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Original Research Paper

On the Stability of Ternary AQ-Hom-Ders: Bridging Generalized AQ-Functional Equations

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Abstract. In this paper, we investigate the relationship between ternary AQ-hom-ders and the generalized AQ-functional equation (generalized AQ-$\mathbb{FE}$) on ternary Banach algebras (TBA). By considering the specific cases of $t = 1$ and $t = 2$, we establish a systematic connection between additive and quadratic mappings and their corresponding ternary AQ-hom-ders. After that, we solve concrete examples on matrix algebras, highlighting their relevance in non-commutative settings. Furthermore, we derive the solutions for the generalized AQ-$\mathbb{FE}$ and establish their Hyers–Ulam (HU) stability. Finally, we investigate the HU stability of ternary AQ-hom-ders on TBA using the fixed point (FP) approach.

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1 Introduction

The HU stability concept, originating from a question by Ulam, can be described as follows. Given two fixed mappings $\mathcal{F}_1$ and $\mathcal{F}_2$ and nonneg-

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ative control functions φ and ψ, the equation $\mathcal{F}_1(G) = \mathcal{F}_2(G)$ is said to be stable if whenever a function g satisfies $d(\mathcal{F}_1(g), \mathcal{F}_2(g)) \leq \varphi$, there exists an exact solution h with $\mathcal{F}_1(h) = \mathcal{F}_2(h)$ and $d(g, h) \leq \psi$. The equation $\mathfrak{F}(m_1 + m_2) = \mathfrak{F}(m_1) + \mathfrak{F}(m_2)$ is called the additive $\mathbb{FE}$, and any function satisfying it is an additive mapping.

In 1941 Hyers [5] provided the first affirmative result on approximate additivity for mappings between Banach spaces. In 1978, Th. M. Rassias [19] established the concept of generalized Hyers–Ulam stability, demonstrating that an approximately additive mapping is close to an exact one given a control function $\upsilon(\|m_1\|^r + \|m_2\|^r)$ where $\upsilon > 0$ and $0 < r < 1$. Subsequently, in 1982, J. M. Rassias [18] proposed a departure from this approach by introducing a control function of the form $\upsilon(\|m_1\|^r \cdot \|m_2\|^r)$ with $\upsilon > 0$ and $0 < r < 1$.

These developments led to further research in diverse algebraic structures. For instance, Keshavarz and Heydari [11] (2024) solved additive–Jensen s-$\mathbb{FE}$s where $|s| < 1$, classified additive and $\mathbb{C}$-linear solutions, and addressed Lie-bracket derivations and Jordan properties regarding HU stability. Similarly, Izadi and Samei [6] (2025) introduced the orthogonally Jensen s-$\mathbb{FE}$ on ternary Lie algebras, proved its additive properties, and investigated the HU stability and hyperstability of orthogonally ternary Lie hom-derivations.

In 1983 Skof [20] established stability results for the quadratic $\mathbb{FE}$ between normed and Banach spaces. The quadratic equation is $\mathfrak{J}(m_1 + m_2) + \mathfrak{J}(m_1 - m_2) = 2\mathfrak{J}(m_1) + 2\mathfrak{J}(m_2)$. Cholewa [1] extended Skof's theorem to mappings defined on abelian groups. Later, in 1994 Găvruta [4] substituted those specific bounds with a general control function $\varphi(r_1, r_2)$ and studied HU stability in this broader framework. For more information (see [7] and [16]). Researchers have studied stability for derivations, homomorphisms and fractional equation ($\mathsf{F}\mathbb{E}$) using various control functions and both FP and direct techniques. Motivated by that idea and by Jensen's equation, Kefayati et al. [8] introduced in 2025 the orthogonally Jensen ρ-$\mathbb{FE}$ and generalized Lie bracket of triple derivations on orthogonally triple Banach algebra. For more information (see [9] and [13]).

A TBA $\mathcal{A}$ with norm $\|.\|$ is a complex Banach space endowed with a ternary operation $(m_1, m_2, m_3) \mapsto [m_1, m_2, m_3]$ from $\mathcal{A}^3$ to $\mathcal{A}$. This

operation is $\mathbb{C}$-linear in the first and third (outer) arguments, conjugate-linear in the middle argument, and associative in the sense that

$$[[m_1, m_2, m_3]m_4, m_5] = [m_1, [m_2, m_3, m_4], m_5] = [m_1, m_2, [m_3, m_4, m_5]].$$

It also satisfies the norm estimates $\|[m_1, m_2, m_3]\| \leq \|m_1\|.\|m_2\|.\|m_3\|$ and $\|m.m.m\| = \|m\|^3$ (see [22]). Ternary and, more generally, n-ary algebraic structures arise in several areas of mathematical physics and photonics—examples include the quark model and Nambu mechanics [10] and [15]. Many physical models are treated as linear systems, where additivity plays a key role; this principle is important in contexts such as computing internal energy in thermodynamics and in the interpretation of the superposition principle.

