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Bounded Linear Operators and Frames in Finitely and Countably Generated Hilbert C^*-modules

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Abstract. We show that in finitely or countably generated Hilbert C^*-module E every frame can be represented as a linear combination of three orthonormal basis and also it can be represented as a linear combination of two orthonormal bases if it is a Riesz basis.

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1. Introduction

Hilbert C^*-modules arose as generalization of notion of Hilbert space. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the C^*-algebra of coefficients being C^*-(anti)linear in its arguments. We give only a brife introduction to the theory of Hilbert C^*-modules to make our explanations self-contained.

Definition 1.1. *Let A be a C^*-algebra and E be a (left) A-module. Suppose that the linear structures given on A and E are compatible. i.e $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in C$, $a \in A$ and $x \in E$. If exists a mapping $(x, y) \longmapsto \langle x, y\rangle : E \times E \longrightarrow A$ with the properties:*

$$\begin{array}{ll} (i) & \langle x,x\rangle \geqslant 0 \;\; \textit{for every} \;\; x \in E; \\ (ii) & \langle x,x\rangle = 0 \;\; \textit{if and only if} \;\; x = 0; \\ (iii) & \langle x,y\rangle = \langle y,x\rangle^* \;\; \textit{for every} \;\; x,y \in E; \\ (iv) & \langle ax,y\rangle = a\langle x,y\rangle \;\; \textit{for every} \;\; a \in A, x,y \in E; \\ (v) & \langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle \;\; \textit{for every} \;\; x,y,z \in E. \end{array}$$

Then the pair $\{E, \langle .,.\rangle\}$ is called a (left) per-Hilbert A-module.

If the per-Hilbert A-module $\{E, \langle .,.\rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ then it is called a Hilbert A-module. The Hilbert A-module E is said to be finitely generated if there exists a finite set $\{x_j\}_{j=1}^{N} \subset E$ such that $x = \sum_j a_j x_j$ for every $x \in E$ and some coefficients $\{a_j\} \subset A$ and E is called countably generated if there exists a countable set $\{x_j\}_{j\in J} \subset H$ such that the set of $\{\sum_j a_j x_j, a_j \in A\}$ is norm-dense in H.

Example 1.2. *If A is a C^*-algebra , then A itself is a Hilbert A-module* If we define $\langle a, b\rangle = ab^*$ for $a, b \in A$.

Example 1.3. *Denote by $E = A^n, n \in N$, the set of all n-tuples with* *entries of A, where the addition is the position-wise addition derived from A, the action of A on E is the multiplication of every entry by a fixed element of A from the left and on the A-valued inner product is defined by the formula:* $\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i^*$ *for* $a = (a_i, a_2, \cdots, a_n)$, $b = (b_i, b_2, \cdots, b_n) \in E$*, then E is a Hilbert A-module.*

This kind of examples plays a crucial rule in Hilbert C^*-module theory. It allows to characterize finitely generated C^*-modules.

Theorem 1.4. *Let A be a unital C^*-algebra. Every algebraically finitely generated A-module E is an orthogonal summand of some Hilbert A-module A for a finite number n.*

Proof: [1. Cor.15.4.8]. □

Example 1.5. ([**6**]) *Let H be a Hilbert space then the algebric tensor product $A \otimes_{alg} H$ (which is a left A-module) has an A-valued inner product given on simple tensor by* $\langle a \otimes h, b \otimes g\rangle = ab^*\langle h, g\rangle_H$*;* $a, b \in A$*;* $h, g \in H$ *then $A \otimes_{alg} H$ becomes a pre-Hilbert A-module and we denote its completion by* $A \otimes H = H_A$. □

Naturally $A^n \cong A \otimes C^n$ for any $n \in N$, Moreover set $\ell_2(A)$ to be the norm-completed algebraic tensor product $\overline{A \otimes \ell_2(C)}$ that possesses as alternative description as :

$$\ell_2(A) = \{a = \{a_j\}_{j \in N} \quad \text{the sum} \quad \sum_j a_j a_j^* \quad \text{converges w.r.t} \quad \|.\|\}$$

with A-valued inner product $\langle \{a_j\}, \{b_j\} \rangle = \sum_j a_j b_j^*$.

The Hilbert A-module $\ell_2(A)$ serves as an universal environment for countably generated Hilbert A-modules that can be described as orthogonal summands. This fact was first observed by G.G. Kasparov[5].

Theorem 1.6. ([5]). *(The Kasparov Stabilization theorem).*

Let A be a C^*-algebra, unital or not, If E is a countably generated Hilbert A-module, then $E \oplus H_A \cong H_A$.

