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Original Research Paper

The Restricted Rung Wiener Index of Layered Graphs

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Abstract. In many transportation and communication networks, inter-layer traffic is forced through fixed gateways, which leads to increased path lengths. To quantify this routing overhead, we introduce the Restricted Rung Wiener (RRW) index. We derive closed-form expressions for the RRW index across ladder, grid, and circular prism graphs. For ladder and grid topologies, the index is minimized when gateways are placed centrally, and the routing penalty increases as gateways shift toward the boundaries. In grid networks, this excess penalty depends solely on the rail length and is independent of the number of layers. For circular prisms, the RRW index is independent of the gateway's position, reflecting the graph's rotational symmetry.

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1 Introduction

In many real-world networks, traffic between different layers must flow through specific, fixed crossing points. For example, passengers in a

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metro system often have to switch lines at a single shared station, while data packets in communication networks might be routed through one main gateway [3, 12]. Similar constraints arise in spatial infrastructure networks modeled by graphs [1]. While forcing traffic through these central points inevitably makes paths longer, current standard graph indices fail to adequately reflect this additional routing cost.

Since its introduction in 1947 to predict boiling points based on molecular structure, the Wiener index [13], defined as $W(G) = \sum_{\{u,v\}\subset V(G)} d(u,v)$, has become a fundamental tool for analyzing structural properties of graphs and networks [4, 6, 8]. Over the years, numerous variations and generalizations of the Wiener index have been proposed to capture different structural characteristics of graphs. Exact formulas for these indices are known for many standard graph families, including paths, cycles, and Cartesian product graphs [5, 9, 11, 14].

Layered graphs, such as grid graphs $P_K \times P_N$ and ladder graphs $P_2 \times P_n$ are natural models for layered infrastructure networks [12]. Similarly, prism graphs $P_2 \times C_n$ provide simple models of cyclic layered structures and ring-like molecular architectures [7]. For all these families, the Cartesian product structure allows exact distance calculations [9, 11, 14].

Beyond simple single-layer structures, multilayer networks have received considerable attention in network science [2, 10]. In many real-world systems, such as metro networks, hierarchical routers, and hub-based logistics, inter-layer movement is not arbitrary; it is strictly limited to designated coupling nodes or gateways. However, classical invariants like the traditional Wiener index assume unrestricted routing across all edges [8]. Consequently, they fail to capture the structural realities of systems with mandatory gateway constraints, motivating the definition of a new routing-cost invariant.

Motivated by this observation, we define the Restricted Rung Wiener (RRW) index, a distance-based invariant that incorporates mandatory crossover points. For a layered graph with rung sequence $\mathbf{j} = (j_1, \dots, j_{K-1})$, the RRW index $RR_{\mathbf{j}}W(G)$ quantifies pairwise routing costs subject to these gateway constraints. In this paper, we derive the RRW index for several fundamental network topologies. The main contributions of the paper are as follows:

- We derive closed-form expressions for the RRW index of ladder, grid, and circular prism graphs.
- For the ladder L_n, we show that shifting the gateway by $d \geq 0$ positions from the optimal rung increases the index by $2nd^2$ when n is odd, and by $2nd(d-1)$ when n is even.
- For the grid $P_K \times P_N$ with the optimal (centrally located) rung sequence, we prove that the minimum overhead Δ relative to the unrestricted baseline is a cubic polynomial in N, regardless of K.
- For circular prisms CL_n, we demonstrate that the RRW index is independent of the chosen rung, a consequence of the graph's rotational symmetry.

The rest of the paper is structured as follows. Section 2 introduces the RRW index and gateway displacement overhead for ladder graphs. Section 3 extends these results to grid graphs $P_K \times P_N$. Finally, Section 4 analyzes circular prism graphs using symmetry arguments.

2 Restricted Rung Wiener Index of Ladder Graphs

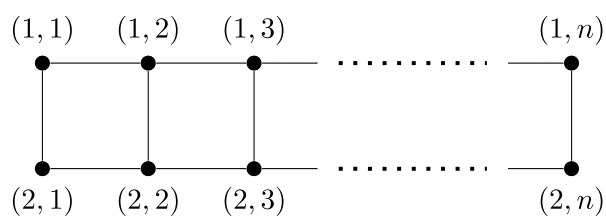

Figure 1: Ladder graph L_n

2.1 Ladder graphs L_n as $P_2 \times P_n$

The ladder graph $L_n = P_2 \times P_n$ (Figure 1) consists of two parallel paths (rails) of length n, connected by n edges (rungs). Formally, the vertex set is $V(L_n) = \{(i,k) : i \in \{1,2\}, k \in \{1,\dots,n\}\}$, where i indexes the rail and k indexes position along the rail. Edges connect consecutive vertices within each rail and corresponding vertices across rails.

The Restricted Rung Wiener (RRW) index measures shortest path distances between vertices on opposite rails, with the constraint that each path must pass through a specific rung. This addresses practical motivations like urban transit with fixed transfer stations or island networks linked by select ferry routes, identifying critical crossover points.