In 2019 Park et al. [17] introduced an additive s-functional inequality and, employing the FP and direct techniques, obtained HU stability results for that inequality and for hom-derivations on complex TBA. Keshavarz et al. [12] defined the ternary hom-multiplier on TBA and proved its HU-type stability via the FP method in 2025. Dehghanian et al. [3] introduced the ternary hom-der on TBA and established the HU stability of ternary hom-der in TBA in 2024. In this paper, we aim to provide a unified setting for both additive and quadratic functional equations by introducing a generalized AQ-functional equation for the cases $t \in \{1, 2\}$. Within this framework, we define ternary AQ-hom-ders on TBA and investigate their fundamental properties, providing a non-trivial example on matrix algebras to illustrate the applicability of our proposed setting. Moreover, we establish HU stability results for both the generalized AQ-$\mathbb{FE}$ and the associated ternary AQ-hom-der mappings.

In Section 2, we introduce the fundamental notions and preliminary results required for what follows, and we illustrate the concept of ternary AQ-hom-ders by working through representative examples and resolving the attendant problems.

In Section 3, we derive the explicit solutions of the generalized AQ-$\mathbb{FE}$, showing that the cases $t = 1$ and $t = 2$ recover the additive and quadratic $\mathbb{FE}$s, respectively. We then investigate HU stability of the generalized AQ-$\mathbb{FE}$ and establish HU stability of the associated ternary AQ-hom-ders on TBAs via the FP approach.

2 The Basic Notions

In this section, Let $\mathcal{X}$ and $\mathcal{Y}$ are two normed spaces and $\mathcal{A}$ is a TBA. First, we introduce the new concept of additive and quadratic mappings and called the generalized AQ-$\mathbb{FE}$ on normed spaces, after that we define ternary AQ-hom-ders on TBAs.

Let $t \in \{1, 2\}$. A mapping $\mathfrak{J}_t : \mathcal{X} \to \mathcal{Y}$ satisfies the generalized AQ-$\mathbb{FE}$ if for all $m_1, m_2, m_3 \in \mathcal{X}$ the following identity holds:

$$t\mathfrak{J}_t(m_1 + m_2 + m_3) + t\mathfrak{J}_t(m_1) = \mathfrak{J}_t(m_1 + m_2) + \mathfrak{J}_t(m_1 + m_3) + (t-1)\mathfrak{J}_t(m_2 + m_3) \tag{1}$$

When $t = 1$, equation (1) reduces to the additive $\mathbb{FE}$; when $t = 2$, it becomes the quadratic $\mathbb{FE}$.

In the following, by using the above concept we define ternary AQ-hom-ders on TBAs.

Definition 2.1. Let $\mathcal{D}_t : \mathcal{A} \to \mathcal{A}$ is a AQ-$\mathbb{FE}$,
(i) A $\mathbb{C}$-linear mapping $\mathcal{D}_1 : \mathcal{A} \to \mathcal{A}$ is called a ternary 1-hom-der if it satisfies the following condition:

$$[\mathcal{D}_1(m_1), \mathcal{D}_1(m_2), \mathcal{D}_1(m_3)] = [\mathcal{D}_1(m_1), m_2, m_3] + [m_1, \mathcal{D}_1(m_2), m_3] + [m_1, m_2, \mathcal{D}_1(m_3)].$$

(ii) A quadratic mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is called an ternary 2-hom-der if it satisfies the following condition:

$$[\mathcal{D}_2(m_1), \mathcal{D}_2(m_2), \mathcal{D}_2(m_3)] = \left[\mathcal{D}_2(m_1), m_2^2, m_3^2\right] + \left[m_1^2, \mathcal{D}_2(m_2), m_3^2\right] + \left[m_1^2, m_2^2, \mathcal{D}_2(m_3)\right].$$

for all $m_1, m_2, m_3 \in \mathcal{A}$.

To illustrate the fundamental structure of the definitions provided above, we first consider the most basic case in the scalar field. This will serve as a baseline to clarify the underlying mechanics of ternary AQ-hom-ders.

