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Original Research Paper

The Cozero Part of the Pointfree Version of $C_c(X)$

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Abstract. We study the cozero part of the ring $\mathcal{C}_c(L)$ in the pointfree setting, that is, the pointfree counterpart of

$$Z_c(X) = \{Z(f) : f \in C_c(X)\}.$$

We introduce the structure $\mathrm{Coz}_c[L]$ and investigate its algebraic and lattice-theoretic properties by way of the idempotent elements of $\mathcal{R}(L)$. A key observation is that for every bounded $\alpha \in \mathcal{R}^*(L)$ and every prime ideal P of L, the pair $(P_u(\alpha), P_l(\alpha))$ produces a Dedekind cut without any additional assumption on P; this fact underlies most of the results that follow. Building on it, we show that $\mathrm{Coz}_c[L]$ forms a σ-frame for every completely regular frame L and that it is consequently regular and an Alexandroff algebra frame, with every cover of $\mathrm{Coz}_c[L]$ shrinkable. We also characterize zero-dimensional frames as precisely the c-completely regular ones, i.e., those for which $\mathrm{Coz}_c[L]$ is a base. These results extend classical theorems on $C_c(X)$ to the locale-theoretic framework.

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1 Introduction

1.1 Motivation and Background

The ring $\mathcal{R}(L)$ of real-valued continuous functions on a frame, which serves as the pointfree analogue of the ring $C(X)$, was studied prior to 1996 by several authors, including Ball and Hager [2]. A systematic investigation of $\mathcal{R}(L)$ in pointfree topology was carried out by Banaschewski in 1996 [5]. For further background on $\mathcal{R}(L)$, we refer to [3, 4, 8, 19, 21].

The largest subring of $C(X)$ consisting of functions with countable range is $C_c(X)$. This motivated the study of its pointfree analogue $\mathcal{R}_c(L)$ by Estaji et al. [14, 20, 22, 12]. In this context, for each $\alpha \in \mathcal{R}(L)$, the set

$$R_\alpha := \{r \in \mathbb{R} : \operatorname{coz}(\alpha - r) \neq \top\}$$

was introduced. Moreover, Aliabadi and Mahmoudi [1] defined

$$\operatorname{pim}(\alpha) := \bigcap\{\omega \in \mathcal{O}(\mathbb{R}) : \alpha(\omega) = \top\},$$

and proved that $\operatorname{pim}(\alpha) = R_\alpha$.

Using the notion of R_α, a further pointfree version of $C_c(X)$, denoted by $\mathcal{C}_c(L)$, was introduced in [23] via the range of functions, providing a framework more compatible with countable-range functions. Several algebraic properties of $\mathcal{C}_c(L)$ have been studied, including the structure of radical ideals and clean elements (see [24, 16]).

It is worth clarifying why the present paper builds the cozero part on $\mathcal{C}_c(L)$ rather than on $\mathcal{R}_c(L)$. Following [20], $\mathcal{R}_c(L)$ consists of those $\alpha \in \mathcal{R}(L)$ for which a countable $S \subseteq \mathbb{R}$ can be found with $\alpha \lhd S$, the pointfree rendering of the classical condition $\operatorname{Im}(\alpha) \subseteq S$. By contrast, $\mathcal{C}_c(L)$ is defined directly through R_α, which is the pointfree translation of the actual image $\operatorname{Im}(\alpha) = \{r \in \mathbb{R} : Z(\alpha - r) \neq \emptyset\}$, rather than of an enclosing countable set. As shown in [20, Example 3.18], the relation $\lhd$ need not pin down the exact image once L has no points, whereas R_α always does; moreover $\mathcal{R}_c(L) \subseteq \mathcal{C}_c(L)$ in general, and the inclusion can be strict [23, Proposition 3.3 and Remark 3.4]. For these reasons $\mathcal{C}_c(L)$, rather than $\mathcal{R}_c(L)$, is taken here as the pointfree counterpart of $C_c(X)$, and $\operatorname{Coz}_c[L]$ is built on top of it.

The cozero map plays a fundamental role in pointfree topology, providing a point-independent approach to arguments that are traditionally point-based. The collection of all cozero elements of a frame L, denoted by $\mathrm{Coz}[L]$, has been extensively studied and is known to form a σ-frame. Cozero elements are central in the study of $\mathcal{R}(L)$, $\mathcal{R}_c(L)$, and $\mathcal{C}_c(L)$.

Despite this, the cozero structure associated with $\mathcal{C}_c(L)$ has not yet been systematically investigated. This paper aims to fill this gap by introducing and studying the cozero part $\mathrm{Coz}_c[L]$ and exploring its algebraic and lattice-theoretic properties.

1.2 Main Results

The main contributions of this paper are as follows:

- We introduce the cozero part $\mathrm{Coz}_c[L]$ associated with $\mathcal{C}_c(L)$.
- We study the algebraic structure of idempotent elements in $\mathcal{R}(L)$ and their relation to Dedekind cuts and prime ideals.
- We prove that $\mathrm{Coz}_c[L]$ is a σ-frame whenever L is completely regular.
- We characterize zero-dimensional frames in terms of c-complete regularity.

1.3 Organization of the Paper

Section 2 recalls preliminary notions on frames and rings of functions. Section 3 studies idempotent elements in $\mathcal{R}(L)$. Section 4 discusses Dedekind cuts and prime ideals. Section 5 introduces bases for $\mathrm{Coz}_c[L]$. Section 6 establishes the σ-frame structure of $\mathrm{Coz}_c[L]$. Section 7 characterizes zero-dimensional frames. Section 8 concludes the paper.

2 Preliminaries and Notation

Throughout this paper, L denotes a frame. Recall that a complete lattice L is called a *frame* if finite meets distribute over arbitrary joins, that is,

$$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b),$$

for every $a \in L$ and every subset $B \subseteq L$. The top and bottom elements of L are denoted by $\top$ and $\bot$, respectively.

For $a \in L$, we write

$$\uparrow a := \{b \in L : a \leq b\}$$

for the principal upset generated by a, and define the implication operation by

$$a \to b := \max\{x \in L : a \wedge x \leq b\} \text{ (if it exists)}, \quad a, b \in L.$$

The *pseudocomplement* of a is defined by $a^* := a \to \bot$. An element a is called *complemented* if $a \vee a^* = \top$, in which case its complement is denoted by a'.

Following [5], the *frame of reals* $\mathcal{L}(\mathbb{R})$ is generated by the ordered pairs (p, q) with $p, q \in \mathbb{Q}$, subject to the relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,

(R4) $\top = \bigvee\{(p, q) : p, q \in \mathbb{Q}\}$.

A *continuous real-valued function* on a frame L is a frame homomorphism

$$\alpha : \mathcal{L}(\mathbb{R}) \to L.$$

The set of all such functions is denoted by $\mathcal{R}(L)$.

For $\alpha, \beta \in \mathcal{R}(L)$ and $\diamond \in \{+, \cdot, \wedge, \vee\}$, the operations on $\mathcal{R}(L)$ are defined by

$$(\alpha \diamond \beta)(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(u, v) : \langle r, s\rangle \diamond \langle u, v\rangle \subseteq \langle p, q\rangle\},$$

where $\langle p, q\rangle := \{r \in \mathbb{R} : p < r < q\}$ for $p, q \in \mathbb{Q}$, and

$$\langle r, s\rangle \diamond \langle u, v\rangle := \{y \diamond z : y \in \langle r, s\rangle,\ z \in \langle u, v\rangle\}.$$

With these operations, $\mathcal{R}(L)$ forms an f-ring.

For convenience, for $p, q \in \mathbb{Q}$, we set

$$(p,-) := \bigvee_{\substack{r,s>p \\ r,s\in\mathbb{Q}}} (r,s), \qquad (-,q) := \bigvee_{\substack{r,s<q \\ r,s\in\mathbb{Q}}} (r,s).$$

For $\alpha \in \mathcal{R}(L)$, the element

$$\operatorname{coz}(\alpha) := \alpha(-,0) \vee \alpha(0,-)$$

is called the *cozero element* of α.