Definition 2.1. For a ladder graph L_n with rungs indexed $j = 1, 2, \ldots, n$, we define the Restricted Rung Wiener index through rung j as follows

$$RR_jW(L_n) = \sum_{\substack{r \in V_1 \\ s \in V_2}} d_j(r, s),$$

where $V_1 = \{(1,k) \mid 1 \leq k \leq n\}$ and $V_2 = \{(2,m) \mid 1 \leq m \leq n\}$ are the two rails, and $d_j(r,s)$ is minimized over all paths from r to s passing through edge $e_j = \{(1,j),(2,j)\}$.
For $r = (1,k)$ and $s = (2,m)$, this distance is

$$d_j(r,s) = |k - j| + 1 + |j - m|. \tag{1}$$

Theorem 2.2. *For a ladder graph L_n and fixed rung $j \in \{1, 2, \ldots, n\}$, the Restricted Rung Wiener index through rung j is*

$$RR_jW(L_n) = n^2 + 2n \sum_{k=1}^{n} |k - j|. \tag{2}$$

Proof. Let L_n be a ladder graph with n rungs, consisting of two parallel rails with vertex sets $V_1 = \{(1,k) \mid k = 1, 2, \ldots, n\}$ and $V_2 = \{(2,m) \mid m = 1, 2, \ldots, n\}$.

By Definition 2.1 and equation (1), we have

$$RR_jW(L_n) = \sum_{k=1}^{n} \sum_{m=1}^{n} d_j((1,k),(2,m)) = \sum_{k=1}^{n} \sum_{m=1}^{n} \big(|k-j| + 1 + |m-j|\big).$$

$$RR_jW(L_n) = \sum_{k=1}^{n} \sum_{m=1}^{n} |k-j| + \sum_{k=1}^{n} \sum_{m=1}^{n} 1 + \sum_{k=1}^{n} \sum_{m=1}^{n} |m-j|.$$

$$\sum_{k=1}^{n}\sum_{m=1}^{n}|k-j| = n\sum_{k=1}^{n}|k-j|.$$

By symmetry, the third term equals $n\sum_{m=1}^{n}|m-j| = n\sum_{k=1}^{n}|k-j|$. Thus,

$$RR_jW(L_n) = n^2 + 2n\sum_{k=1}^{n}|k-j|,$$

which establishes (2). $\square$

Definition 2.3. The *unrestricted cross-rail Wiener index* of L_n is defined as

$$W_0(L_n) = \sum_{\substack{r\in V_1\\ s\in V_2}} d(r,s),$$

where $d(r,s)$ is the standard shortest-path distance in L_n, and V_1, V_2 are the two rails as in Definition 2.1.

Proposition 2.4. *For the ladder graph $L_n = P_2 \times P_n$, the unrestricted cross-rail Wiener index is*

$$W_0(L_n) = n^2 + \frac{n(n^2-1)}{3}.$$

Proof. For vertices $(1,i) \in V_1$ and $(2,j) \in V_2$, the unrestricted distance is given by

$$d((1,i),(2,j)) = 1 + |i-j|.$$

Thus,

$$W_0(L_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(1+|i-j|) = n^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}|i-j|.$$

The double sum equals

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|i-j| = 2\sum_{1\le i<j\le n}(j-i) = \frac{n(n^2-1)}{3},$$

which is twice the Wiener index of the path P_n. Substituting this value into the previous expression completes the proof. $\square$

Corollary 2.5. *The $RR_jW(L_n)$ index, $1 \leq j \leq n$, satisfies*

(i) Symmetry: $RR_jW(L_n) = RR_{n-j+1}W(L_n)$;

(ii) Minimized at center: $j = \lceil n/2 \rceil$ (or $\lfloor n/2 \rfloor + 1$ when n is even);

(iii) Maximized at extremes: $j = 1$ or $j = n$.

Note 2.6. *We define two measures of overhead for the restricted rung Wiener index. The* absolute overhead *is*

$$\Delta = RR_jW - W_0,$$

$$\Delta_d = RR_{m+d}W - RR_mW,$$

where $m = \lceil n/2 \rceil$ is the optimal rung position (by Corollary 2.5; for n even, $m+1$ is equally optimal by symmetry, and m is fixed as the reference throughout).

Thus, Δ measures the increase relative to the unrestricted baseline W_0, whereas Δ_d measures the deviation from the optimal restricted position m when displaced by d rungs. In Section 3, Δ_d extends naturally to $\Delta_{\mathbf{d}}$ for displacement vectors $\mathbf{d} = (d_1, \ldots, d_{K-1})$ in grid graphs.

Remark 2.7. The overhead relative to unrestricted routing is

$$\Delta = RR_jW(L_n) - W_0(L_n) = 2n \sum_{k=1}^{n} |k - j| - \frac{n(n^2-1)}{3},$$

minimized when $j = \lceil n/2 \rceil$ and maximized at $j \in \{1, n\}$, quantifying the penalty of constrained routing.