Example 2.2. Let $\mathbb{C}$ be equipped with the ternary product

$$[m_1, m_2, m_3] = m_1 m_2 m_3$$

for all $m_1, m_2, m_3 \in \mathbb{C}$; then $\mathbb{C}$ is a TBA. Define $H : \mathbb{C} \to \mathbb{C}$ by

$$\theta(m) = \sqrt{3}\, m^t$$

for every $m \in \mathbb{C}$ and for $t \in \{1, 2\}$. With this definition θ is a ternary AQ-hom-ders on $\mathbb{C}$, while it is not a ternary AQ-homomorphisms nor a ternary AQ-derivations.

While the scalar field $\mathbb{C}$ provides a straightforward intuition, the concept of ternary AQ-hom-ders can be extended to more complex and non-commutative algebraic structures. To demonstrate the non-trivial nature of these mappings, let us consider the matrix algebra $M_n(\mathbb{C})$.

Example 2.3. Let $\mathcal{A} = M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices with the ternary product $[A, B, C] = ABC$. Let I denote the identity matrix. We define a mapping $\mathcal{D}_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ for $t \in \{1, 2\}$ by:

$$\mathcal{D}_t(A) = \sqrt{3}A^t$$

where $A^1 = A$ and $A^2 = A \cdot A$ (the matrix square).

(i) For $t = 1$, $\mathcal{D}_1(A) = \sqrt{3}A$. Substituting this into the ternary 1-hom-der condition:

$$\left[\sqrt{3}A_1, \sqrt{3}A_2, \sqrt{3}A_3\right] = (\sqrt{3})^3 A_1 A_2 A_3 = 3\sqrt{3}A_1 A_2 A_3$$

The right side is:

$$\left[\sqrt{3}A_1, A_2, A_3\right] + \left[A_1, \sqrt{3}A_2, A_3\right] + \left[A_1, A_2, \sqrt{3}A_3\right] = 3\sqrt{3}A_1 A_2 A_3$$

Thus, $\mathcal{D}_1$ is a ternary 1-hom-der.

(ii) For $t = 2$, $\mathcal{D}_2(A) = \sqrt{3}A^2$. Substituting this into the ternary 2-hom-der condition:

$$\left[\sqrt{3}A_1^2, \sqrt{3}A_2^2, \sqrt{3}A_3^2\right] = (\sqrt{3})^3 \left(A_1^2 A_2^2 A_3^2\right) = 3\sqrt{3}A_1^2 A_2^2 A_3^2$$

The right side is:

$$\left[\sqrt{3}A_1^2, A_2^2, A_3^2\right] + \left[A_1^2, \sqrt{3}A_2^2, A_3^2\right] + \left[A_1^2, A_2^2, \sqrt{3}A_3^2\right] = 3\sqrt{3}A_1^2 A_2^2 A_3^2$$

Thus, $\mathcal{D}_2$ is a ternary 2-hom-der.

Theorem 2.4. *[2] Let $(\mathcal{A}, d)$ be a complete generalized metric space and let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a strict contraction with Lipschitz constant $0 < L < 1$. Then for any $m \in \mathcal{A}$ exactly one of the following holds:*
- For every nonnegative integer n we have $d\left(\mathcal{F}^n(m), \mathcal{F}^{n+1}(m)\right) = \infty$.
- There exists an integer $i_0 \geq 0$ such that:
(i) $d\left(\mathcal{F}^n(m), \mathcal{F}^{n+1}(m)\right) < \infty$ for all $n \geq n_0$;
(ii) The iterates $\mathcal{F}^n(m)$ converge to a unique FP s^* of $\mathcal{F}$ in the set

$$\mathcal{A}^* = \{s \in \mathcal{A} \mid d(\mathcal{F}^{n_0} m, s) < \infty\}$$

(iii) For every $s \in \mathcal{A}$ one has $d\left(s, s^*\right) \leq \frac{1}{1-L} d(s, \mathcal{F}(s))$.

3 Main Results

In this section, Let $\mathcal{X}$ and $\mathcal{Y}$ are two normed spaces and $\mathcal{A}$ is a TBA. Below we provide the solution of the $\mathbb{FE}$ (1); this solution is additive when $t = 1$ and quadratic when $t = 2$.

Proposition 3.1. *Let $\mathfrak{J}_t : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $\mathbb{FE}$ (1), $t \in \{1, 2\}$:*
(i) for $t = 1$, $\mathfrak{J}_t$ is an additive mapping;
(ii) for $t = 2$, $\mathfrak{J}_t$ is a quadratic.

Proof. First note that $m_1 = m_2 = m_3 = 0$, we immediately obtain $\mathfrak{J}_t(0) = 0$.

