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Cyclicity of the Shift Operator on Analytic Function Spaces

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Abstract. In this paper we characterize some sufficient conditions for a vector in the Hilbert spaces of analytic functions to be cyclic for the backward shift operator.

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1. Introduction

Let H be a Hilbert space of complex-valued analytic functions on the open unit disc $\mathbb{D}$ such that point evaluations are bounded linear functionals on H. Then for every $w \in \mathbb{D}$ there exists a function k_w in H such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ by $K(z, w) = k_w(z)$, then K is a positive definite function with the reproducing property

$$f(w) = \langle f(\cdot), K(\cdot, w)\rangle$$

for every $w \in \mathbb{D}$ and $f \in H$. The function K is called the *reproducing kernel* for H.

Recall that a function $K : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ is *positive definite* (denoted $K \gg 0$) provided

$$\sum_{j,k=1}^{n} a_j \bar{a}_k K(w_j, w_k) \geqslant 0$$

for any finite set of complex numbers $a_1, \ldots, a_n$ and any finite subset $w_1, \ldots, w_n$ of $\mathbb{D}$. Conversely, if $K : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ is positive definite then

$$\{\sum_{j=1}^{n} a_j K(\cdot, w_j) : a_1, \ldots, a_n \in \mathbb{C} \text{ and } w_1, \ldots, w_n \in \mathbb{D}\}$$

has dense linear span in a Hilbert space $H(K)$ of functions with

$$\|\sum_{j=1}^{n} a_j K(\cdot, w_j)\|^2 = \sum_{j,k=0}^{n} a_j \bar{a}_k K(w_j, w_k)$$

and

$$f(w) = \langle f(\cdot), K(\cdot, w)\rangle$$

for every w in $\mathbb{D}$ and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in $\mathbb{D}$. Note also that convergence in $H(K)$ implies uniform convergence on compact subsets of $\mathbb{D}$.

Now if K is a kernel on $\mathbb{D} \times \mathbb{D}$ which is analytic in the first variable and consequently coanalytic in the second variable, then $K(z, \bar{w})$ is an analytic function on $\mathbb{D} \times \mathbb{D}$ in the two variables z and w. Hence $K(z, w)$ can be represented by the double power series

$$\sum_{j,k=0}^{\infty} a_{jk} z^j \bar{w}^k.$$

If C denotes the matrix $[a_{jk}]$, then such a K can be written more compactly in the form

$$K(z, w) = \bar{\mathbf{Z}}^* C \bar{\mathbf{W}} = \langle C\bar{\mathbf{W}}, \bar{\mathbf{Z}} \rangle_{l^2_+},$$

where $\mathbf{Z}$ denotes the column vector whose transpose is $(1, z, z^2, \ldots)$. (Here l^2_+ denotes the usual space of all square sumable sequences.) It is well known that $K \gg 0$ if and only if $C > 0$. Henceforth for positive matrices $C, H(C)$ will denote the space $H(K)$ where $K = \bar{\mathbf{Z}}^* C \bar{\mathbf{W}}$. For more information about reproducing kernels the reader is referred to [2]. Some sources on spaces of analytic functions are [3;4;5;7;8;9;10;11].

Theorem 1.1([1]). *If $C = B^* B$ for some bounded operator B on l^2_+, then the operator V from $(\ker B^*)^\perp$ into $H(C)$ defined by*

$$V(f)(z) = \langle B^* f, \bar{\mathbf{Z}} \rangle_{l^2_+}$$

is unitary.

Corollary 1.2. *If $C = B^* B$ and $\{f_n\}$ is an orthonormal basis for $(\ker B^*)^\perp$, then $\{\langle B^* f_n, \bar{\mathbf{Z}} \rangle_{l^2_+}\}$ is an orthonormal basis for $H(C)$.*

We can construct a basis for $H(C)$ by using the Cholesky decomposition of the nonnegative matrix C into the product U^*U, where U is upper triangular. For more details the reader is referred to [6].

Throughout this paper H is a Hilbert space of analytic functions on $\mathbb{D}$ such that $1 \in H$, $zH \subset H$ and point evaluations are bounded for every $w \in \mathbb{D}$. If the set $\{1, z, z^2, \ldots\}$ is an orthogonal basis for H and

$$f = \sum_{n=0}^{\infty} f_n z^n \in H,$$

then by boundedness of point evaluations, the power series expansion of f can be written as

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

The backward shift operator on H is denoted by L that is defined by

$$L(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} f_{n+1} z^n.$$

We assume that L is a bounded operator on H.