A frame L is called *completely regular* if it is generated by its cozero elements, i.e., for every $x \in L$ there exists a family $\{\alpha_i\}_{i\in I} \subseteq \mathcal{R}(L)$ such that

$$x = \bigvee_{i\in I} \operatorname{coz}(\alpha_i).$$

A *σ-frame* is a distributive lattice in which every countable subset has a join and finite meets distribute over countable joins, that is,

$$a \wedge \bigvee S = \bigvee_{s\in S} (a \wedge s),$$

for all $a \in L$ and every countable set $S \subseteq L$.

The set

$$\operatorname{Coz}[L] := \{\operatorname{coz}(\alpha) : \alpha \in \mathcal{R}(L)\}$$

is a sub-σ-frame of L [8, Corollary 2].

For each $\alpha \in \mathcal{R}(L)$, define

$$R_\alpha := \{r \in \mathbb{R} : \operatorname{coz}(\alpha - r) \neq \top\}.$$

The pointfree analogue of the ring $C_c(X)$ is defined by

$$\mathcal{C}_c(L) := \{\alpha \in \mathcal{R}(L) : R_\alpha \text{ is countable}\}$$

as the pointfree topology version of the ring $C_c(X)$ (for more details see [23, 24, 16]). Finally, we define the *cozero part associated with* $\mathcal{C}_c(L)$ by

$$\operatorname{Coz}_c[L] := \{\operatorname{coz}(\alpha) : \alpha \in \mathcal{C}_c(L)\}.$$

3 Algebraic Structure of Idempotents in $\mathcal{R}(L)$

We write $B(A)$ for the set of all the idempotent elements of a given ring A. If $e \in B(\mathcal{R}(L))$, then $\operatorname{coz}(e)$ is complemented. For $e_1, e_2 \in B(\mathcal{R}(L))$ in this section, we check $(e_1 \diamond e_2)(-, x)$ and $(e_1 \diamond e_2)(x, -)$ for every $x \in \mathbb{Q}$, where $\diamond \in \{+, -, \cdot, \vee, \wedge\}$. The following proposition collects the basic order-theoretic and range-theoretic properties of an idempotent element of $\mathcal{R}(L)$; it makes Remark 3.2 below completely transparent.

Proposition 3.1. *Let $e \in B(\mathcal{R}(L))$. Then:*

(1) $\operatorname{coz}(e) = \big(\operatorname{coz}(\mathbf{1} - e)\big)'$.

(2) $e(-, 0) = e(0, 1) = e(1, -) = \bot$, *and* $e(r, -) = \top = e(-, s)$ *for every* $r < 0 < 1 < s$.

(3) $\mathbf{0} \leq e \leq \mathbf{1}$.

(4) $e = \mathbf{0}$ *if and only if* $R_e = \{0\}$, *if and only if* $e(0, -) = \bot$.

(5) $e = \mathbf{1}$ *if and only if* $R_e = \{1\}$, *if and only if* $e(-, 1) = \bot$.

(6) $R_e = \{0, 1\}$ *if and only if* $e \notin \{\mathbf{0}, \mathbf{1}\}$.

Proof. (1) Since $\mathbf{1} - e$ is idempotent along with e, we have

$$\operatorname{coz}(e) \wedge \operatorname{coz}(\mathbf{1} - e) = \operatorname{coz}\big(e(\mathbf{1} - e)\big) = \operatorname{coz}(\mathbf{0}) = \bot,$$

$$\operatorname{coz}(e) \vee \operatorname{coz}(\mathbf{1} - e) = \operatorname{coz}\big(e^2 + (\mathbf{1} - e)^2\big) = \operatorname{coz}\big(e + (\mathbf{1} - e)\big) = e(\mathbf{1}) = \top,$$

so $\operatorname{coz}(\mathbf{1} - e)$ is the complement of $\operatorname{coz}(e)$.

(2) Since $e^2 = e \geq \mathbf{0}$, $e(-, 0) = \bot$; and since $\mathbf{1} - e$ is idempotent, $(\mathbf{1} - e)(-, 0) = \bot$ gives $e(1, -) = \bot$. By (1),

$$\begin{aligned} \bot &= e(R \setminus \{0\}) \wedge e(R \setminus \{1\}) \\ &= e\big[(-, 0) \cup (0, 1) \cup (1, -)\big] \\ &= e(-, 0) \vee e(0, 1) \vee e(1, -), \end{aligned}$$

so $e(0, 1) = \bot$ as well. Finally, for $r < 0 < 1 < s$,

$$e(r, -) = e(-, 0) \vee e(r, -) = e(R) = \top,$$

and similarly $e(-,s)=\top$.

(3) By (2), $e(-,0)=\bot$ gives $e\geq \mathbf{0}$, and $(\mathbf{1}-e)(-,0)=\bot$ gives $\mathbf{1}-e\geq\mathbf{0}$, i.e., $e\leq\mathbf{1}$.

(4) If $e=\mathbf{0}$, clearly $R_e=\{0\}$ and $e(0,-)=\bot$. Conversely, if $e(0,-)=\bot$, then for every open $w\in\mathcal{O}(\mathbb{R})$ with $0\in w$ we have $(-r,r)\subseteq w$ for some $r>0$, so

$$e(w)\geq e(-r,r)=e(-,r)\wedge e(r,-)=\top$$

(using (2) applied at the point r in place of 1, since the argument in (2) works for any pair of distinct reals), which forces $e=\mathbf{0}$.

(5) Symmetric to (4), applied to $\mathbf{1}-e$.

(6) Immediate from (3)-(5): if $e\notin\{\mathbf{0},\mathbf{1}\}$, then $0,1\in R_e$ by (4)-(5), while $R_e\subseteq\{0,1\}$ by (5). $\square$

Remark 3.2. Let e be an idempotent element of $\mathcal{R}(L)$. Then the following statements are true:

(1) For every $x\in\mathbb{Q}$,

$$e(x,-)=\begin{cases}\top, & x<0,\\ \operatorname{coz}(e), & 0\leq x<1,\\ \bot, & x\geq 1,\end{cases}$$

and

$$e(-,x)=\begin{cases}\bot, & x\leq 0,\\ (\operatorname{coz}(e))', & 0<x\leq 1,\\ \top, & x>1.\end{cases}$$

(2) If $x,k\in\mathbb{Q}$ with $k>0$, then

$$ke(x,-)=\begin{cases}\top, & x<0,\\ \operatorname{coz}(e), & 0\leq x<k,\\ \bot, & x\geq k,\end{cases}$$

and

$$ke(-,x)=\begin{cases}\bot, & x\leq 0,\\ (\operatorname{coz}(e))', & 0<x\leq k,\\ \top, & x>k.\end{cases}$$

(3) If $x, k \in \mathbb{Q}$ with $k < 0$, then

$$ke(x,-) = \begin{cases} \bot, & x \geq 0, \\ (\mathrm{coz}(e))', & k \leq x < 0, \\ \top, & x < k, \end{cases}$$

and

$$ke(-,x) = \begin{cases} \top, & x > 0, \\ \mathrm{coz}(e), & k < x \leq 0, \\ \bot, & x \leq k. \end{cases}$$

Lemma 3.3. *If $e_1, e_2 \in B(\mathcal{R}(L))$, then for every $x \in \mathbb{R}$, we have*

$$(e_1+e_2)(x,-) = \begin{cases} \top & x < 0, \\ \mathrm{coz}(e_1+e_2), & 0 \leq x < 1, \\ \mathrm{coz}(e_1) \wedge \mathrm{coz}(e_2), & 1 \leq x < 2, \\ \bot, & x \geq 2, \end{cases}$$

and

$$(e_1+e_2)(-,x) = \begin{cases} \bot, & x \leq 0, \\ \big(\mathrm{coz}(e_1)\big)' \wedge \big(\mathrm{coz}(e_2)\big)', & 0 < x \leq 1, \\ \big(\mathrm{coz}(e_1)\big)' \vee \big(\mathrm{coz}(e_2)\big)', & 1 < x \leq 2, \\ \bot, & x > 2. \end{cases}$$

Proof. Since $\mathbf{0} \leq e_1 + e_2 \leq \mathbf{2}$, it suffices to establish the first identity for $0 \leq x < 2$; the cases $x < 0$ and $x \geq 2$ are then immediate from $e_1 + e_2 \geq \mathbf{0}$ and $e_1 + e_2 \leq \mathbf{2}$, respectively, and the second identity follows from the first by taking complements as in Remark 3.2. For $x, r \in \mathbb{R}$ put $H_{(x,r)} = e_1(r,-) \wedge e_2(x-r,-)$, so that $(e_1+e_2)(x,-) = \bigvee_r H_{(x,r)}$.