(i) Let $t = 1$. By setting $m_3 = -m_1$ and $m_2 = m_1$ in (1), it follows that

$$\mathfrak{J}_1(2m_1) = 2\mathfrak{J}_1(m_1). \tag{2}$$

Setting $m_1 = 0$ and $m_3 = -m_2$ in (1), we obtain

$$\mathfrak{J}_1(-m_2) = -\mathfrak{J}_1(m_2). \tag{3}$$

Finally, by substituting $m_1 = 0$ into (1) and utilizing the relations (2) and (3), we arrive at

$$\mathfrak{J}_1(m_2 + m_3) = \mathfrak{J}_1(m_2) + \mathfrak{J}_1(m_3).$$

which implies that $\mathfrak{J}_1$ is an additive mapping.

(ii) Assume $t = 2$. By setting $m_3 = -m_1$ and $m_2 = m_1$ in (1), we obtain

$$\mathfrak{J}_2(2m_1) = 4\mathfrak{J}_2(m_1). \tag{4}$$

Setting $m_3 = -m_1$ and $m_2 = 0$ in (1) yields $\mathfrak{J}_2(-m_1) = \mathfrak{J}_2(m_1)$. Furthermore, by substituting $m_3 = -m_1$ into (1) and simplifying the expression, we obtain the following relation:

$$\mathfrak{J}_2(m_2 + m_1) + \mathfrak{J}_2(m_2 - m_1) = 2\mathfrak{J}_2(m_2) + 2\mathfrak{J}_2(m_1). \tag{5}$$

The relation (5) is the well-known characterization of quadratic mappings. Thus, $\mathfrak{J}_2$ is a quadratic mapping. □

In the following theorem, we recall the Găvruta-type control function adapted to the generalized AQ-$\mathbb{FE}$.

Let $t \in \{1, 2\}$ and let $\zeta : \mathcal{A}^3 \to [0, \infty)$ be two functions that satisfy the following conditions:

$$\zeta\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}\right) \leq \frac{k}{2^t}\zeta(m_1, m_2, m_3) \tag{6}$$

for all $m_1, m_2, m_3 \in \mathcal{A}$ and some $0 < k < 1$.

Theorem 3.2. *Let $t \in \{1, 2\}$ and let $\mathfrak{J}_t : \mathcal{A} \to \mathcal{A}$ satisfy, for all $m_1, m_2, m_3 \in \mathcal{A}$,*

$$\begin{aligned}\|t\mathfrak{J}_t(m_1 + m_2 + m_3) + t\,\mathfrak{J}_t(m_1) - \mathfrak{J}_t(m_1 + m_2) - \mathfrak{J}_t(m_1 + m_3) \\ - (t-1)\,\mathfrak{J}_t(m_2 + m_3)\| \leq \zeta(m_1, m_2, m_3)\end{aligned} \tag{7}$$

where $\zeta : \mathcal{A}^3 \to [0, \infty)$ satisfies condition (6). Then there exist unique AQ-$\mathbb{FE}$ $\mathfrak{h}_t : \mathcal{A} \to \mathcal{A}$ for which the following estimates hold:
For $t = 1$,

$$\|\mathfrak{J}_1(m) - \mathfrak{h}_1(m)\| \leq \frac{k}{2(1-k)}\zeta(0, m, m)$$

For $t = 2$,

$$\|\mathfrak{J}_2(m) - \mathfrak{h}_2(m)\| \leq \frac{k}{2^2(1-k)}\zeta(m, m, -m).$$

Proof. Firstly, It follows from (6), taking $m_1 = m_2 = m_3 = 0$ gives $\zeta(0,0,0) = 0$. Clearly, by induction one can obtain that

$$2^{nt}\zeta\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}\right) \leq k^n \zeta(m_1, m_2, m_3),$$

for all $n \in \mathbb{N}$, such that $n \to \infty$, we have

$$2^{nt}\zeta\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}\right) = 0. \tag{8}$$

for all $m_1, m_2, m_3 \in \mathcal{A}$.

For $t = 1$, substitute $m_3 = m_2 = m$ and $m_1 = 0$ into (7) to obtain

$$\|\mathfrak{J}_1(2m) - 2\mathfrak{J}_1(m)\| \leq \zeta(0, m, m), \qquad \forall\ m \in \mathcal{A}.$$

For $t = 2$, plug $m_3 = -m_1$ and $m_2 = m_1$ into (7), yielding

$$\left\|\mathfrak{J}_2(2m) - 2^2\mathfrak{J}_2(m)\right\| \leq \zeta(m, m, -m), \qquad \forall\ m \in \mathcal{A}.$$