It follows from Stabilization Theorem that every countably generated Hilbert A-module E possesses an embedding into $\ell_2(A)$ as an orthogonal summand in such a way that the orthogonal complement of it is isometrically isomorphic to $\ell_2(A)$ again, i.e. $E \oplus \ell_2(A) \cong \ell_2(A)$.

The Hilbert A-module $A^n, n \in N$ and $\ell_2(A)$ possess canonical orthonormal bases.

2. Perturbation of Operators

Definition 2.1. *Let E and F be Hilbert A-modules. A map $T : E \longrightarrow F$ is adjointable if there is a map $T^* : F \longrightarrow E$ such that*

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \text{ for all} x \in E, y \in F$$

Let A be a C^*-algebra and E, F are Hilbert C^*-modules. The set of bounded adjointable A-linear operators from E onto F is denoted by

$L(E,F)$, or $L(E)$ if $E=F$.

Proposition 2.2. *Let $T \in L(E)$ with $\|T\| < 1$ then for every unitary operator U of E there are unitaries operators V and W such that $T+U = V+W$.*

Proof: Replacing, if necessary, T with U^*T we may assume that $U = I$. Since $\|T\| < 1$, both operators $I+T$ and $I+T^*$ are bounded away from zero, so $I+T$ is invertible and so by polar decomposition, $I+T = R|I+T|$ with R unitary operator (see [7]) and since $\|I+T\| \leqslant 2$ so $\||I+T|\| \leqslant 2$, it follows $|T+I| = S+S^*$ for some unitary operators ([6]). With $U = RS$ and $V = RS^*$ the lemma follows. □

Proposition 2.3. *If $T \in End_A^*(E)$ is surjective and possesses a polar decomposition then it is a multiple of the sum the of two partial isometries W_1 and W_2. i.e. $T = \frac{\|T\|}{2}(W_1+W_2)$.*

Proof: If T is adjointable bounded $A-$linear map of E onto itself that has a polar decomposition $T = V|T|$ inside $End_A^*(E)$ which, V is a partial isometry. Since $\left|\frac{T}{\|T\|}\right| \leqslant 1$ so $\left|\frac{T}{\|T\|}\right| = \frac{1}{2}(U+U^*)$ where, $U = \left|\frac{T}{\|T\|}\right| + i\sqrt{1-\left|\frac{T}{\|T\|}\right|^2}$ is unitary ([4]). Now we have the representation $T = \frac{\|T\|}{2}(VU+VU^*)$ with partial isometries VU, VU^*. □

3. Application to Frames

Frames serve as a replacement for bases in Hilbert spaces that guarantee canonical reconstruction of every element of the Hilbert space by the reconstruction formula and the following result is satisfies in Hilbert spaces. Let us being with a short introduction to the part of frame theory which we need. For more information about general frame theory we refer to ([2]).

Definition 3.1. *Let A be a unital C^*- algebra and J be a finite or countable index set . A sequence $\{x_j, j \in J\}$ of elements in a Hilbert A-module E is said to be frame if there are real canstants $C, D > 0$ such that*

$$C.\langle x, x\rangle \leqslant \sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle \leqslant D.\langle x, x\rangle$$

for every $x \in E$. The optimal constants (i.e. maximal for C and minimal for D) are called frame bounds. The frame $\{x_j, j \in J\}$ is said to be a tight frame if $C = D$ and said to be normalized tight frame if $C = D = 1$.

If $\{x_i\}_{i=1}^{\infty}$ is a frame , one can define bounded operator (frame transform) $\theta : E \longrightarrow \ell_2(A)$; $\theta(x) = \{\langle x, x_i\rangle\}_{i=1}\infty$. The adjoint operator is $\theta^* : \ell_2(A) \longrightarrow E$; $\theta^*(e_j) = x_j$ [1], which, $\ell_2(A) = \{a = \{a_j\}_{j\in N}$ the sum $\sum_j a_j a_j^*$ converges w.r.t $\|.\|_A\}$ and $\{e_i\}_{i=1}\infty$ is an orthonormal basis of

$\ell_2(A)$. The operator $S = \theta^*\theta$ is called frame operator. Conversely , the image of a normalized tight frame $\{x_j, j \in J\}$ of E under an invertible adjointable bounded A-linear operator T on E is frame of E[1].

So since there exist a equivalence between frames and onto operators that is, if we have a theorem about onto operator on Hilbert C^*-modules then we have theorem about frames.