Case $0 \leq x < 1$. If $r < 0$, then $H_{(x,r)} \in \{\bot, \mathrm{coz}(e_2)\}$ according as $x - r \geq 1$ or $0 \leq x - r < 1$. If $0 \leq r \leq x$, then $H_{(x,r)} = \mathrm{coz}(e_1) \wedge \mathrm{coz}(e_2)$. If $x < r < 1$, then $H_{(x,r)} = \mathrm{coz}(e_1)$. Hence

$$\begin{aligned} (e_1+e_2)(x,-) &= \mathrm{coz}(e_2) \vee \mathrm{coz}(e_1) \vee \big(\mathrm{coz}(e_1) \wedge \mathrm{coz}(e_2)\big) \\ &= \mathrm{coz}(e_1) \vee \mathrm{coz}(e_2) \\ &= \mathrm{coz}(e_1+e_2). \end{aligned}$$

Case $1 \leq x < 2$. If $r \geq 1$ or $x - r \geq 1$, then $H_{(x,r)} = \bot$; so we may assume $0 \leq r < 1$ and $0 \leq x - r < 1$, in which case $H_{(x,r)} = \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2)$. Hence

$$(e_1 + e_2)(x, -) = \bigvee_{0 \leq r,\, x-r<1} H_{(x,r)} = \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2).$$

This proves the first identity for every $x \in \mathbb{R}$. □

Lemma 3.4. *If* $e_1, e_2 \in B(\mathcal{R}(L))$*, then for every* $x \in \mathbb{R}$*, it holds that*

$$(e_1e_2)(x, -) = \begin{cases} \top, & x < 0, \\ \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2), & 0 \leq x < 1, \\ \bot, & x \geq 1, \end{cases}$$

and

$$(e_1e_2)(-, x) = \begin{cases} \bot, & x \leq 0, \\ \big(\operatorname{coz}(e_1)\big)' \vee \big(\operatorname{coz}(e_2)\big)', & 0 < x \leq 1, \\ \top, & x > 1. \end{cases}$$

Proof. Since e_1, e_2 are commuting idempotents of the f-ring $\mathcal{R}(L)$, so is their product e_1e_2, and $(e_1e_2)^2 = e_1^2e_2^2 = e_1e_2$. Moreover $e_1e_2 \geq \mathbf{0}$ and $e_1e_2 \leq e_1 \wedge e_2 \leq \mathbf{1}$, since $e_1, e_2 \in [\mathbf{0}, \mathbf{1}]$. Also,

$$\operatorname{coz}(e_1e_2) = \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2),$$

because $(e_1e_2)(0, -) = e_1(0, -) \wedge e_2(0, -) = \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2)$ by definition of the product on $\mathcal{R}(L)$ and Remark 3.2. Thus $e_1e_2 \in B(\mathcal{R}(L))$ is an idempotent element with $\operatorname{coz}(e_1e_2) = \operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2)$, and both stated identities are simply the case $e = e_1e_2$ of Remark 3.2(1). □

The next lemma is an immediate consequence of [1, Proposition 1.4], according to which, for every $\alpha, \beta \in \mathcal{R}(L)$ and every $x \in \mathbb{R}$,

$$(\alpha \vee \beta)(x, -) = \alpha(x, -) \vee \beta(x, -), \quad (\alpha \vee \beta)(-, x) = \alpha(-, x) \wedge \beta(-, x),$$

$$(\alpha \wedge \beta)(x, -) = \alpha(x, -) \wedge \beta(x, -), \quad (\alpha \wedge \beta)(-, x) = \alpha(-, x) \vee \beta(-, x);$$

together with the fact, noted in Lemma 3.3, that

$$\operatorname{coz}(e_1 + e_2) = \operatorname{coz}(e_1) \vee \operatorname{coz}(e_2).$$

Lemma 3.5. *If $e_1, e_2 \in B(\mathcal{R}(L))$, then for every $x \in \mathbb{R}$, we have*

$$(e_1 \vee e_2)(x,-) = \begin{cases} \top, & x < 0, \\ \operatorname{coz}(e_1 + e_2), & 0 \leq x < 1, \\ \bot, & x \geq 1, \end{cases}$$

$$(e_1 \vee e_2)(-,x) = \begin{cases} \bot, & x \leq 0, \\ \big(\operatorname{coz}(e_1)\big)' \wedge \big(\operatorname{coz}(e_2)\big)', & 0 < x \leq 1, \\ \top, & x > 1, \end{cases}$$

$$(e_1 \wedge e_2)(x,-) = \begin{cases} \top, & x < 0, \\ \operatorname{coz}(e_1 \wedge e_2), & 0 \leq x < 1, \\ \bot, & x \geq 1, \end{cases}$$

and

$$(e_1 \wedge e_2)(-,x) = \begin{cases} \bot, & x \leq 0, \\ \big(\operatorname{coz}(e_1)\big)' \vee \big(\operatorname{coz}(e_2)\big)', & 0 < x \leq 1, \\ \top, & x > 1. \end{cases}$$

4 Dedekind Cuts and Prime Ideals

In the article [1], $A_p(\alpha)$ and $B_p(\alpha)$ were introduced. We found that it is better to use $P_u(\alpha)$ and $P_l(\alpha)$ symbols instead of them, which would be easier for the reader to understand the concepts. They are given below. For every $\alpha \in \mathcal{R}(L)$ and every prime ideal P in L, let

$$P_u(\alpha) := \{x \in \mathbb{Q} : \alpha(x,-) \in P\} \quad \text{and}$$
$$P_l(\alpha) := \{x \in \mathbb{Q} : \alpha(-,x) \in P\}.$$

To express our arguments, we need the following lemma and result from [1].

Lemma 4.1. [1] *For every $\alpha \in \mathcal{R}(L)$ and every prime ideal P in L, the following statements are true:*

(1) $P_u(\alpha) \cup P_l(\alpha) = \mathbb{Q}$.

(2) *Any element of* $P_u(\alpha)$ *is an upper bound of* $P_l(\alpha)$ *and any element of* $P_l(\alpha)$ *is a lower bound of* $P_u(\alpha)$.

(3) $\uparrow P_u(\alpha) = P_u(\alpha)$ *and* $\downarrow P_l(\alpha) = P_l(\alpha)$, *where, for* $S \subseteq \mathbb{Q}$, $\uparrow S := \{x \in \mathbb{Q} : \exists\, s \in S,\ s \leq x\}$ *and* $\downarrow S := \{x \in \mathbb{Q} : \exists\, s \in S,\ x \leq s\}$; *that is,* $P_u(\alpha)$ *is an up-set and* $P_l(\alpha)$ *is a down-set of* $\mathbb{Q}$.

Corollary 4.2. [1] *Let* P *be a prime ideal of* L *and let* $\alpha \in \mathcal{R}(L)$ *be given. Then the following statements are equivalent:*

(1) $\inf P_u(\alpha) \in \mathbb{R}$.

(2) $P_l(\alpha) \neq \emptyset \neq P_u(\alpha)$.

(3) $\sup P_l(\alpha) \in \mathbb{R}$.

(4) $\inf P_u(\alpha) = \sup P_l(\alpha) \in \mathbb{R}$.

Although $P_u(\alpha)$ and $P_l(\alpha)$ are subsets of $\mathbb{Q}$, the infimum and supremum above are computed in the extended real line $\mathbb{R} \cup \{\pm\infty\}$; the corollary asserts that the two agree and are finite exactly when $(P_l(\alpha), P_u(\alpha))$ is (the rational trace of) a genuine Dedekind cut of $\mathbb{R}$. This point, which is due to [1], is used repeatedly in what follows.