We say that a vector f in a Hilbert space H is a cyclic vector of a bounded operator A on H if

$$H = \overline{span}\{A^n f : \ n = 0, 1, 2, \ldots\}.$$

Here $\overline{span}\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$.

2. Main Result

In the main theorem of this paper we give sufficient conditions for a vector f in $H(K)$ to be cyclic for the backward shift operator on $H(K)$.

Theorem 2.1. *Let H have the reproducing kernel*

$$K(z,w)=\frac{1-z\bar{w}}{(1-z)(1-w)}\sum_{i=0}^{\infty}a_i(z\bar{w})^i,$$

where $\{a_i\}_i$ is a nondecreasing sequence of positive numbers. Suppose that for sufficiently large positive integer N,

(i) $J_m=\sup\{(a_{k+n}-a_{k+n-1})/[(a_{m+n}-a_{m+n-1})(a_{k+N}-a_{k+N-1})]:k\geqslant N+m,n\geqslant 1\}<\infty$ for any positive integer m and

(ii) $\{(k+N+1)(a_{k+N}-a_{k+N-1})/(a_k-a_{k-1})\}_k\in\ell^1$.

If f is a vector in H with infinitely many $f^{(n)}(0)\neq 0$, then f is a cyclic vector of L.

Proof. We have

$$\begin{aligned}K(z,w) &= \left(\frac{z}{1-z}+\frac{1}{1-\bar{w}}\right)\sum_{i=0}^{\infty}a_i(z\bar{w})^i\\ &= \sum_{i=0}^{\infty}\sum_{m=1}^{\infty}a_iz^i\bar{w}^{i+m}+\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}a_iz^{i+n}\bar{w}^i.\end{aligned}$$

If we denote the matrix of $K(z,w)$ by $A=(a_{ij})_{i,j=0}^{\infty}$, then

$$a_{i,i+m}=a_i\ ,\quad i=0,1,2,\dots\ ,\quad m=1,2,\dots\ ,$$

$$a_{i+n,i}=a_i\ ,\quad i=0,1,2,\dots\ ,\quad n=0,1,2,\dots\ .$$

Hence $a_{ij}=a_i$ for $j\geqslant i$ and A is symmetric. Now by Corollary 1.2 we can see that the set $\{e_n\}_{n=0}^{\infty}$ is an orthogonal basis for H where $e_n=z^nw_0$ and

$$w_0=\sum_{i=0}^{\infty}z^i.$$

Let

$$f = \sum_{n=0}^{\infty} f_n e_n$$

and put

$$M = \overline{span}\{L^n f : \ n = 0, 1, 2, \ldots\}.$$

If $M \neq H$, then there is a nonzero

$$g = \sum_{n=0}^{\infty} g_n e_n$$

in H such that $\langle L^n f, g \rangle = 0$ for all $n = 0, 1, 2, \ldots$. Put

$$m = \min\{k : g_k \neq 0\}.$$

Since

$$L^n f = \sum_{k=0}^{\infty} f_{n+k} e_k,$$

we get

$$0 = \langle L^n f, g \rangle = \sum_{k=m}^{\infty} f_{n+k} \bar{g}_k \|e_k\|^2.$$

Therefore

$$f_{m+n} \bar{g}_m \|e_m\|^2 = - \sum_{k=m+1}^{\infty} f_{n+k} \bar{g}_k \|e_k\|^2.$$

Now choosing N as in conditions of the theorem, we can choose g such that $g_k = 0$ for $m < k < m + 2N$. Hence we obtain

$$|f_{m+n}| \|e_{m+n}\| \leqslant \frac{1}{\|e_m\|^2 |\bar{g}_m|} \sum_{k \geqslant m+2N} [(|f_{n+k}| \|e_{k+n}\|)(|\bar{g}_k| \|e_k\|) \frac{\|e_k\| \|e_{m+n}\|}{\|e_{k+n}\|}].$$

Since $\|e_n\|^{-2} = a_n - a_{n-1}$, by condition (i) we have

$$\frac{\|e_{m+n}\|\|e_k\|}{\|e_{k+n-N}\|} = \left(\frac{a_{k+n-N} - a_{k+n-N-1}}{(a_k - a_{k-1})(a_{m+n} - a_{m+n-1})}\right)^{\frac{1}{2}} \leqslant J_m^{\frac{1}{2}}$$

for $k \geqslant m + 2N$. Therefore

$$|f_{m+n}|\|e_{m+n}\| \leqslant \frac{J_m^{\frac{1}{2}}\|f\|}{|\bar{g}_m|\|e_m\|^2} \sum_{k\geqslant m+2N} \left[(|\bar{g}_k|\|e_k\|)\frac{\|e_{k+n-N}\|}{\|e_{k+n}\|}\right].$$

