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Dini Lipschitz Functions for the Generalized Fourier-Bessel Transform in the Space $L^2_{\alpha,n}$

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Abstract. In this paper, using a generalized translation operator, we obtain an analog of Younis [8, Theorem 2.5] in for the generalized Fourier-Bessel transform for functions satisfying the Fourier-Bessel Dini Lipschitz condition in the space $L^2_{\alpha,n}$.

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1. Introduction

Integral transforms and their inverses, the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [7] and [6]).

Younis [8, Theorem 2.5] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

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Theorem 1.1. ([8]) *Let $f \in L^2(\mathbb{R})$. Then the following are equivalents:*

(a) $\|f(x+h) - f(x)\| = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right)$, *as* $h \to 0, 0 < \eta < 1, \gamma \geqslant 0$

(b) $\displaystyle\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, *as* $r \to \infty$,

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half line which generalizes the Bessel operator $\mathcal{B}_\alpha$. We obtain an analog of Theorem 1.1 in the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^2_{\alpha,n}$. For this purpose, we use a generalized translation operator.

We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see [2] and [3]):

Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, ...$. For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 \leqslant p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0, \infty[, x^{2\alpha+1})$.

For $\alpha \geqslant \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}, \tag{1}$$

where $\Gamma(x)$ is the gamma-function (see [4]). The function $y = j_\alpha(z)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0,$$

with the initial condition $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(z)$ is infinitely differentiable and indeed is an entire analytic.

From (1) we see that

$$\lim_{z\to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0, \tag{2}$$

hence , there exist $c > 0$ and $\nu > 0$ satisfying

$$|z| \leqslant \nu \Rightarrow |j_\alpha(z) - 1| \geqslant c|z|^2. \tag{3}$$

From [2], we have

$$|j_\alpha(x)| \leqslant 1, \tag{4}$$

$$1 - j_\alpha(x) = O(x^2), \quad 0 \leqslant x \leqslant 1. \tag{5}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x). \tag{6}$$

From [1] and [6] recall the following properties.

Proposition 1.2.
(**c**) φ_λ *satisfies the differential equation*

$$\mathcal{B}\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

(d) *For all* $\lambda \in \mathbb{C}$, *and* $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leqslant x^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform which we call it the integral transform, is defined by

$$\mathcal{F}_{\mathcal{B}} f(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \lambda \geqslant 0, f \in L^1_{\alpha,n}$$

(see [1]).

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_{\mathcal{B}}(f) \in L^1_{\alpha+2n} = L^1([0,\infty[, x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^\infty \mathcal{F}_{\mathcal{B}} f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}$$

(see [1]). From [1] and [6], we have the following proposition.

Proposition 1.3.
(**e**) *For every* $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ *we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(**f**) *The generalized Fourier-Bessel transform* $\mathcal{F}_{\mathcal{B}}$ *extends uniquely to an isometric isomorphism from* $L^2_{\alpha,n}$ *onto* $L^2([0,+\infty[,\mu_{\alpha+2n})$.

Define the generalized translation operator T^h, $h \geqslant 0$, by the relation

$$T^h f(x) = (xh)^{2n}\tau^h_{\alpha+2n}(M^{-1}f)(x), x \geqslant 0,$$

where τ^h_α is the Bessel translation operator of order α defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos t}) \sin^{2\alpha} t dt,$$

where
$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} tdt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f \in L^2_{\alpha,n}$, we have

$$\mathcal{F}_\mathcal{B}(T^h f)(\lambda) = \varphi_\lambda(h)\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{7}$$
$$\mathcal{F}_\mathcal{B}(\mathcal{B}f)(\lambda) = -\lambda^2\mathcal{F}_\mathcal{B}(f)(\lambda). \tag{8}$$

(see [2] for details). Denote by $W_2^m(\mathcal{B}), m = 0, 1, 2...$, the class of functions $f \in L^2_{\alpha,n}$ that have on $\mathbb{R}^+$ generalized derivatives $f'(x), f''(x), ..., f^{(2m)}(x)$ in the sense of Levi (see [5]) and belong to $L^2_{\alpha,n}$ with $\mathcal{B}^m f \in L^2_{\alpha,n}$, i.e.,

$$W_2^m(\mathcal{B}) = \left\{f \in L^2_{\alpha,n}/\mathcal{B}^m f \in L^2_{\alpha,n}\right\},$$

where $\mathcal{B}^0 f = f$, $\mathcal{B}^m = \mathcal{B}(\mathcal{B}^{m-1}f), m = 0, 1, 2....$

2. Dini Lipschitz Condition

In the rest of these papers, we give the main results. For this objective, we first need to define the Fourier-Bessel Dini Lipschitz class.