Example 3.2. ([2]). *Suppose that $\{x_j, j \in J\}$ is a sequence in Hilbert space H such that the equation $x = \sum_{j\in J}\langle x, x_j\rangle x_j$ holds for all $x \in H$. Then $\{x_j, j \in J\}$ is a normalized tight frame H since every $x \in H$ we have:*

$$\begin{aligned}
\langle x, x\rangle &= \lim_{n\longrightarrow\infty} \left\langle \sum_{j=1}^{n}\langle x, x_j\rangle x_j, x \right\rangle \\
&= \lim_{n\longrightarrow\infty} \sum_{j=1}^{n}\langle\langle x, x_j\rangle x_j, x\rangle \\
&= \lim_{n\longrightarrow\infty} \sum_{j=1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle = \sum_{j=1}^{\infty}\langle x, x_j\rangle\langle x_j, x\rangle.
\end{aligned}$$

Example 3.3. ([2]). *For a special case let $\{e_1, e_2, e_3\}$ be an orthonormal basis for a 3- dimensional Hilbert space K. Another orthonormal basis for K is then*

$$\{\frac{1}{\sqrt{3}}(e_1 + e_2 + e_3), \frac{1}{\sqrt{6}}(e_1 - 2e_2 + e_3), \frac{1}{\sqrt{2}}(e_1 - e_3)\}$$

thus from example 3.2. the set

$$\{\frac{1}{\sqrt{3}}(e_1+e_2), \frac{1}{\sqrt{6}}(e_1-2e_2), \frac{1}{\sqrt{2}}(e_1)\}$$

is a normalized tight frame for $H = span\{e_1, e_2\}$.

Example 3.4. *Every sequence* $\{x_j, j \in J\}$ *of a finitely or countably generated Hilbert A-module for which every element* $x \in E$ *can be represented as* $x = \sum_j \langle x, x_j\rangle x_j$ *is a normalized tight frame in* E.

The decomposition of elements of E is norm-convergent if and only if it is a standard normalized tight frame . Indeed,

$$\begin{aligned} \langle x, x\rangle &= \mathcal{W} - \lim_{n \longrightarrow \infty} \left\langle \sum_{j=1}^{n} \langle x, x_j\rangle x_j, x \right\rangle \\ &= \mathcal{W} - \lim_{n \longrightarrow \infty} \sum_{j=1}^{n} \langle \langle x, x_j\rangle x_j, x\rangle \\ &= \mathcal{W} - \lim_{n \longrightarrow \infty} \sum_{j=1}^{n} \langle x, x_j\rangle\langle x_j, x\rangle = \sum_{j=1}^{\infty} \langle x, x_j\rangle\langle x_j, x\rangle. \end{aligned}$$

Example 3.5. *Let* H *be an infinite-dimensional Hilbert space and* $\{P_j, j \in J\}$ *be a maximal set of pair wise orthogonal minimal orthogonal projections on* H. *Consider the* C^**-algebra* $A = B(H)$ *of all bounded linear operators on* H *and Hilbert A-modules* $E = B(H)$. *The set* $\{P_j, j \in J\}$ *is a normalized tight frame for* E, *since* $\{P_j, j \in J\}$ *is a basis for* E *so for each* $x \in E$ *we have* $x = \sum_{j\in} \langle x, P_j\rangle P_j$ *in the a sense of* w^**- convergence in* A.

Recalling the standard identifications $A^n \cong A \otimes C^n$ and $\ell_2(A) \cong \overline{A \otimes \ell_2(C)}$.We observe that for every(normalized tight) frame $\{x_j\}_j$ of a Hilbert space H the sequence $\{1_A \otimes x_j\}$, is a standard (normalized tight) module frame of the Hilbert A-module $E = \overline{A \otimes \ell_2(C)}$ with the same frame bounds. So standard module frames exist in abundance in the canonical Hilbert A-modules. To show the existence of module frame in arbitrary finitely or countably generated Hilbert A-modules we have the following fact:

Proposition 3.6. *Let $\{x_j, j \in J\}$ normalized tight frame of Hilbert A-module E. For every partial isometry $V \in End^*_A(E)$ the sequence $\{V(x_j), j \in J\}$ becomes a normalized tight frame of $V(E)$4.*

Proof: see [4]. $\square$

Proposition 3.7. *Let A be a unital C^*-algebra and $E = \ell_2(A)$. If $\{x_j, j \in J\}$ is a frame for E then there exist orthonormal bases $\{f_j, j \in J\}, \{g_j, j \in J\}, \{h_j, j \in J\}$ for E such that $x_j = \lambda(f_j + g_j + h_j)$ for every $j \in J$ and some $\lambda \in R$.*

