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t-Best Approximation in Fuzzy Quotient Space

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Abstract. The main aim of this paper is to define and investigate the fuzzy quotient spaces and t-best approximation in fuzzy quotient space and prove some theorems on quotient spaces. Finally, we present an application.

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1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh [8] in 1965. Since then, many mathematicians have studied fuzzy normed spaces from several angles ([2], [5], [6]) and, in [7], P. Veeramani introduced the concept of t-best approximations in fuzzy metric spaces. In this paper we consider the set of t-best approximations on fuzzy quotient spaces and prove several theorems pertaining to this set.

Definition 1.1. [7] *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if* $*$ *satisfies:*

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(1) $*$ *is commutative and associative;*
(2) $*$ *is continuous;*
(3) $a * 1 = a$ *for all* $a \in [0, 1]$*;*
(4) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$*, and* $a; b; c; d \in [0, 1]$.

Definition 1.2. [7] *The 3-tuple* $(X, N, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* N *is a fuzzy set on* $X \times X \times (0, \infty)$ *satisfying the following conditions for all* $x, y \in X$ *and* $t, s > 0$
(1) $N(x, y, t) > 0$,
(2) $N(x, y, t) = 1$ *if and only if* $x = y$,
(3) $N(x, y, t) = N(y, x, t)$,
(4) $N(x, y, s + t) \geqslant N(x, z, s) * N(z, y, t)$,
(5) $N(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous.*

Definition 1.3. [7] *The 3-tuple* $(X, N, *)$ *is said to be a fuzzy normed space if* X *is a vector space,* $*$ *is a continuous t-norm and* N *is a fuzzy set on* $X \times (0, \infty)$ *satisfying the following conditions for every* $x, y \in X$ *and* $t, s > 0$,

(1)$N(x, t) > 0$,
(2)$N(x, t) = 1 \Leftrightarrow x = 0$,
(3)$N(\alpha x, t) = N(x, t/|\alpha|)$, *for all* $\alpha \neq 0$,
(4)$N(x, t) * N(y, s) \leqslant N(x + y, t + s)$,
(5)$N(x, .) : (0, \infty) \longrightarrow [0, 1]$ *is continuous* ,
(6) $\lim_{t \longrightarrow 0} N(x, t) = 1$.

Definition 1.4. [7] *Let* $(X, N, *)$ *be a fuzzy normed space. The open ball B(x,r,t) and the closed ball* $B\,[x, r, t]$ *with the center* $x \in X$ *and radius* $0 < r < 1, t > 0$ *are defined as follows:*

$$B\,(x, r, t) = \{y \in X : N(x - y, t) > 1 - r\},$$

$$B\,[x, r, t] = \{y \in X : N(x - y, t) \geqslant 1 - r\}.$$

Definition 1.5. *Let* A *be a nonempty subset of a fuzzy normed space* $(X, N, *)$ *for* $x \in X, t > 0$*, let*

$$d(A, x, t) = \bigvee\nolimits_{y \in X} \{N(y - x, t)\}.$$

An element $y_0 \in A$ is said to be a t-best approximation of x from A if

$$N(y_0 - x, t) = d(A, x, t).$$

We shall denote the set of all elements of t-best approximation of x from A by $P_A^t(x)$, i.e.,

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t-best approximation in A, then A is called a t-proximinal (respectively t-Chebyshev) set.

Definition 1.6. *Let $(X, N, *)$ be a fuzzy normed space. A subset X is called F-bounded, if there exist $t > 0$ and $0 < r < 1$ such that N(x, t)$> 1 - r$ for all $x \in X$.*

Lemma 1.7.

Let $(X, N, *)$ be a fuzzy normed space. Then

(1) N(x, t) is nondecreasing with respect to t for each $x \in X$,

(2) $N(x - y, t) = N(y - x, t)$.

2. Main Results

In this section, we define the fuzzy quotient spaces, and we prove some theorems on these spaces.