Definition 2.1. *Let $f \in W_2^m(\mathcal{B})$, and define*

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \leqslant C\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}, \quad \gamma \geqslant 0, m = 0, 1, 2...;$$

i.e.,

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}\right),$$

for all x in $\mathbb{R}^+$ and for all sufficiently small h, C being a positive constant and I is the unit operator in $L^2_{\alpha,n}$. Then we say that f satisfies a Fourier-Bessel Dini Lipschitz of order η, or f belongs to $Lip(\eta,\gamma)$.

Definition 2.2. *If*

$$\frac{\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n}}{\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}} \to 0, \quad as \quad h \to 0, \gamma \geqslant 0, m = 0, 1, 2...;$$

i.e.,

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}\right),$$

then f is said to be belong to the little Fourier-Bessel Dini Lipschitz class $lip(\eta,\gamma)$.

Remark 2.3. *It follows immediately from these definitions that*

$$lip(\eta,\gamma) \subset Lip(\eta,\gamma).$$

Theorem 2.4. *Let $\eta > 1$. If $f \in Lip(\eta,\gamma)$, then $f \in lip(1,\gamma)$.*

Proof. For $x \in \mathbb{R}^+$, h small enough and $f \in Lip(\eta,\gamma)$ we have

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \leqslant C\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}.$$

Then

$$(\log\frac{1}{h})^\gamma \|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \leqslant Ch^{\eta+2n}.$$

Therefore

$$\frac{(\log\frac{1}{h})^\gamma}{h^{1+2n}}\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \leqslant Ch^{\eta-1},$$

which tends to zero with $h \to 0$. Thus

$$\frac{(\log\frac{1}{h})^\gamma}{h^{1+2n}}\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \to 0, \quad h \to 0.$$

Then $f \in lip(1,\gamma)$. $\square$

Theorem 2.5. *If $\eta < \nu$, then $Lip(\eta,0) \supset Lip(\nu,0)$ and $lip(\eta,0) \supset lip(\nu,0)$.*

Proof. We have $0 \leqslant h \leqslant 1$ and $\eta < \nu$, then $h^\nu \leqslant h^\eta$.
So the proof of theorem is complete. $\square$

3. New Results on Dini Lipschitz Class

Lemma 3.1. *For $f \in W_2^m(\mathcal{B})$, we have*

$$\left(h^{4n}\int_0^\infty \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2}} = \|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n},$$

where $m = 0, 1, 2...$

Proof. From formula (8), we obtain

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}^m f)(\lambda) = (-1)^m \lambda^{2m}\mathcal{F}_{\mathcal{B}}f(\lambda); m = 0, 1, ... \tag{9}$$

By using the formulas (6), (7) and (9), we conclude that

$$\mathcal{F}_{\mathcal{B}}(T^h\mathcal{B}^m f)(\lambda) = (-1)^m h^{2n} j_{\alpha+2n}(\lambda h)\lambda^{2m}\mathcal{F}_{\mathcal{B}}f(\lambda).$$

thus

$$\mathcal{F}_{\mathcal{B}}((T^h - h^{2n}I)\mathcal{B}^m f)(\lambda) = (-1)^m h^{2nk}(j_{\alpha+2n}(\lambda h) - 1)\lambda^{2m}\mathcal{F}_{\mathcal{B}}f(\lambda).$$

Now by proposition 1.3, we have the result. □

Theorem 3.2. *Let $\eta > 2$. If f belongs to the Fourier-Bessel Dini Lipschitz class, i.e.,*

$$f \in Lip(\eta, \gamma), \quad \eta > 2, \gamma \geqslant 0,$$

then f is equal to the null function in $\mathbb{R}^+$.

Proof. Assume that $f \in Lip(\eta, \gamma)$. Then we have

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} \leqslant C\frac{h^{\eta+2n}}{(\log\frac{1}{h})^\gamma}, \quad \gamma \geqslant 0.$$

From Lemma 3.1, we get

$$h^{4n}\int_0^\infty \lambda^{4m}|j_{\alpha+2n}(\lambda h) - 1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leqslant C^2\frac{h^{2\eta+4n}}{(\log\frac{1}{h})^{2\gamma}}.$$

Therefore

$$\int_0^{\infty} \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leqslant C^2\frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}.$$

Then

$$\frac{\int_0^{\infty} \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)}{h^4} \leqslant C^2\frac{h^{2\eta-4}}{(\log\frac{1}{h})^{2\gamma}}.$$

Since $\eta > 2$ we have

$$\lim_{h\to 0}\frac{h^{2\eta-4}}{(\log\frac{1}{h})^{2\gamma}} = 0.$$

Thus

$$\lim_{h\to 0}\int_0^{\infty}\left(\frac{|1-j_{\alpha}(\lambda h)|}{\lambda^2 h^2}\right)^2 \lambda^{4+4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = 0,$$

and also from the formula (2) and Fatou theorem, we obtain

$$\int_0^{\infty} \lambda^{4+4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = 0.$$

Hence $\lambda^{2+2m}\mathcal{F}_{\mathcal{B}}f(\lambda) = 0$ for all $\lambda \in \mathbb{R}^+$ and so $f(x)$ is the null function. □

Analogous to the Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *Let $f \in W_2^m(\mathcal{B})$. If f belong to $lip(2,0)$, i.e.,*

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} = O(h^{2+2n}), \quad \text{as} \quad h \to 0,$$

then f is equal to null function in $\mathbb{R}^+$.