Corollary 4.3. *Let* P *be a prime ideal of* L *and let* $\alpha \in \mathcal{R}(L)$ *be given. Then* $\inf P_u(\alpha) \in \mathbb{Q}$ *if and only if* $\inf P_u(\alpha) \in P_u(\alpha)$ *if and only if* $|P_u(\alpha) \cap P_l(\alpha)| = 1$.

Proof. By Corollary 4.2, it is evident. □

Lemma 4.4. *Let* P *be a prime ideal of* L. *If* e *is an idempotent element of* $\mathcal{R}(L)$, *then the following statements are true:*

(1) $\inf P_u(e) = \sup P_l(e) \in P_u(e) \cap P_l(e)$ *and*

$$\inf P_u(e) = \begin{cases} 0, & \operatorname{coz}(e) \in P, \\ 1, & \operatorname{coz}(e) \notin P. \end{cases}$$

(2) *For every* $k \in \mathbb{Q}$, $\inf P_u(ke) = k \inf P_u(e)$.

Proof. (1). By Proposition 3.1(1), $\mathrm{coz}(e)$ and $\big(\mathrm{coz}(e)\big)'$ are complementary elements of the prime ideal's ambient frame, so exactly one of them belongs to P; this, together with Proposition 3.1(2), yields the two cases below. If $\mathrm{coz}(e) \in P$, then

$$P_u(e) = \{x \in \mathbb{Q} : x \geq 0\} \quad \text{and} \quad P_l(e) = \{x \in \mathbb{Q} : x \leq 0\}.$$

Therefore,

$$\inf P_u(e) = \sup P_l(e) = 0 \in P_u(e) \cap P_l(e).$$

If $\big(\mathrm{coz}(e)\big)' \in P$, then

$$P_u(e) = \{x \in \mathbb{Q} : x \geq 1\} \text{ and } P_l(e) = \{x \in \mathbb{Q} : x \leq 1\}.$$

Hence,

$$\inf P_u(e) = \sup P_l(e) = 1 \in P_u(e) \cap P_l(e).$$

(2). If $k = 0$, then $\mathbf{k}e = \mathbf{0}$, so $\inf P_u(\mathbf{k}e) = 0 = k \inf P_u(e)$ trivially. Suppose $k \in \mathbb{Q}$ with $k > 0$. Then

$$x \in P_u(ke) \Leftrightarrow ke(x, -) \in P \Leftrightarrow e(\frac{x}{k}, -) \in P \Leftrightarrow x \in kP_u(e),$$

which implies that $\inf P_u(ke) = k \inf P_u(e)$.

Let $k \in \mathbb{Q}$ with $k < 0$ be given. Then

$$x \in P_u(ke) \Leftrightarrow ke(x, -) \in P \Leftrightarrow e(-, \frac{x}{k}) \in P \Leftrightarrow x \in kP_l(e),$$

which implies that

$$\inf P_u(ke) = \inf kP_l(e) = k \sup P_l(e) = k \inf P_u(e).$$

□

In [1, Theorem 2.23], the prime ideal P is considered as countably $\bigvee$-complete, which here, since our discussion is about idempotent elements, we are able to remove this condition and express it as follows.

Proposition 4.5. *Let P be a prime ideal of L. If $e_1, e_2 \in B(\mathcal{R}(L))$, then*

$$\inf P_u(e_1 \diamond e_2) = \sup P_l(e_1 \diamond e_2) \in P_u(e_1 \diamond e_2) \cap P_l(e_1 \diamond e_2)$$

and

$$\inf P_u(e_1 \diamond e_2) = \inf P_u(e_1) \diamond \inf P_u(e_2)$$

for every $\diamond \in \{+, \vee, \wedge, \cdot\}$.

Proof. We only prove it for $\diamond = +$; the rest will be proved similarly. Since $\mathbf{0} \leq e_1 + e_2 \leq \mathbf{2}$, it is evident that $P_u(e_1 + e_2) \cap (-\infty, 0) = \emptyset$ and $P_l(e_1 + e_2) \cap (2, +\infty) = \emptyset$.

Case (1): If $\operatorname{coz}(e_1) \in P$ and $\operatorname{coz}(e_2) \in P$, then $\operatorname{coz}(e_1 + e_2) = \operatorname{coz}(e_1) \vee \operatorname{coz}(e_2) \in P$, which implies that $P_u(e_1 + e_2) = \{x \in \mathbb{Q} : x \geq 0\}$ and $P_l(e_1 + e_2) = \{x \in \mathbb{Q} : x \leq 0\}$. Therefore,

$$\inf P_u(e_1 + e_2) = \sup P_l(e_1 + e_2) = 0 \in P_u(e_1 + e_2) \cap P_l(e_1 + e_2)$$

and $\inf P_u(e_1 + e_2) = \inf P_u(e_1) + \inf P_u(e_2)$.

Case (2): If $\operatorname{coz}(e_1) \in P$ and $\big(\operatorname{coz}(e_2)\big)' \in P$, then $\operatorname{coz}(e_1 + e_2) \vee \big(\operatorname{coz}(e_2)\big)' = \top$, which implies that $\operatorname{coz}(e_1 + e_2) \notin P$. Hence, $P_u(e_1 + e_2) \cap [0, 1) \cap \mathbb{Q} = \emptyset$. Since $\operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2) \leq \operatorname{coz}(e_1) \in P$, we conclude that $[1, 2) \cap \mathbb{Q} \subseteq P_u(e_1 + e_2)$. We clearly have $[2, +\infty) \subseteq P_u(e_1 + e_2)$ and implies that $P_u(e_1 + e_2) = [1, +\infty) \cap \mathbb{Q}$. A similar proof shows that $P_l(e_1 + e_2) = (-\infty, 1] \cap \mathbb{Q}$. Therefore,

$$\inf P_u(e_1 + e_2) = \sup P_l(e_1 + e_2) = 1 \in P_u(e_1 + e_2) \cap P_l(e_1 + e_2)$$

and $\inf P_u(e_1 + e_2) = \inf P_u(e_1) + \inf P_u(e_2)$.

Case (3): Let $\big(\operatorname{coz}(e_1)\big)' \in P$ and let $\operatorname{coz}(e_2) \in P$. Similar to the proof of Case (2), we have

$$\inf P_u(e_1 + e_2) = \sup P_l(e_1 + e_2) = 1 \in P_u(e_1 + e_2) \cap P_l(e_1 + e_2)$$

and $\inf P_u(e_1 + e_2) = \inf P_u(e_1) + \inf P_u(e_2)$.

Case (4): Let $\big(\operatorname{coz}(e_1)\big)' \in P$ and let $\big(\operatorname{coz}(e_2)\big)' \in P$. We have

$$\begin{aligned} &\big(\operatorname{coz}(e_1 + e_2)\big)' = \big(\operatorname{coz}(e_1)\big)' \wedge \big(\operatorname{coz}(e_2)\big)' \in P \Rightarrow \operatorname{coz}(e_1 + e_2) \notin P \\ &\Rightarrow P_u(e_1 + e_2) \cap [0, 1) \cap \mathbb{Q} = \emptyset. \end{aligned}$$

If $\operatorname{coz}(e_1) \wedge \operatorname{coz}(e_2) \in P$, then $\operatorname{coz}(e_1) \in P$ or $\operatorname{coz}(e_2) \in P$, which implies that $\top = \operatorname{coz}(e_1) \vee \big(\operatorname{coz}(e_1)\big)' \in P$ or $\top = \operatorname{coz}(e_2) \vee \big(\operatorname{coz}(e_2)\big)' \in P$, a contradiction. Hence, $P_u(e_1 + e_2) \cap [1, 2) \cap \mathbb{Q} = \emptyset$, which follows that