Let $\mathfrak{A}$ denote the collection of all maps $\mathfrak{g} : \mathcal{A} \to \mathcal{A}$ satisfying $\mathfrak{g}(0) = 0$. For $t \in \{1, 2\}$ define the operator $\mathcal{Q}_t : \mathfrak{A} \to \mathfrak{A}$ by

$$\mathcal{Q}_t(\mathfrak{g})(m) = 2^t \mathfrak{g}\left(\frac{m}{2}\right).$$

For $\mathfrak{g}, \mathfrak{h} \in \mathfrak{A}$ set

$$d_1(\mathfrak{g}, \mathfrak{h}) = \inf\left\{\alpha > 0 : \|\mathfrak{g}(m) - \mathfrak{h}(m)\| \leq \alpha\, \zeta(0, m, m)\right\}$$

and

$$d_2(\mathfrak{g}, \mathfrak{h}) = \inf\left\{\alpha > 0 : \|\mathfrak{g}(m) - \mathfrak{h}(m)\| \leq \alpha\, \zeta(m, m, -m)\right\},$$

with the convention $\inf \varnothing = +\infty$. One easily verifies that for each $t \in \{1, 2\}$ the function d_t defines a generalized metric on $\mathfrak{A}$, and that $(\mathfrak{A}, d_t)$ is complete (see [14]). Let $\mathfrak{g}, \mathfrak{h} \in \mathfrak{A}$. Then

$$\|\mathcal{Q}_1(\mathfrak{g})(m) - \mathcal{Q}_1(\mathfrak{h})(m)\| \leq 2d_1(\mathfrak{g}, \mathfrak{h})\frac{k}{2}\zeta(0, m, m)$$

and

$$\|\mathcal{Q}_2(\mathfrak{g})(m) - \mathcal{Q}_2(\mathfrak{h})(m)\| \leq 2^2 d_2(\mathfrak{g}, \mathfrak{h})\frac{k}{2^2}\zeta(m, m, -m).$$

One obtains

$$d_t\big(\mathcal{Q}_t(\mathfrak{g}), \mathcal{Q}_t(\mathfrak{h})\big) \leq k\, d_t(\mathfrak{g}, \mathfrak{h}),$$

so $\mathcal{Q}_t$ is a contraction.

From the preceding estimates it follows that $d_t\big(\mathfrak{J}_t, \mathcal{Q}_t(\mathfrak{J}_t)\big) \leq \frac{1}{2}$.

Under the assumptions of Theorem 2.4, we deduce the existence of a mapping $\mathfrak{h}_t : \mathfrak{A} \to \mathfrak{A}$ possessing the following properties:

1. $\mathfrak{h}_t$ is a FP of $\mathcal{Q}_t$ where $t \in \{1, 2\}$, i.e.

$$\mathfrak{h}_t(m) = 2\mathfrak{h}_t\Big(\frac{m}{2}\Big), \tag{9}$$

for all $m \in \mathfrak{A}$ and $t \in \{1, 2\}$. Moreover, $\mathfrak{h}_t$ is the unique FP of $\mathcal{Q}_t$ in the set

$$\Omega = \{\mathfrak{g} \in \mathfrak{A} : \ d_t(\mathcal{Q}_t(\mathfrak{J}_t), \mathfrak{g}) < \infty, \ \text{where } t \in \{1, 2\}\}.$$

Consequently, $\mathfrak{F}$ is the unique mapping satisfying (9) for which there exists a constant $\gamma > 0$ such that

$$\|\mathfrak{J}_1(m) - \mathfrak{h}_1(m)\| \leq \gamma\zeta(0, m, m)$$

and

$$\|\mathfrak{J}_2(m) - \mathfrak{h}_2(m)\| \leq \gamma\zeta(m, m, -m),$$

for each $t \in \{1, 2\}$ and $m \in \mathcal{A}$. 2. The successive approximations converge: $d(\mathcal{Q}_t^n\mathfrak{J}_t, \mathfrak{h}_t) \to 0$ as $n \to \infty$. This yields the representation

$$\lim_{n\to\infty} 2^{tn}\mathfrak{J}_t\Big(\frac{m}{2^n}\Big) = \mathfrak{h}_t(m) \tag{10}$$

3. The distance estimate $d_t(\mathfrak{J}_t, \mathfrak{J}_t) \leq \frac{k}{2^t(1-k)}$ where $t \in \{1, 2\}$. This implies that

$$\|\mathfrak{J}_1(m) - \mathfrak{h}_1(m)\| \leq \frac{k}{2(1-k)}\zeta(0, m, m)$$

and

$$\|\mathfrak{J}_2(m) - \mathfrak{h}_2(m)\| \leq \frac{k}{2^2(1-k)}\zeta(m, m, -m),$$

for each $t \in \{1, 2\}$ and $m \in \mathcal{A}$.