By the Hölder inequality we have

$$|f_{m+n}|\|e_{m+n}\| \leqslant \frac{J_m^{\frac{1}{2}}\|f\|\|g\|}{|\bar{g}_m|\|e_m\|^2} \left(\sum_{k\geqslant m+2N} (\frac{\|e_{k+n-N}\|}{\|e_{k+n}\|})^2\right)^{\frac{1}{2}}.$$

Now if

$$h = L^i f = \sum_{n=0}^{\infty} h_n e_n,$$

then

$$h_n = (L^i f)_n = f_{n+i}$$

and $\langle L^n h, g\rangle = 0$ for all $n = 0, 1, 2, \ldots$. By the same manner as we used in the above calculations, by replacing f by h, we obtain

$$|h_{m+n}|\|e_{m+n}\| \leqslant c_{m+n}\|h\|,$$

where

$$c_{m+n} = \frac{J_m^{\frac{1}{2}}\|g\|}{|\bar{g}_m|\|e_m\|^2} \left(\sum_{k\geqslant m+N} (\frac{\|e_{k+n-N}\|}{\|e_{k+n}\|})^2\right)^{\frac{1}{2}}.$$

So for any vector

$$h = \sum_{n=0}^{\infty} h_n e_n$$

in M, we have

$$|h_{m+n}|\|e_{m+n}\| \leqslant c_{m+n}\|h\|,$$

where the constants c_{m+n} do not depend upon the choice of h in M. If

$$\alpha_i = \begin{cases} c_i & i \geqslant m \\ 1 & i < m \end{cases},$$

then

$$|h_i|\|e_i\| \leqslant \alpha_i\|h\| \qquad (*)$$

for all

$$h = \sum_{i=0}^{\infty} h_i e_i$$

in M. Now we prove that $\{\alpha_i\} \in \ell^2$. Note that for $i \geqslant m$,

$$\alpha_i = \frac{J_m^{\frac{1}{2}}\|g\|}{|\bar{g}_m|\|e_m\|^2} \left(\sum_{k \geqslant 2N} (\frac{\|e_{k+i-N}\|}{\|e_{k+i}\|})^2 \right)^{\frac{1}{2}}.$$

Put

$$\gamma_i = \left(\sum_{k \geqslant 2N} (\frac{\|e_{k+i-N}\|}{\|e_{k+i}\|})^2 \right)^{\frac{1}{2}}, \qquad i \geqslant m.$$

It is sufficient to show that

$$\sum_{i \geqslant m} \gamma_i^2 < \infty.$$

We have

$$\begin{aligned}
\sum_{i\geqslant m}\gamma_i^2 &= \sum_{i\geqslant m}\sum_{k\geqslant 2N}\Big(\frac{\|e_{k+i-N}\|}{\|e_{k+i}\|}\Big)^2\\
&= \sum_{k=0}^{\infty}(k+1)\Big(\frac{\|e_{m+k+N}\|}{\|e_{2N+m+k}\|}\Big)^2\\
&= \sum_{k\geqslant 2N+m}(k+1-2N-m)\Big(\frac{\|e_{k-N}\|}{\|e_k\|}\Big)^2\\
&\leqslant \sum_{k\geqslant 2N}(k+1)\Big(\frac{\|e_{k-N}\|}{\|e_k\|}\Big)^2.
\end{aligned}$$

But $\|e_n\|^{-2} = a_n - a_{n-1}$, thus

$$\begin{aligned}
\sum_{i\geqslant m}\gamma_i^2 &\leqslant \sum_{k\geqslant N}(k+1)\frac{a_k - a_{k-1}}{a_{k-N} - a_{k-N-1}}\\
&= \sum_{k=0}^{\infty}(k+N+1)\frac{a_{k+N} - a_{k+N-1}}{a_k - a_{k-1}}
\end{aligned}$$

that is finite by condition (ii). Now, by inequality $(*)$, we can see that M is finite dimensional, which contradicts our assumption that $f^{(n)}(0) \neq 0$ for infinitely many n. This implies that $M = H$ and so f is a cyclic vector of L. This completes the proof. □

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