Now, we give another main result of this paper analogous to the Theorem 1.1.

Theorem 3.4. *Let $f \in W_2^m(\mathcal{B})$. Then the following conditions are equivalent*

(i) $f \in Lip(\eta,\gamma)$, $0 < \eta < 1, \gamma \geqslant 0,$

(ii) $\int_r^{\infty} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$

Proof. $(i) \Rightarrow (ii)$. Suppose that $f \in Lip(\eta,\gamma)$. Then

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log \frac{1}{h})^{\gamma}}\right), \quad h \to 0.$$

From Lemma 3.1, we have

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|^2_{2,\alpha,n} = h^{4n}\int_0^{\infty} \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

By formula (3) , we get

$$\int_{\nu/2h}^{\nu/h} \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \geqslant \frac{c^2\nu^4}{2^4}\int_{\nu/2h}^{\nu/h} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Then there exists a positive constant C such that

$$\begin{aligned}\int_{\nu/2h}^{\nu/h} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &\leqslant C\int_{\nu/2h}^{\nu/h} \lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\leqslant \frac{C}{h^{4n}}\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|^2_{2,\alpha,n}\\ &= O\left(\frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}}\right).\end{aligned}$$

So we obtain

$$\int_r^{2r} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leqslant C'\frac{r^{-2\eta}}{(\log r)^{2\gamma}},$$

where C' is a positive constant. Now, we have

$$\begin{aligned}\int_r^{\infty} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r}^{2^{i+1}r} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\leqslant C'\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}} + \frac{(2r)^{-2\eta}}{(\log 2r)^{2\gamma}} + \frac{(4r)^{-2\eta}}{(\log 4r)^{2\gamma}} + \cdots\right)\\ &\leqslant C'\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\left(1 + 2^{-2\eta} + (2^{-2\eta})^2 + (2^{-2\eta})^3 + \cdots\right)\\ &\leqslant K_\eta\frac{r^{-2\eta}}{(\log r)^{2\gamma}},\end{aligned}$$

where $K_\eta = C'(1-2^{-2\eta})^{-1}$ since $2^{-2\eta} < 1$.
Consequently,

$$\int_r^\infty \lambda^{4m}|\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$$

$(ii) \Rightarrow (i)$. Suppose that

$$\int_r^\infty \lambda^{4m}|\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty,$$

and write

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|^2_{2,\alpha,n} = h^{4n}(I_1 + I_2),$$

where

$$I_1 = \int_0^{1/h} \lambda^{4m}|j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

and

$$I_2 = \int_{1/h}^\infty \lambda^{4m}|j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Using the inequality (4), we get

$$I_2 \leqslant 4\int_{1/h}^\infty \lambda^{4m}|\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}}\right), \quad as \quad h \to 0.$$

Set

$$\phi(\lambda) = \int_\lambda^\infty x^{2m}|\mathcal{F}_\mathcal{B} f(x)|^2 d\mu_{\alpha+2n}(x).$$

From formula (5) and integration by parts, we have

$$\begin{aligned}
I_1 &= -\int_0^{1/h} |j_{\alpha+2n}(\lambda h)-1|^2 |\phi'(\lambda) d\lambda \\
&\leqslant -C_1 h^2 \int_0^{1/h} \lambda^2 \phi'(\lambda) d\lambda \\
&\leqslant -C_1 \phi(\frac{1}{h}) + 2C_1 h^2 \int_0^{1/h} \lambda\phi(\lambda) d\lambda \\
&\leqslant C_2 h^2 \int_0^{1/h} \lambda^{1-2\eta} (\log\lambda)^{-2\gamma} d\lambda \\
&\leqslant C_2 \frac{h^{2\eta}}{(\log\frac{1}{h})^{2\gamma}}
\end{aligned}$$

where C_1 and C_2 are positive constants, and this completes the proof. □

Corollary 3.5. *Let* $f \in W_2^m(\mathcal{B})$*. If*

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{2,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log\frac{1}{h})^{\gamma}}\right), \quad as \quad h \to 0,$$

then

$$\int_r^{\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-4m-2\eta}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$$

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