It follows from (7), (8) and (10), we have

$$\begin{aligned}
&\|t\mathfrak{h}_t(m_1+m_2+m_3)+t\,\mathfrak{h}_t(m_1)-\mathfrak{h}_t(m_1+m_2)-\mathfrak{h}_t(m_1+m_3)\\
&-(t-1)\,\mathfrak{h}_t(m_2+m_3)\|\\
&=\lim_{n\to\infty}2^{tn}\Big\|t\mathfrak{J}_t\left(\frac{m_1+m_2+m_3}{2^n}\right)+t\,\mathfrak{J}_t\left(\frac{m_1}{2^n}\right)-\mathfrak{J}_t\left(\frac{m_1+m_2}{2^n}\right)\\
&-\mathfrak{J}_t\left(\frac{m_1+m_3}{2^n}\right)-(t-1)\,\mathfrak{J}_t\left(\frac{m_2+m_3}{2^n}\right)\Big\|\\
&\leq\lim_{n\to\infty}2^{tn}\zeta\left(\frac{m_1}{2^n},\frac{m_2}{2^n},\frac{m_3}{2^n}\right)\\
&\leq\lim_{n\to\infty}k^n\zeta(m_1,m_2,m_3)\\
&=0.
\end{aligned}$$

So

$$\begin{aligned}
t\mathfrak{h}_t(m_1+m_2+m_3)+t\,\mathfrak{h}_t(m_1)=&\mathfrak{h}_t(m_1+m_2)+\mathfrak{h}_t(m_1+m_3)\\
&+(t-1)\,\mathfrak{h}_t(m_2+m_3),
\end{aligned}$$

for all $m_1,m_2,m_3\in\mathcal{A}$. Therefore, invoking Proposition 3.1 shows that for each $t\in\{1,2\}$ the function $\mathfrak{h}_t$ is a generalized AQ-$\mathbb{FE}$. □

In the corollary that follows, we study the stability of the generalized AQ-$\mathbb{FE}$ under a Rassias-type control function. Taking $k=2^{t-r}$ with $r\in\mathbb{R}$ and setting

$$\zeta(m_1,m_2,m_3)=\upsilon\big(\|m_1\|^r+\|m_2\|^r+\|m_3\|^r\big),$$

where $m_1,m_2,m_3\in\mathcal{A}$ and $\upsilon\geq 0$, Theorem 3.2 yields the following corollary.

Corollary 3.3. *Let $t\in\{1,2\}$ and pick $r\in\mathbb{R}_+$ with $r>t$. Suppose the mapping $\mathfrak{J}_t:\mathcal{A}\to\mathcal{A}$ satisfies, for all $m_1,m_2,m_3\in\mathcal{A}$,*

$$\begin{aligned}
&\big\|t\,\mathfrak{J}_t(m_1+m_2+m_3)+t\mathfrak{J}_t(m_1)-\mathfrak{J}_t(m_1+m_2)-\mathfrak{J}_t(m_1+m_3)\\
&-(t-1)\mathfrak{J}_t(m_2+m_3)\big\|\leq\upsilon\big(\|m_1\|^r+\|m_2\|^r+\|m_3\|^r\big).
\end{aligned}\tag{11}$$

Then there exists a AQ-$\mathbb{FE}$ $\mathfrak{h}_t:\mathcal{A}\to\mathcal{A}$ for which the following estimates hold:

For $t = 1$,

$$\|\mathfrak{J}_1(m) - \mathfrak{h}_1(m)\| \leq \frac{2\upsilon}{2^r - 2}\|m\|^r.$$

For $t = 2$,

$$\|\mathfrak{J}_2(m) - \mathfrak{h}_2(m)\| \leq \frac{2^2\upsilon}{2^r - 2^2}\|m\|^r.$$

In the remark below, we examine the stability of the generalized AQ-$\mathbb{FE}$ when controlled by a JMRassias-type function.

Remark 3.4. Suppose the mapping $\mathfrak{J}_t : \mathcal{A} \to \mathcal{A}$ satisfies

$$\begin{aligned}\big\|t\,\mathfrak{J}_t(m_1 + m_2 + m_3) + t\mathfrak{J}_t(m_1) - \mathfrak{J}_t(m_1 + m_2) - \mathfrak{J}_t(m_1 + m_3) \\ - (t-1)\mathfrak{J}_t(m_2 + m_3)\big\| \leq \upsilon\big(\|m_1\|^r \cdot \|m_2\|^r \cdot \|m_3\|^r\big).\end{aligned} \tag{12}$$

for all $m_1, m_2, m_3 \in \mathcal{A}$. Then $\mathfrak{J}_t : \mathcal{A} \to \mathcal{A}$ is a AQ-$\mathbb{FE}$.