$P_u(e_1+e_2) = [2,+\infty)\cap\mathbb{Q}$. A similar proof shows that $P_l(e_1+e_2) = (-\infty,2]\cap\mathbb{Q}$. Therefore,

$$\inf P_u(e_1+e_2) = \sup P_l(e_1+e_2) = 2 \in P_u(e_1+e_2)\cap P_l(e_1+e_2)$$

and $\inf P_u(e_1+e_2) = \inf P_u(e_1) + \inf P_u(e_2)$. □

Corollary 4.6. *Let P be a prime ideal of L. Suppose that $e_1, e_2 \in B(\mathcal{R}(L))$. If $x = \inf P_u(e_1 \diamond e_2)$, then $x \in R_{e_1\diamond e_2}$ for every $\diamond \in \{+,\cdot,\vee,\wedge\}$.*

Proof. By Proposition 4.5, it is obvious. □

We recall that for every $p \in \mathbb{R}$,

$$\mathbf{p}(-,r) = \begin{cases} \bot, & r \le p, \\ \top, & r > p, \end{cases} \quad \text{and} \quad \mathbf{p}(r,-) = \begin{cases} \top, & r < p, \\ \bot, & r \ge p. \end{cases}$$

Remark 4.7. If $\alpha, \beta \in \mathcal{R}(L)$ satisfy $\alpha \le \beta$, then $\alpha(x,-) \le \beta(x,-)$ and $\beta(-,x) \le \alpha(-,x)$ for every $x \in \mathbb{R}$; since every ideal P is a down-set containing $\bot$, this gives $P_u(\beta) \subseteq P_u(\alpha)$ and $P_l(\alpha) \subseteq P_l(\beta)$.

We write $\mathcal{R}^*(L)$ for the set of bounded elements of $\mathcal{R}(L)$, that is, those $\alpha \in \mathcal{R}(L)$ for which $\mathbf{m} \le \alpha \le \mathbf{n}$ for some $m, n \in \mathbb{R}$.

Remark 4.8. For an arbitrary prime ideal P of L and an arbitrary $\alpha \in \mathcal{R}(L)$, the pair $(P_u(\alpha), P_l(\alpha))$ need not determine a Dedekind cut of $\mathbb{R}$: it is only guaranteed that $P_u(\alpha) \cup P_l(\alpha) = \mathbb{Q}$ and that every element of $P_u(\alpha)$ bounds $P_l(\alpha)$ above (Lemma 4.1), but $\inf P_u(\alpha)$ and $\sup P_l(\alpha)$ may fail to be finite, or may fail to coincide. A prime ideal P for which the pair $(P_u(\alpha), P_l(\alpha))$ *does* yield a Dedekind cut for every $\alpha \in \mathcal{R}(L)$ is called *real*; for every prime (equivalently, real) element p of L, a corresponding evaluation $\widehat{p} : \mathcal{R}(L) \to \mathbb{R}$ is constructed via Dedekind cuts in [10], and this construction has since been extended to σ-frames and σ-points in [11]. The lemma below isolates a situation in which reality of P is not needed at all: if α is merely *bounded*, then $(P_u(\alpha), P_l(\alpha))$ is automatically a Dedekind cut, for every prime ideal P whatsoever. This is the technical fact on which the countable-join arguments of Sections 5 and 6 ultimately rest.

Lemma 4.9. *Let P be a prime ideal of L. If $\alpha \in \mathcal{R}^*(L)$, then* $\inf P_u(\alpha) = \sup P_l(\alpha)$.

Proof. There exists $n \in \mathbb{N}$ such that $-\mathbf{n} \leq \alpha \leq \mathbf{n}$. For every $x > n$, since $\alpha \leq \mathbf{n}$ and $\mathbf{n}(x,-) = \bot$, monotonicity gives $\alpha(x,-) \leq \bot$, so $\alpha(x,-) = \bot \in P$, i.e., $x \in P_u(\alpha)$. Likewise, for every $y < -n$, since $-\mathbf{n} \leq \alpha$ and $-\mathbf{n}(-,y) = \bot$, monotonicity gives $\alpha(-,y) \leq \bot$, so $\alpha(-,y) = \bot \in P$, i.e., $y \in P_l(\alpha)$. Hence, $(n,+\infty) \cap \mathbb{Q} \subseteq P_u(\alpha)$ and $(-\infty,-n) \cap \mathbb{Q} \subseteq P_l(\alpha)$. It follows from Corollary 4.2 that $\inf P_u(\alpha) = \sup P_l(\alpha)$. $\square$

Proposition 4.10. *Let P be a prime ideal of L and let $n \in \mathbb{N}$. If $e_1, \ldots, e_n \in B(\mathcal{R}(L))$, then*

$$\inf P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n) = \sup P_l(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n)$$

belongs to

$$P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n) \cap P_l(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n)$$

and

$$\inf P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n) = r_1 \inf P_u(e_1) + \cdots + r_n \inf P_u(e_n)$$

for every $r_1, \ldots, r_n \in \mathbb{Q}$.

Proof. We prove it by induction on $n \in \mathbb{N}$. For $n = 1$, the result is trivial. Let $r_1, \ldots, r_n \in \mathbb{Q}$ be given and let $n \geq 2$. We put

$$\alpha = \mathbf{r_1}e_1 + \cdots + \mathbf{r_{n-1}}e_{n-1}.$$

Let $x \in P_u(\alpha + \mathbf{r_n}e_n)$ be given. Then

$$\bigvee_{a \in \mathbb{Q}} \alpha(a,-) \wedge \mathbf{r_n}e_n(x-a,-) = (\alpha + \mathbf{r_n}e_n)(x,-) \in P,$$

which implies that $\alpha(a,-) \wedge \mathbf{r_n}e_n(x-a,-) \in P$ for every $a \in \mathbb{Q}$, and so $\alpha(a,-) \in P$ or $\mathbf{r_n}e_n(x-a,-) \in P$ for every $a \in \mathbb{Q}$. If $a < \inf P_u(\alpha)$, then $\alpha(a,-) \notin P$, which implies from Remark 3.2 that

$$\bot = \mathbf{r_n}e_n(x-a,-) \in P \text{ or } \operatorname{coz}(e_n)' = \mathbf{r_n}e_n(x-a,-) \in P.$$

From

$$v := \mathbf{r_n} e_n(x - \inf P_u(\alpha), -) = \bigvee_{\substack{a\in\mathbb{Q} \\ a<\inf P_u(\alpha)}} r_n e_n(x - a, -),$$

we conclude that $v = \bot \in P$ or $v = \operatorname{coz}(e_n)' \in P$, which implies that $\inf P_u(\mathbf{r_n} e_n) \leq x - \inf P_u(\alpha)$ and hence,

$$\inf P_u(\alpha) + \inf P_u(\mathbf{r_n} e_n) \leq \inf P_u(\alpha + \mathbf{r_n} e_n).$$

Let $x \in P_l(\alpha + \mathbf{r_n} e_n)$ be given. We set

$$w := \mathbf{r_n} e_n(-, x - \sup P_l(\alpha)) = \bigvee_{\substack{s\in\mathbb{Q} \\ s>\sup P_l(\alpha)}} r_n e_n(-, x - s).$$

Therefore, similar to the above, we conclude that for every $s > \sup P_l(\alpha)$, which we conclude from Remark 3.2 that $w = \bot \in P$ or $w = \operatorname{coz}(e_n) \in P$. Thus we have $x - \sup P_l(\alpha) \leq \sup P_l(\mathbf{r_n} e_n)$ and hence,

$$\sup P_l(\alpha + \mathbf{r_n} e_n) \leq \sup P_l(\alpha) + \sup P_l(\mathbf{r_n} e_n),$$

and finally, by Lemma 4.9,

$$\inf P_u(\alpha) + \inf P_u(\mathbf{r_n} e_n) = \inf P_u(\alpha + \mathbf{r_n} e_n).$$

The rest follows with Lemma 4.4. □

5 Bases of the Cozero Part $\mathrm{Coz}_c[L]$

In [24], for every $r \in \mathbb{R}$, it was defined that

$$u_r := \bigvee_{n\in\mathbb{N}} (-, a_n) \text{ and } l_r := \bigvee_{n\in\mathbb{N}} (b_n, -),$$

where $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{Q}$ such that $a_n, b_n \to r$ and for every $n \in \mathbb{N}$,

$$a_n < a_{n+1} < r < b_{n+1} < b_n.$$

It was also shown that, for every $\alpha \in \mathcal{R}(L)$ and every $r \notin R_\alpha$, $\alpha(u_r)$ and $\alpha(l_r)$ are complemented in L and that the complement of $\alpha(l_r)$ is $\alpha(u_r)$; see also Lemma 4.1 in [15].