Proof: Let $\{e_j, j \in J\}$ be an orthonormal basis of E. By Proposition 2.2 for the operator θ^* defined by $\theta^*(e_j) = x_j$ we can write $\theta^* = \|\theta^*\|(U_1 + U_2 + U_3)$ where, each U_j; $1 \leqslant j \leqslant 3$ is unitary operator. Setting $f_j = U_1(e_j)$, $g_j = U_2(e_j)$, $h_j = U_3(e_j)$ we are done. $\square$

Example 3.8. *There is a normalized tight frame for a Hilbert A-module $E = \ell_2(A)$ which can not be written as any linear combination of two orthonormal basis in E.*

Solution: Let $\{e_j, j \in J\}$ be an orthonormal basis for $\ell_2(C)$ and consider the normalized tight frame $x_1 = 0_A$, and for all $1 \leqslant i$, $x_{i+1} = 1_A \otimes e_i$ for $\ell_2(A) \cong \overline{A \otimes \ell_2(C)}$. We proceed by way of contradiction.

If we can find orthonormal sequences $\{f_j, j \in J\}$, $\{g_j, j \in J\}$ in $\overline{A \otimes \ell_2(C)}$ and numbers α, β so that $x_j = \alpha f_i + \beta g_j$, for all $j \in J$ then

$$x_1 = 0 = \alpha f_1 + \beta g_1$$

Hence, if $\alpha \neq 0 \neq \beta$, then $span(f_1) = span(g_1)$ and orthogonality imply

$$\text{span}\{f_i\}_{j=2}^{\infty} = \text{span}\{g_j\}_{j=2}^{\infty} \neq \overline{A \otimes \ell_2(C)}$$

while

$$\text{span}\{\alpha f_i \beta g_j\}_{j=2}^{\infty} = \text{span}\overline{\{x_j\}_{j=2}^{\infty} = A \otimes \ell_2(C)}$$

This contradiction completes the proof of the example . □

Proposition 3.9. *Let A be a unital C^*-algebra and E be a finitely or countably generated Hilbert A-module . Every frame of E is (a multiple of) the sum of two normalized tight frames of E.*

Proof: Let $\{x_j, j \in J\}$ is a frame for E and consider the adjoint $\theta^* : \ell_2(A) \longrightarrow E$ of the frame transform θ.

By Theorem 4.1 in [1], θ^* is surjective and possesses a polar decomposition and $x_j = \theta^*(e_j)$, where $\{e_j, j \in J\}$ is a standard orthonormal basis of $\ell_2(A)$. So by Proposition 2.3, $\theta^* = \frac{\|\theta^*\|}{2}(W_1 + W_2)$ where, W_1 and W_2 are partial isometry. Setting $f_j = W_1(e_j)$, $h_j = W_2(e_j)$ by Proposition 3.6 we get the desired result. □

Definition 3.10. *A Riesz basis is a family of the form* $\{x_j\}_{j=1}^{\infty} = \{T(f_j)\}_{j=1}^{\infty}$*, where* $\{f_j\}_{j=1}\infty$ *is a orthonormal basis for* E *and* T *is a bounded adjointable invertible* A*-linear operator on* E*.*

In fact we can to prove that([7]): $\{x_j\}_{j=1}^{\infty}$ *is a Riesz basis* $\Longleftrightarrow$ $\{x_j\}_{j=1}^{\infty}$ *is a frame and if* $\sum_{i=j} a_j x_j = 0$ *with* $\{a_j, j \in J\} \subset A$ *and* $J \in N$*, then* $a_j x_j = 0$*,* $\forall j \in J$*.*

Proposition 3.11. *A frame* $\{x_j, j \in J\}$ *of a Hilbert* A*-module* $E = \ell_2(A)$ *can be written as a linear combination of two orthonormal bases of* E *if it is a Riesz basis.*

Proof: By Definition 3.10, there is adjointable invertible bounded A-linear operator T on E and the orthonormal basis $\{e_j, j \in J\}$ of E such that $T(e_j) = x_j$ (in fact $T = \theta^*$). Since T is invertible so it is a linear combination of two unitary operators U_1 and U_2, i.e. $T = \frac{\|T\|}{2}(U_1 + U_2)$

by Proposition 2.3 Setting $f_j = U_1(e_j)$, $h_j = U_2(e_j)$ we get the desired result. $\Box$

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