Definition 2.1. *Let $(X, N, *)$ be a fuzzy norm space, M be a linear manifold in X and let $Q : X \longrightarrow X/M$ be the natural map, $Qx = x$ $+M$. We define*

$$N(x + M, t) = \bigvee_{y \in X}\{N(x + y, t) : y \in M\}.$$

Definition 2.2. *The 3-tuple $(X/M, N, *)$ is said to be fuzzy normed quotient space if $(X, N, *)$ is a fuzzy norm space and N is fuzzy norm on fuzzy set X/M, that $X/M = \{x + M : x \in X\}$ and M is a linear manifold in X satisfying the following conditions for every $x, y \in X$ and $\lambda > 0$*

(1) $N((x + M) + (y + M), t) = N((x + y) + M, t)$

(2) $N(\lambda(x+M),t) = N(\lambda x + M,t)$.

Definition 2.3. *Let* $(X/M, N, *)$ *be a fuzzy normed quotient space. The open ball B(x+M,r,t) and the closed ball* $B[x+M,r,t]$ *with the center* $x+M \in X/M$ *and radius* $0 < r < 1, t > 0$ *are defined as follows:*

$$B(x+M,r,t) = \{y+M \in X/M : N((x+M)-(y+M),t) > 1-r\},$$

$$B[x+M,r,t] = \{y+M \in X/M : N((x+M)-(y+M),t) \geqslant 1-r\}.$$

Theorem 2.4. [4] *The following assertions hold for* $t > 0$

(1) $d(x+M, x+M, t) = d(M, W, t), \forall x \in C$,

(2) $d(\lambda M, \lambda W, t) = d(M, W, t/\mid\lambda\mid), \forall \lambda \in C$,

(3) $P^t_{W+x}(M+x) = P^t_W(M) + x, \forall x \in X$,

(4) $P^{|\lambda|t}_{\lambda W}(\lambda M) = \lambda P^t_W(M), \forall \lambda \in C$

Definition 2.5. *The fuzzy normed space* $(X, N, *)$ *is said to be a fuzzy Banach space whenever* X *is complete with respect to the fuzzy metric induced by fuzzy norm.*

Theorem 2.6. *If* M *is a closed manifold of fuzzy normed space* X *and N(x+M,t) is defined as above then:*

(a) N *is a fuzzy norm on* X/M.

(b) $N(Qx,t) \geqslant N(x,t)$.

(c) If $(X, N, *)$ *is a fuzzy Banach space, then so is* $(X/M, N, *)$.

Proof. It is clear that $N(x+M,t) \geqslant 0$.
Let $N(x+M,t) = 1$. By definition there is a sequence $\{x_n\}$ in M such that $N(x+x_n,t) \longrightarrow 1$. So $x+x_n \longrightarrow 0$ or equivalently $x_n \longrightarrow (-x)$ and since M is closed, $x \in X$ and $x+M = M$, the zero element of X/M.

$$\begin{aligned} N((x+M)+(y+M),t) &= N((x+y)+M,t) \\ &\geqslant N((x+m)+(y+m),t) \\ &\geqslant N(x+m,t_1) * N(y+m,t_2), \end{aligned}$$

for $m, n \in M, x, y \in X$ and $t_1 + t_2 = t$. Now if we take sup on both sides, we have,

$$N((x+M)+(y+M),t) \geqslant N(x+M,t_1) * N(y+M,t_2).$$

Also we have,

$$\begin{aligned} N(\alpha(x+M),t) &= N(\alpha x+M,t) \\ &= \bigvee_{y\in M}\{N(\alpha x+\alpha y,t)\} \\ &= \bigvee_{y\in M}\{N(x+y,t/|\alpha|)\} \\ &= N(x+M,t/|\alpha|). \end{aligned}$$