The following result is a counterpart of [15, Proposition 5.1].

Proposition 5.1. *If $\alpha \in \mathcal{C}_c(L)$, then $\mathrm{coz}(\alpha)$ is a countable join of complemented elements of L.*

Proof. Since R_α is countable, we may choose, for each $n \in \mathbb{N}$, by [24, Remark 7.2], $r_n, s_n \in \mathbb{R} \setminus R_\alpha$ satisfying all of the following conditions:

(1) $\frac{-1}{n} < r_n < 0 < s_n < \frac{1}{n}$,

(2) $u_{r_n}, u_{s_n}, l_{r_n}$ and l_{s_n} are complemented,

(3) $\alpha(\frac{1}{n}, -) \leqslant \alpha(l_{s_n}) \leqslant \mathrm{coz}(\alpha)$ and $\alpha(-, \frac{-1}{n}) \leqslant \alpha(u_{r_n}) \leqslant \mathrm{coz}(\alpha)$,

(4) $\alpha(u_{r_n} \vee l_{s_n})$ is complemented.

Therefore,

$$\mathrm{coz}(\alpha) = \bigvee_{n\in\mathbb{N}} \alpha(-, \frac{-1}{n}) \vee \alpha(\frac{1}{n}, -) \leqslant \alpha(u_{r_n}) \vee \alpha(l_{s_n}) \leqslant \mathrm{coz}(\alpha),$$

which shows that $\mathrm{coz}(\alpha) = \bigvee_{n\in\mathbb{N}} \alpha(u_{r_n} \vee l_{s_n})$. □

Remark 5.2. It is easy to see that if $a \in L$ is complemented, then $e_a : \mathcal{L}(\mathbb{R}) \longrightarrow L$ given by

$$e_a(u) = \begin{cases} a' & \text{if } \ 0 \in \tau(u), \\ a & \text{if } \ 0 \notin \tau(u), \end{cases}$$

is a continuous real-valued function, and by [13, Proposition 4.1], $e_a^2 = e_a$ and $\mathrm{coz}(e_a) = a$. Conversely, it is easily seen that e is an idempotent element of $\mathcal{R}(L)$ if and only if $e = e_{\mathrm{coz}(e)}$; this recovers Proposition 3.1(1)-(3) and shows that $a \mapsto e_a$ is a bijection between the complemented elements of L and $B(\mathcal{R}(L))$, with inverse $e \mapsto \mathrm{coz}(e)$.

The following result is a counterpart of [15, Theorem 5.2].

Proposition 5.3. *For every $\alpha \in \mathcal{C}_c(L)$, there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq B(\mathcal{C}_c(L))$ with $\mathrm{coz}(\alpha) = \bigvee_{n\in\mathbb{N}} \mathrm{coz}(\alpha_n)$.*

Proof. By Proposition 5.1 and Remark 5.2, it is obvious. □

6 σ-Frame Structure of $\mathrm{Coz}_c[L]$

In this section, the main purpose of writing this article is going to be proved, that is, $\mathrm{Coz}_c[L]$ is a σ-frame for every completely regular frame L. To achieve this purpose, we need some lemmas and propositions, which we will express and prove them at the beginning.

Lemma 6.1. *Let α be an element of $\mathcal{R}(L)$ and let $x \in \mathbb{R}$. Then the following statements are true:*

(1) *If $x \in R_\alpha$, then there exists a prime ideal P in L such that $x = \inf P_u(\alpha)$.*

(2) *If there exists a countably $\vee$-complete prime ideal P in L such that $x = \inf P_u(\alpha)$, then $x \in R_\alpha$.*

(3) *Let P be a prime ideal of L and let $n \in \mathbb{N}$ be given. Suppose that $e_1, \ldots, e_n \in B(\mathcal{R}(L))$. If $r_1, \ldots, r_n \in \mathbb{Q}$ and*

$$x = \inf P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n),$$

then $x \in R_{\mathbf{r_1}e_1+\cdots+\mathbf{r_n}e_n}$.

Proof. (1). By the hypothesis we have $\mathrm{coz}(\alpha - \mathbf{x}) \neq \top$. Since every element of a frame that is different from $\top$ is contained in some prime ideal (a standard consequence of the Prime Ideal Theorem for frames; see, e.g., [21, Ch. II]), there exists a prime ideal P in L such that $\mathrm{coz}(\alpha - \mathbf{x}) \in P$, which implies that

$$\bigvee_{\substack{r\in\mathbb{Q}\\ r<x}} \alpha(-,r) \vee \bigvee_{\substack{r\in\mathbb{Q}\\ r>x}} \alpha(r,-) \in P.$$

Therefore,

$$\forall r \in \mathbb{Q} \ \big(r < x \Rightarrow \alpha(-,r) \in P \Rightarrow r \in P_l(\alpha)\big) \Rightarrow x \leq \sup P_l(\alpha),$$

and

$$\forall r \in \mathbb{Q} \ \big(r > x \Rightarrow \alpha(-,r) \in P \Rightarrow r \in P_u(\alpha)\big) \Rightarrow x \geq \inf P_u(\alpha).$$

Since, by Corollary 4.2, $\sup P_l(\alpha) = \inf P_u(\alpha)$, we conclude that $x = \inf P_u(\alpha)$.

(2). Let P be a prime ideal of L such that $x = \inf P_u(\alpha)$. Then, by Lemma 4.1,

$$\forall r \in \mathbb{Q} \ \big(r < x \Rightarrow r \in P_l(\alpha) \Rightarrow \alpha(-, r) \in P\big)$$

and

$$\forall r \in \mathbb{Q} \ \big(r > x \Rightarrow r \in P_u(\alpha) \Rightarrow \alpha(r, -) \in P\big).$$

By the hypothesis we have

$$\operatorname{coz}(\alpha - \mathbf{x}) = \bigvee_{\substack{r\in\mathbb{Q}\\ r<x}} \alpha(-, r) \vee \bigvee_{\substack{r\in\mathbb{Q}\\ r>x}} \alpha(r, -) \in P,$$

and so $x \in R_\alpha$.

(3). By Lemma 6.1,

$$x \in P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n) \cap P_l(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n).$$

Then $\operatorname{coz}(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n - x)$ is equal to

$$(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n)(-, x) \vee (\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n)(x, -)$$

and belongs to P, which implies that $\operatorname{coz}(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n - \mathbf{x}) \neq \top$. Therefore, $x \in R_{\mathbf{r_1}e_1+\cdots+\mathbf{r_n}e_n}$. □

For every $A, B \subseteq \mathbb{R}$, we put $A \diamond B := \{a \diamond b : a \in A \text{ and } b \in B\}$ for every $\diamond \in \{+, \cdot, \wedge, \vee\}$.

Proposition 6.2. *Let n be an element of $\mathbb{N}$. Suppose that $e_1, \ldots, e_n \in B(\mathcal{R}(L))$. If $r_1, \ldots, r_n \in \mathbb{Q}$, then*

$$R_{\mathbf{r_1}e_1+\cdots+\mathbf{r_n}e_n} \subseteq R_{\mathbf{r_1}e_1} + \cdots + R_{\mathbf{r_n}e_n}.$$

Proof. Suppose that $x \in R_{\mathbf{r_1}e_1+\cdots+\mathbf{r_n}e_n}$. Thus, by Lemma 6.1(1), there exists a prime ideal P in L such that $x = \inf P_u(\mathbf{r_1}e_1 + \cdots + \mathbf{r_n}e_n)$. Hence, by Proposition 4.10, $x = x_1 + \cdots + x_n$, where $x_i = \inf P_u(\mathbf{r_i}e_i)$ for every $1 \leq i \leq n$. Then, by Lemma 6.1(3), $x_i \in R_{\mathbf{r_i}e_i}$. Therefore,

$$R_{\mathbf{r_1}e_1+\cdots+\mathbf{r_n}e_n} \subseteq R_{\mathbf{r_1}e_1} + \cdots + R_{\mathbf{r_n}e_n}.$$

□

Proposition 6.3. *Suppose that $e_1, e_2 \in B(\mathcal{R}(L))$. Then,*

$$R_{e_1 \diamond e_2} \subseteq R_{e_1} \diamond R_{e_2}$$

for every $\diamond \in \{+, \cdot, \vee, \wedge\}$.