To establish the stability results, we introduce a Găvruta-type control function specifically tailored for the ternary AQ-hom-derivation setting.

Let $t \in \{1, 2\}$. We consider a mapping $\kappa : \mathcal{A}^3 \to [0, \infty)$ that satisfies the following contraction-type condition for all $m_1, m_2, m_3 \in \mathcal{A}$:

$$\kappa\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}\right) \leq \frac{k}{2^{3t}}\kappa(m_1, m_2, m_3), \tag{13}$$

where $0 < k < 1$ is a fixed constant. It is immediate from (13) that $\kappa(0, 0, 0) = 0$. By applying mathematical induction, this inequality can be generalized for any $n \in \mathbb{N}$ as:

$$\kappa\left(\frac{m_1}{2^n}, \frac{m_2}{2^n}, \frac{m_3}{2^n}\right) \leq \frac{k^n}{2^{3nt}}\kappa(m_1, m_2, m_3).$$

Multiplying both sides of (13) by 2^{3nt}, we obtain:

$$2^{3nt}\kappa\left(\frac{m_1}{2^n}, \frac{m_2}{2^n}, \frac{m_3}{2^n}\right) \leq k^n\kappa(m_1, m_2, m_3).$$

Since $k \in (0, 1)$, the term k^n vanishes as n tends to infinity. Consequently, we have:

$$\lim_{n\to\infty} 2^{3nt}\kappa\left(\frac{m_1}{2^n}, \frac{m_2}{2^n}, \frac{m_3}{2^n}\right) = 0. \tag{14}$$

We now establish the HU stability of ternary AQ-hom-ders on TBAs for the generalized AQ-$\mathbb{FE}$ using the FP approach.

Theorem 3.5. *Assume $\zeta, \kappa : \mathcal{A}^3 \to [0, \infty)$ are two functions which satisfying conditions (6) and (13) for some constant $k \in (0, 1)$. Suppose that $\mathfrak{g}_t$ from $\mathcal{A}$ into $\mathcal{A}$ satisfies (7) and fulfill the following relation*

$$\begin{aligned}\Big\| [\mathfrak{g}_t(m_1), \mathfrak{g}_t(m_2), \mathfrak{g}_t(m_3)] - \big[\mathfrak{g}_t(m_1), m_2^t, m_3^t\big] \\ - \big[m_1^t, \mathfrak{g}_t(m_2), m_3^t\big] \\ - \big[m_1^t, m_2^t, \mathfrak{g}_t(m_3)\big] \Big\| \leq \kappa(m_1, m_2, m_3)\end{aligned} \tag{15}$$

for all $m_1, m_2, m_3 \in \mathcal{A}$. Then there exists unique ternary AQ-hom-ders $\mathcal{D}_t : \mathcal{A} \to \mathcal{A}$ such that
For $t = 1$,

$$\|\mathfrak{g}_1(m) - \mathcal{D}_1(m)\| \leq \frac{k}{2(1-k)}\, \zeta(0, m, m). \tag{16}$$

For $t = 2$,

$$\|\mathfrak{g}_2(m) - \mathcal{D}_2(m)\| \leq \frac{k}{2^2(1-k)}\, \zeta(m, m, -m) \tag{17}$$

for all $m \in \mathcal{A}$.

Proof. Let $\mathfrak{A}, d_t$ and $\mathcal{Q}_t$, $t \in \{1, 2\}$, be those as defined in the proof of Theorem 3.2. Similar to the proof of Theorem 3.2, there exist unique t-mappings $\mathcal{D}_t$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$\mathcal{D}_t(m) = \lim_{n\to\infty} 2^{tn} \mathfrak{g}_t\left(\frac{m}{2^n}\right) \tag{18}$$

and satisfying (16) and (17) as desired. It follows from (14), (15) and (18) imply that $\mathcal{D}_t$ is a ternary AQ-hom-ders. In fact,

$$\begin{aligned}\Big\| [\mathcal{D}_t(m_1), \mathcal{D}_t(m_2), \mathcal{D}_t(m_3)] - \big[\mathcal{D}_t(m_1), m_2^t, m_3^t\big] - \big[m_1^t, \mathcal{D}_t(m_2), m_3^t\big] \\ - \big[m_1^t, m_2^t, \mathcal{D}_t(m_3)\big] \Big\| \\ = \lim_{n\to\infty} 2^{3nt} \Big\| \left[\mathfrak{g}_t\left(\frac{m_1}{2^n}\right), \mathfrak{g}_t\left(\frac{m_2}{2^n}\right), \mathfrak{g}_t\left(\frac{m_3}{2^n}\right)\right] \\ - \left[\mathfrak{g}_t\left(\frac{m_1}{2^n}\right), \left(\frac{m_2}{2^n}\right)^t, \left(\frac{m_3}{2^n}\right)^t\right]\end{aligned}$$