Therefore $(X,N,*)$ is a fuzzy normed space.
To prove (b) we have,

$$\begin{aligned} N(Qx,t) &= N(x+M,t) \\ &= \bigvee_{y\in M}\{N(x+y,t)\} \\ &\geqslant N(x,t). \end{aligned}$$

Let $\{x_n+M\}$ be a Cauchy sequence in X/M. Then there exists $\epsilon_n>0$ such that $\epsilon_n \longrightarrow 0$ and,

$$N((x_n+M)-(x_{n+1}+M),t) \geqslant 1-\epsilon_n \ .$$

Let $y_1=0$. We choose $y_2\in M$ such that,

$$N(x_1-(x_2-y_2),t) \geqslant N((x_1-x_2)+M,t) * (1-\epsilon_n).$$

but $N((x_1-x_2)+M,t) \geqslant (1-\epsilon_n)$. Therefore,

$$N(x_1-(x_2-y_2),t) \geqslant (1-\epsilon_1)(1-\epsilon_1).$$

Now suppose y_{n-1} has been chosen,$y_n\in M$ can be chosen such that

$$N((x_{n-1}+y_{n-1})-(x_n+y_n),t) \geqslant N((x_{n-1}-x_n)+M,t) * (1-\epsilon_{n-1}),$$

and therefore,

$$N((x_{n-1}+y_{n-1})-(x_n+y_n),t) \geqslant (1-\epsilon_{n-1}) * (1-\epsilon_{n-1}).$$

Thus, $\{x_n+y_n\}$ is a Cauchy sequence in X. Since X is complete, there is an x_0 in X such that $x_n + y_n \longrightarrow x_0$ in X. On the other hand

$$x_n + M = Q(x_n + y_n) \longrightarrow Q(x_0) = x_0 + M.$$

Therefore every Cauchy sequence $\{x_n + M\}$ is convergent in $X\diagup M$ and so $X\diagup M$ is complete and $(X\diagup M, N, *)$ is a fuzzy Banach space. $\square$

Lemma 2.7. [4] *Let $(X, N, *)$ be a fuzzy normed space, M a t-proximinal subspace of X and S be an arbitrary subset of X. The following assertions are equivalent:*
(1) S is a F-bounded subset of X.
(2) $S\diagup M$ is a F-bounded subset of $X\diagup M$.

Theorem 2.8. *Let M be a closed subspace of a fuzzy normed space X. Let $Q : X \longrightarrow X\diagup M$ if $x \in X$ and $\epsilon \in [0, N(Qx, t))$, then there is an x_0 in X such that, x_0+M=x+M and $N(x_0, t) > N(Qx, t) * \epsilon$.*

Proof. There always exists a $y \in M$ such that,

$$N(x + y, t) > N(x + M, t) * \epsilon = N(Qx, t) * \epsilon.$$

Now it is enough to put $x_0 = x + y$. $\square$

Theorem 2.9. *Let M be a t-Chebyshev subspace of $(X, N, *)$ and $W \supseteq M$ a subspace of X.If $W\diagup M$ is t-Chebyshev with X/M, then W is Chebyshev with X.*

Proof. Suppose $W\diagup M$ is t-Chebyshev. Then some F-bounded subset K of X has distinct t-best approximations such as x_1 and x_2 in W/M. Thus we have,

$$x_1, x_2 \in P^t_W(K).$$

It is clear that,

$$x_1 + M, x_2 + M \in P^t_{W\diagup M}(K\diagup M).$$

Since $W\diagup M$ is t-Chebyshev, $x_1 + M = x_2 + M$ and $x_1 - x_2 \in M$. Now since

$$x_1, x_2 \in P^t_W(M),$$

there exists $w - x_1$ and $w - x_2$ in W and $W \supseteq M$; therefore $w - x_1$ and $w - x_2$ are in M. So there exists

$$0 \in P_W^t(M),$$

and also

$$x_1, x_2 \in P_W^t(w - x_2).$$

Since M is t-Chebyshev, $x_1 = x_2$. $\square$

Corollary 2.10. [4] *Let M be a t-proximinal subspace of $(X, N, *)$ and $W \supseteq M$ a subspace of X. If W is t-proximinal then $W \diagup M$ is a t-proximinal subspace of $X \diagup M$.*

