e_2^2 \cdot e_1e_3 + e_1e_3 \cdot e_1e_3 \\ &+ e_1^2 \cdot e_2e_4 + e_1e_2 \cdot e_1e_4 + e_1e_2 \cdot e_2e_3 + e_3e_5 \end{aligned}$$

$$\begin{aligned} Sq^1_*(C) &= e_1^2 \cdot e_1^2 \cdot e_1^2 \cdot e_1^2 + e_1^2 \cdot e_1e_2 \cdot e_1e_2 + e_1^2 \cdot e_1^2 \cdot e_1e_3 + e_2^2 \cdot e_1e_3 + \\ & e_1^2 \cdot e_2e_4 + e_1e_2 \cdot e_1e_4 + e_1e_2 \cdot e_2e_3 + e_3e_5 \end{aligned}$$

So from the above it possible the elements $Q^7e_1^2, Q^5e_1e_3$ to be spherical. But since

$$Sq^2_*(Q^7e_1^2) = Sq^4_*(Q^7e_1^2) = 0, \quad Sq^2_*(Q^5e_1e_3) = Sq^4_*(Q^5e_1e_3) = 0.$$

So we have the following corollary.

corollary 3.2. *The primitive $\mathcal{A}_2$-annihilated sub-module of $H_9QMO(2)$ is generated by*

$$Q^7e_1^2, \quad Q^5e_1e_3.$$

Proof of Theorem 1.1. The un- oriented cobordism group of $7-$ dimensional manifolds is generated by boundaries and $P^2 \times V^5$. From which the manifold $P^2 \times V^5$ does not immerse in $\mathbb{R}^9$, but the boundaries immerse up to cobordism as we know the manifold S^7 immerse in $\mathbb{R}^8$ so in $\mathbb{R}^9$. There is an immersion of S^7 in $\mathbb{R}^8 \subset \mathbb{R}^9$ known as Hopf immersion which has non trivial Hurewicz image. Represent this immersion in $\pi_9QMO(2)$ by α. Since $w_1^2[S^7] = 0$, so necessarily we have $h(\alpha) =$

$Q^7e_1^2$. Therefore by Theorem 3.1 of [1] we conclude that $\xi_*P_2h(\alpha) = 0$. Therefore the multiple point manifolds of this immersion are boundary. Note that e_1e_3 is dual to $w_2w_1^2$, so any manifold with Hurewicz image $Q^5e_1e_3$ must have the condition $w_1^2[M] \neq 0$. On the other hand e_1e_3 is spherical, in fact there is an immersion of $P^2 \looparrowright \mathbb{R}^4$ such that its Hurewicz image is e_1e_3. Now by using the homotopy and stable Hopf invariant description we can show that $Q^5e_1e_3$ is spherical. This means that there is an immersion of a boundary with Hurewicz image $Q^5e_1e_3$. Let β represent this immersion in $\pi_9QMO(2)$ then we have $h(\beta) = Q^5e_1e_3$. So by Theorem 3.1 of [1]

$$\xi_*P_2h(\beta) = e_1^2e_2e_5 + e_1^3e_6 + e_1e_2e_3^2 + e_1^2e_3e_4.$$

This demonstrate that the double point manifold of this immersion is cobordant to 5-dimensional Dold manifold V^5. To prove this let us see what is the Hurewicz image of the immersion $V^5 \looparrowright \mathbb{R}^9$. The Homology group $H_9QMO(4)$ is generated by the following elements.

$$e_1^4 \cdot e_1^3e_2, \quad e_1^3e_6, \quad e_1^2e_2e_5, \quad e_1^2e_3e_4, \quad e_1e_2^2e_4, \quad e_1e_2e_3^2, \quad e_2^3e_3$$

The primitive sub-module is generated by

$$e_1^4 \cdot e_1^3e_2 + e_2^3e_3, \quad e_1^3e_6, \quad e_1^2e_2e_5, \quad e_1^2e_3e_4, \quad e_1e_2^2e_4, \quad e_1e_2e_3^2$$

Now the action of Sq^i_* on these elements shows that the primitive $\mathcal{A}_2$-annihilated sub-module of $H_9QMO(4)$ is generated by a single element

$$e_1^3e_6 + e_1^2e_2e_5 + e_1^2e_3e_4 + e_1e_2e_3^2.$$

As a result if γ denotes the immersion $V^5 \looparrowright \mathbb{R}^9$ then

$$h(\gamma) = e_1^3e_6 + e_1^2e_2e_5 + e_1^2e_3e_4 + e_1e_2e_3^2.$$

This is the same as $\xi_*P_2h(\beta)$. This proves our claim.□

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