Proof. By Proposition 4.5, Corollary 4.6, and Lemma 6.1, this proposition will be easily proved. □

Remark 6.4. Propositions 6.2 and 6.3 may also be deduced as special cases of the general inclusion $R_{\alpha \diamond \beta} \subseteq R_\alpha \diamond R_\beta$ established, for arbitrary $\alpha, \beta \in \mathcal{R}(L)$ satisfying suitable countability hypotheses, in [1, Theorem 2.26]; here they are obtained directly for idempotents, without invoking that more general theorem.

Let $\{c_n\}_{n=0}^{\infty} \subseteq L$ and $\{\alpha_n\}_{n=0}^{\infty} \subseteq \mathcal{R}(L)$ be such that $\mathbf{0} \leq \alpha_n \leq \mathbf{1}$ and $\operatorname{coz}(\alpha_n) = c_n$ for every n. Banaschewski [6] showed that $\alpha := \sum_{i=0}^{\infty} \frac{\alpha_i}{\mathbf{2}^i} \in \mathcal{R}(L)$, $\operatorname{coz}(\alpha) = \bigvee_{i=0}^{\infty} c_i$ and $\alpha \leq \sum_{i=0}^{n} \frac{\alpha_i}{\mathbf{2}^i} + \frac{\mathbf{1}}{\mathbf{2}^n}$ for every n.

Given any countable family $(e_n)_{n=0}^{\infty}$ in $B\big(\mathcal{R}(L)\big)$, consider $\alpha = \sum_{n=0}^{\infty} \frac{e_n}{\mathbf{2}^n}$. Then for every $n \in \mathbb{N}$, we have

$$\alpha \leq \sum_{i=0}^{n} \frac{e_i}{\mathbf{2}^i} + \frac{\mathbf{1}}{\mathbf{2}^n} \Rightarrow P_u\Big(\sum_{i=0}^{n} \frac{e_i}{\mathbf{2}^i} + \frac{\mathbf{1}}{\mathbf{2}^n}\Big) \subseteq P_u(\alpha)$$
$$\Rightarrow \inf P_u(\alpha) \leq \sum_{i=0}^{n} \inf P_u\Big(\frac{e_i}{\mathbf{2}^i}\Big) + \frac{\mathbf{1}}{\mathbf{2}^n},$$

which implies that $\inf P_u(\alpha) \leq \sum_{i=0}^{\infty} \inf P_u(\frac{e_i}{\mathbf{2}^i})$. On the other hand, for every $n \in \mathbb{N}$, it follows that

$$\sum_{i=0}^{n} \frac{e_i}{\mathbf{2}^i} \leq \alpha \Rightarrow P_u(\alpha) \subseteq P_u\Big(\sum_{i=0}^{n} \frac{e_i}{\mathbf{2}^i}\Big) \Rightarrow \sum_{i=0}^{n} \inf P_u\Big(\frac{e_i}{\mathbf{2}^i}\Big) \leq \inf P_u(\alpha),$$

which implies that $\sum_{i=0}^{\infty} \inf P_u(\frac{e_i}{\mathbf{2}^i}) \leq \inf P_u(\alpha)$. Hence, $\inf P_u(\alpha) = \sum_{i=0}^{\infty} \inf P_u(\frac{e_i}{\mathbf{2}^i})$.

Proposition 6.5. *Let $(e_n)_{n=0}^{\infty}$ be a countable family of $B\big(\mathcal{R}(L)\big)$. If $\alpha = \sum_{n=0}^{\infty} \frac{e_n}{\mathbf{2}^n}$, then $R_\alpha \subseteq \sum_{n=0}^{\infty} R_{\frac{e_n}{\mathbf{2}^n}}$, where, for subsets $A_0, A_1, \ldots \subseteq \mathbb{R}$, the notation $\sum_{n=0}^{\infty} A_n$ denotes the set of all finite sums $\sum_{n=0}^{\infty} x_n$ such that $x_n \in A_n$ for every n, and $x_n = 0$ for all but finitely many n.*

Proof. Suppose that $x \in R_\alpha$. Thus, by Lemma 6.1(1), there exists a prime ideal P in L such that $x = \inf P_u(\alpha)$. Hence, it follows from Remark 4.7 that $x = \sum\limits_{n=0}^{\infty} x_i$, where $x_i = \inf P_u(\frac{e_i}{\mathbf{2}^i})$ for every $i \in \mathbb{N} \cup \{0\}$. Then, by Lemma 6.1(3), we have $x_i \in R_{\frac{e_i}{\mathbf{2}^i}}$. Therefore, $R_\alpha \subseteq \sum\limits_{n=0}^{\infty} R_{\frac{e_n}{\mathbf{2}^n}}$. □

Theorem 6.6. *For every completely regular frame L, $\mathrm{Coz}_c[L]$ is a σ-frame.*

Proof. By Proposition 5.3, it suffices to show that if $(e_n)_{n=0}^{\infty}$ is a countable family of $B\big(\mathcal{R}(L)\big)$, then $\bigvee\limits_{n=0}^{\infty} \mathrm{coz}(e_n) \in \mathrm{Coz}_c[L]$. We put $\alpha = \sum\limits_{n=0}^{\infty} \frac{e_n}{\mathbf{2}^n}$. It is well known that $\mathrm{coz}(\alpha) = \bigvee\limits_{n=0}^{\infty} \mathrm{coz}(e_n)$. Since $\sum\limits_{n=0}^{\infty} R_{\frac{e_n}{\mathbf{2}^n}}$ is a countable set, we conclude from Proposition 6.5 that R_α is too, which implies that $\alpha \in C_c(L)$. Therefore, $\bigvee\limits_{n=0}^{\infty} \mathrm{coz}(e_n) \in \mathrm{Coz}_c[L]$.

The hypothesis that L is completely regular enters through Proposition 5.3 (via Proposition 5.1): it is precisely what guarantees that every $\mathrm{coz}(\alpha)$, $\alpha \in \mathcal{C}_c(L)$, can be written as a countable join $\bigvee\limits_{n} \mathrm{coz}(e_n)$ of idempotent-cozero elements in the first place, so that the reduction above applies to an arbitrary element of $\mathrm{Coz}_c[L]$. □

An element a of frame L is said to *rather below* an element b or a is *well inside* b, written as $a \prec b$, if there is an element x such that $a \wedge x = \bot$ and $x \vee b = \top$. In [9], a σ-frame L is *regular* if for every $a \in L$, there exists $(a_n)_{n=1}^{\infty}$ such that, for every n, $a_n \prec a$, and $a = \bigvee\limits_{n=1}^{\infty} a_n$.

Corollary 6.7. *For every completely regular frame L, $\mathrm{Coz}_c[L]$ is regular.*

Proof. Let $a \in \mathrm{Coz}_c[L]$. By Proposition 5.3, which relies on the complete regularity of L, there is a sequence $(e_n)_{n=1}^{\infty} \subseteq B(\mathcal{R}(L))$ with

$a = \bigvee_{n=1}^{\infty} \operatorname{coz}(e_n)$. Setting $a_n := \operatorname{coz}(e_n)$, we have $a_n \leq a$ and, since a_n is complemented with complement $\operatorname{coz}(e_n)'$, the element $x = \operatorname{coz}(e_n)'$ satisfies $a_n \wedge x = \bot$ and $x \vee a \geq x \vee a_n = \top$; hence $a_n \prec a$ for every n, and $a = \bigvee_n a_n$ by construction. Thus $\operatorname{Coz}_c[L]$ is regular. $\square$

A bounded lattice L is called *normal* if for every $a, b \in L$ with $a \vee b = \top$, there exist $u, v \in L$ such that $a \vee u = \top = v \vee b$ and $u \wedge v = \bot$. Also, a normal σ-frame L is called *perfectly normal* if for each $a \in L$ there exists $(a_n)_{n=1}^{\infty}$ such that, for every $b, c \in L$, $a \wedge b = a \wedge c$ if and only if $a_n \vee b = a_n \vee c$ for every n (see [9]). In the classical setting, a topological space X is perfectly normal precisely when every closed set is a cozero set, i.e., when $\operatorname{Coz}(X)$ generates the closed sets, equivalently $\operatorname{Coz}[L] = L$ for the frame $L = \mathcal{O}(X)$ of open sets. The pointfree definition recalled above, due to [9], is the natural σ-frame-theoretic strengthening of this: it does not literally require $\operatorname{Coz}[L] = L$, but it is designed so that this equality holds whenever L is itself a σ-frame such as $\operatorname{Coz}[L]$ or $\operatorname{Coz}_c[L]$, recovering the classical picture in that case.