Theorem 2.11. [4] *Let M be a t-proximinal subspace of $(X, N, *)$, $W \supseteq M$ a t-proximinal subspace of X. Then for each F-bounded set K in X,*

$$Q(P_W^t(K)) = P_{W \diagup M}^t(K \diagup M).$$

Theorem 2.12. *Let M and W be subspaces of a fuzzy normed space $(X, N, *)$ such that $M \subset W$ and let $x \in X \diagup W$ and $w_1 \in W$. f w_1 is a t-best approximation to x from W, then $w_1 + M$ is a t-best approximation to $x + M$ from the quotient space W/M.*

Proof. Assume that $w_1 + M$ is not a t-best approximation to $x + W$ from $W \diagup M$. Then there exists a $w_2 + M \in W \diagup M$ such that

$$N(w_2 + M - (w_1 + M), t) < N(x + M - (w_2 + M), t).$$

That is,

$$N(w_2 - w_1 + M, t) < N(x - w_2 + M, t).$$

That is,

$$N(x - w_2, M, t) > N(w_2 - w_1, M, t).$$

This implies that there exists a $m \in M$ such that

$$N(x - w_2 - m, t) > d(w_2 - w_1, M, t) > N(w_2 - w_1 + m, t).$$

That is,

$$N((m+w_2)-w_1,t) < N(x-(m+w_2),t).$$

Thus w_1 is not a t-best approximation to x from W, a contradiction. $\square$

Definition 2.13. *Let $(X/M, N, *)$ be a fuzzy quotient normed space. A subset W/M of X/M is said to be t-convex if $\lambda(x+M)+(1-\lambda)(y+M) \in W/M$ whenever $x+M, y+M \in W/M$ and $0<\lambda<1$.*

Theorem 2.14. *Let M be a t-proximinal subspace of $(X/M, N, *)$ and $W \supseteq M$ a subspace of X/M. Let K/M be F-bounded in X/M. If W/M is t-convex of X/M, then $P^t_{W/M}(K/M)$ is t-convex.*

Proof. Suppose that W/M is t-convex and is a subset of X/M. We show that $P^t_{W/M}(K/M)$ is t-convex. Since W/M is t-convex, there exists $\lambda(x+M)+(1-\lambda)(y+M) \in W/M$, for all $x+M, y+M \in W/M$ and $0<\lambda<1$. Now for $t>0$ we have,

$$\bigwedge_{k+M\in\frac{K}{M}} N(\lambda(x+M)+(1-\lambda)(y+M)-(k+M),t) \leqslant d(K/M, W/M, t).$$

On the other hand, for a given $t>0$, take the natural number n such that $t>\frac{1}{n}$. We have,

$$\begin{aligned}
\bigwedge_{k+M\in\frac{K}{M}} N(\lambda(x+M) &+ (1-\lambda)(y+M)-(k+M),t) \\
&= \bigwedge_{k+M\in\frac{K}{M}} N(\lambda(x-y)+y+M-(k+M),t) \\
&= \bigwedge_{k+M\in\frac{K}{M}} N((x-y)+M,\frac{1}{\lambda n}) * N(y+M-(k+M),t-\frac{1}{n}) \\
&= N((x-y)+M,\frac{1}{\lambda n}) * \bigwedge_{k+M\in\frac{K}{M}} N(y+M-(k+M),t-\frac{1}{n}) \\
&\geqslant lim_{n\longrightarrow\infty}(\bigwedge_{k+M\in\frac{K}{M}} N(y+M-(k+M),t-\frac{1}{n})) \\
&= d(K/M, W/M, t).
\end{aligned}$$

So $P^t_{W/M}(K/M)$ is convex. $\square$

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