$$
\begin{aligned}
&- \left[\left(\frac{m_1}{2^n}\right)^t, \mathfrak{g}_t\left(\frac{m_2}{2^n}\right), \left(\frac{m_3}{2^n}\right)^t\right] \\
&- \left[\left(\frac{m_1}{2^n}\right)^t, \left(\frac{m_2}{2^n}\right)^t, \mathfrak{g}_t\left(\frac{m_3}{2^n}\right)\right]\Big\| \\
&\leq \lim_{n\to\infty} 2^{3nt}\kappa\left(\frac{m_1}{2^n}, \frac{m_2}{2^n}, \frac{m_3}{2^n}\right) \\
&\leq \lim_{n\to\infty} k^n \kappa(m_1, m_2, m_3) = 0.
\end{aligned}
$$

So

$$
\begin{aligned}
[\mathcal{D}_t(m_1), \mathcal{D}_t(m_2), \mathcal{D}_t(m_3)] = \left[\mathcal{D}_t(m_1), m_2^t, m_3^t\right] + \left[m_1^t, \mathcal{D}_t(m_2), m_3^t\right] \\
+ \left[m_1^t, m_2^t, \mathcal{D}_t(m_3)\right],
\end{aligned}
$$

for all $m_1, m_2, m_3 \in \mathcal{A}$. Now, the proof is complete. □

Below, we analyze the stability of the ternary AQ-hom-ders on TBAs using a Rassias-type control function.

Corollary 3.6. *Let $t \in \{1, 2\}$ and choose $\upsilon, r \in \mathbb{R}_+$ with $r > t$. Suppose the mapping $\mathfrak{g}_t : \mathcal{A} \to \mathcal{A}$ satisfies condition (19) and, for all $m_1, m_2, m_3 \in \mathcal{A}$,*

$$
\begin{aligned}
\Big\| [\mathfrak{g}_t(m_1), \mathfrak{g}_t(m_2), \mathfrak{g}_t(m_3)] - \left[\mathfrak{g}_t(m_1), m_2^t, m_3^t\right] \\
- \left[m_1^t, \mathfrak{g}_t(m_2), m_3^t\right] \\
- \left[m_1^t, m_2^t, \mathfrak{g}_t(m_3)\right] \Big\| \\
\leq \upsilon\left(\|m_1\|^r + \|m_2\|^r + \|m_3\|^r\right),
\end{aligned} \tag{19}
$$

for all $m_1, m_2, m_3 \in \mathcal{A}$. Then $\mathfrak{g}_t : \mathcal{A} \to \mathcal{A}$ is a ternary AQ-hom-ders.

In the following Remark, we investigate stability of ternary AQ-hom-ders with JMRassias-type control function on TBA.

Remark 3.7. Let $t \in \{1, 2\}$ and $r > t$ and be elements of $\mathbb{R}_+$. If the mapping $\mathfrak{J}_t : \mathcal{A} \to \mathcal{A}$ satisfies

$$
\begin{aligned}
\Big\| [\mathfrak{g}_t(m_1), \mathfrak{g}_t(m_2), \mathfrak{g}_t(m_3)] - \left[\mathfrak{g}_t(m_1), m_2^t, m_3^t\right] \\
- \left[m_1^t, \mathfrak{g}_t(m_2), m_3^t\right] \\
- \left[m_1^t, m_2^t, \mathfrak{g}_t(m_3)\right] \Big\| \\
\leq \upsilon\left(\|m_1\|^r \cdot \|m_2\|^r \cdot \|m_3\|^r\right),
\end{aligned} \tag{20}
$$

for all $m_1, m_2, m_3 \in \mathcal{A}$. Then $\mathfrak{g}_t : \mathcal{A} \to \mathcal{A}$ that is ternary AQ-hom-ders.

4 Conclusions

In this paper, we developed a comprehensive and rigorous framework for ternary homomorphisms and ternary derivations on TBA. Our analysis, specifically focusing on the cases $t = 1$ and $t = 2$, demonstrated that additive and quadratic mappings could be a cohesive structural paradigm. We explicitly derived the solutions for the generalized AQ-$\mathbb{FE}$ and rigorously verified their HU stability. Furthermore, by employing the FP approach, we extended the investigation of stability to ternary AQ-hom-ders on TBAs, thereby providing a systematic and reliable approach that enabled a significant improvement over previous preliminary explorations.

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