By [18, Proposition 1-1], it is obvious that $\operatorname{Coz}_c[L]$ is normal and perfectly normal for every completely regular frame L.

An *Alexandroff algebra* is a normal σ-lattice L that for every $a \in L$, there exist $(a_n, b_n)_{n=1}^{\infty}$ such that $b_n \vee a = \top$, $b_n \wedge a_n = \bot$ and $\bigvee_{n=1}^{\infty} a_n = a$.

Corollary 6.8. *For every completely regular frame L, $\operatorname{Coz}_c[L]$ is an Alexandroff algebra frame.*

Proof. Let $a \in \operatorname{Coz}_c[L]$ be given, say $a = \operatorname{coz}(\alpha)$ for some $\alpha \in \mathcal{C}_c(L)$. By Proposition 5.3, there exists $(e_n)_{n=1}^{\infty} \subseteq B\big(\mathcal{R}(L)\big)$ such that $a = \bigvee_{n=1}^{\infty} \operatorname{coz}(e_n)$. We put

$$(a_n, b_n)_{n=1}^{\infty} = \big(\operatorname{coz}(e_n), \operatorname{coz}(e_n)'\big)_{n=1}^{\infty},$$

then we have

(1) $\bigvee_{n=1}^{\infty} a_n = a$,

(2) $b_n \wedge a_n = \bot$ for every $n \in \mathbb{N}$, and

(3) $b_m \vee a = \mathrm{coz}(e_m)' \vee \bigvee_{n=1}^{\infty} \mathrm{coz}(e_n) = \top$ for every $m \in \mathbb{N}$.

Therefore, the proof is complete. □

A *cover* of σ-frame L is a family $(a_n)_{n=1}^{\infty} \subseteq L$ such that $\bigvee_{n=1}^{\infty} a_n = \top$. A cover $(b_n)_{n=1}^{\infty}$ of σ-frame L is *locally finite* if there is a cover $(c_n)_{n=1}^{\infty}$ of L and, for each m, a finite subset $p(m)$ of $\mathbb{N}$ such that $c_m \wedge b_n = \bot$ if $n \notin p(m)$. This σ-frame formulation, taken from [9], is the countable-indexing counterpart of the usual lattice-theoretic notion: a subset S of a bounded lattice L is locally finite if there is a cover $\{a_\lambda\}_{\lambda\in\Lambda}$ of L such that, for every $\lambda \in \Lambda$, some finite $F_\lambda \subseteq S$ satisfies $a_\lambda \wedge s = \bot$ for all $s \in S \setminus F_\lambda$; one recovers the definition above by restricting attention to countable covers and covers indexed by $\mathbb{N}$. To keep this distinction visible, one could name the σ-frame notions above *locally σ-finite*, *σ-paracompact*, and *σ-shrinkable*, respectively; we retain the terminology of [9, 7] for consistency with the sources cited below. A σ-frame is *paracompact* if for each countable cover $\mathcal{A} = (a_n)_{n=1}^{\infty}$ of L there is a locally finite cover $\mathcal{B} = (b_m)_{m=1}^{\infty}$ of L such that for each m there is some n with $b_m \leq a_n$. A cover $(a_n)_{n=1}^{\infty}$ of σ-frame L is *shrinkable* if there is a cover $(d_n)_{n=1}^{\infty}$ of L with $d_n \prec a_n$ for every n.

Proposition 6.9. *For every completely regular frame L, the following statements are true:*

(1) $\mathrm{Coz}_c[L]$ *is paracompact.*

(2) *Each cover of* $\mathrm{Coz}_c[L]$ *is shrinkable.*

Proof. By [7, Proposition 4], every regular σ-frame is paracompact, and every cover of a regular σ-frame is shrinkable. Since $\mathrm{Coz}_c[L]$ is regular by Corollary 6.7, both (1) and (2) follow at once. □

7 Zero-Dimensional Frames and c-Complete Regularity

We now study the connection between zero-dimensional frames and the cozero part $\mathrm{Coz}_c[L]$. We introduce the notion of c-complete regularity

and prove that it characterizes zero-dimensional frames in the pointfree setting.

Definition 7.1. A frame L is a *c-completely regular frame* if $\mathrm{Coz}_c[L]$ is a base for L.

The following result is a counterpart of [17, Proposition 4.4].

Proposition 7.2. *Let L be a frame. Then L is a zero-dimensional frame if and only if L is a c-completely regular frame.*

Proof. *Necessity.* Let $a \in L$. Then there exists a family $\{a_\lambda\}_{\lambda\in\Lambda}$ of complemented elements of L such that $a = \bigvee_{\lambda\in\Lambda} a_\lambda$. Also, we know $\mathrm{coz}(e_{a_\lambda}) = a_\lambda$, which implies that $a = \bigvee_{\lambda\in\Lambda} \mathrm{coz}(e_{a_\lambda})$. Since, for every $\lambda \in \Lambda$, e_{a_λ} is an idempotent element and every idempotent element of $\mathcal{R}(L)$ belongs to $\mathcal{C}_c(L)$ (its range being finite), it follows that each $a_\lambda = \mathrm{coz}(e_{a_\lambda}) \in \mathrm{Coz}_c[L]$, and therefore, the proof is complete.

Sufficiency. By Definition 7.1, $\mathrm{Coz}_c[L]$ is a base for L, and by Proposition 5.1, for every $\alpha \in \mathcal{C}_c(L)$, $\mathrm{coz}(\alpha)$ is a countable join of complemented elements of L. Then L is a zero-dimensional frame. □

8 Conclusion

We have introduced the cozero part $\mathrm{Coz}_c[L]$ of $\mathcal{C}_c(L)$ and analyzed its algebraic and lattice-theoretic properties, using the idempotent elements of $\mathcal{R}(L)$ as our main technical tool. A central role was played by the observation, made precise in Lemma 4.9, that for every *bounded* $\alpha \in \mathcal{R}^*(L)$ and every prime ideal P of L, the pair $(P_u(\alpha), P_l(\alpha))$ always determines a Dedekind cut, with no further hypothesis on P (such as P being real or countably $\bigvee$-complete) required. This boundedness argument, together with the algebra of idempotents developed in Section 3, allowed us to show that for every completely regular frame L, the structure $\mathrm{Coz}_c[L]$ forms a σ-frame (Theorem 6.6), is regular (Corollary 6.7), is an Alexandroff algebra frame, and has the property that every one of its covers is shrinkable (Proposition 6.9). We further characterized zero-dimensional frames as exactly the c-completely regular frames, that is, those frames for which $\mathrm{Coz}_c[L]$ is a base (Theorem 7.2).

A comparison with the classical theory of $C_c(X)$ shows that, while in the point-set framework the cozero sets of functions in $C_c(X)$ form a σ-algebra, in the pointfree setting the structure $\mathrm{Coz}_c[L]$ naturally constitutes a σ-frame.

This work opens several avenues for further investigation, including the study of prime and maximal spectra of $\mathcal{C}_c(L)$, the relationship between $\mathrm{Coz}_c[L]$ and the pointfree theory of real prime ideals and σ-points developed in [10, 11], and deeper interactions with locale theory and other aspects of pointfree topology.

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