Theorem 2.8. *Let L_n denote the ladder graph with n rungs, and let $m = \lceil n/2 \rceil$ represent the central rung minimizing the restricted rung Wiener index $RR_mW(L_n)$. For a displacement $d \geq 0$ such that $m + d \leq n$, the relative overhead Δ_d (as defined in Note 2.6)*

satisfies

$$\Delta_d = \begin{cases} 2nd^2, & \text{if } n \text{ is odd,} \\ 2nd(d-1), & \text{if } n \text{ is even.} \end{cases}$$

Proof. From Theorem 2.2, let $S_j = \sum_{k=1}^{n} |k-j|$, so

$$\Delta_d = 2n(S_{m+d} - S_m).$$

The closed form of S_j is

$$S_j = \frac{(j-1)j}{2} + \frac{(n-j)(n-j+1)}{2}.$$

Case 1: $n = 2\ell + 1$ (odd), $m = \ell + 1$.

$$S_{m+d} = \frac{(\ell+d)(\ell+d+1)}{2} + \frac{(\ell-d)(\ell-d+1)}{2},$$

$$S_m = \frac{\ell(\ell+1)}{2} + \frac{\ell(\ell+1)}{2} = \ell(\ell+1).$$

$$S_{m+d} - S_m = d^2, \quad \Delta_d = 2nd^2.$$

Case 2: $n = 2\ell$ (even), $m = \ell$.

$$S_{m+d} = \frac{(\ell+d-1)(\ell+d)}{2} + \frac{(\ell-d)(\ell-d+1)}{2},$$

$$S_m = \frac{(\ell-1)\ell}{2} + \frac{\ell(\ell+1)}{2} = \ell^2.$$

$$S_{m+d} - S_m = d(d-1), \quad \Delta_d = 2nd(d-1).$$

□

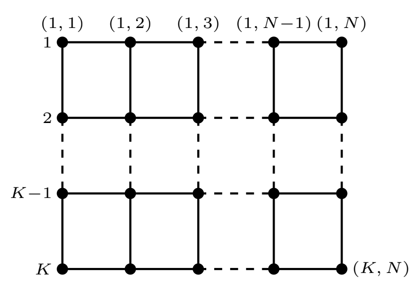

Figure 2: Grid Graph $P_K \times P_N$

3 Restricted Rung Wiener Index of Grid Graphs $P_K \times P_N$

The RRW index can be generalized to higher dimensional grid graphs, where paths are constrained to pass through selected rungs across multiple rails. Let $G = P_K \times P_N$ be a grid graph with vertex sets $V_i = \{(i, \ell) : 1 \le \ell \le N\}$, $i = 1, \dots, K$, where edges connect consecutive vertices within each rail and corresponding vertices across adjacent rails.

Definition 3.1. Let $G = P_K \times P_N$ be a grid graph (Fig 2) and $\mathbf{j} = (j_1, j_2, \dots, j_{K-1})$ a rung sequence where $1 \le j_i \le N$ for each i.

For vertices $r = (1, \ell) \in V_1$ and $s = (K, m) \in V_K$ with $1 \le \ell, m \le N$, the restricted distance $d_{\mathbf{j}}(r, s)$ is the length of the shortest path from r to s that passes through rung (i, j_i) to $(i+1, j_i)$ for each $i = 1, 2, \dots, K-1$:

$$d_{\mathbf{j}}(r, s) = |\ell - j_1| + \left(\sum_{i=1}^{K-2} |j_i - j_{i+1}| \right) + |j_{K-1} - m| + (K-1). \tag{3}$$

The Restricted Rung Wiener (RRW) index with respect to $\mathbf{j}$ is defined by

$$RR_{\mathbf{j}}W(P_K \times P_N) = \sum_{r \in V_1} \sum_{s \in V_K} d_{\mathbf{j}}(r, s).$$

Theorem 3.2. *For a grid graph $G = P_K \times P_N$ with $K \ge 2$ rails and $N \ge 1$ vertices per rail, and a rung sequence $\mathbf{j} = (j_1, \dots, j_{K-1})$, we have*

$$RR_{\mathbf{j}}W(P_K \times P_N) = N^2(K-1) + N^2 \sum_{i=1}^{K-2} |j_i - j_{i+1}| + N \left(\sum_{\ell=1}^{N} |\ell - j_1| + \sum_{m=1}^{N} |j_{K-1} - m| \right).$$

Proof. From Definition 3.1, we have

$$RR_{\mathbf{j}}W(P_K \times P_N) = \sum_{\ell=1}^{N} \sum_{m=1}^{N} d_{\mathbf{j}}\big((1, \ell), (K, m)\big).$$

Substituting the distance equation (3), we obtain

$$RR_{\mathbf{j}}W(P_K \times P_N) = \sum_{\ell=1}^{N} \sum_{m=1}^{N} \Big(|\ell - j_1| + \sum_{i=1}^{K-2} |j_i - j_{i+1}| + |j_{K-1} - m| + (K-1) \Big).$$

Evaluating each contribution:

$$\sum_{\ell=1}^{N}\sum_{m=1}^{N}(K-1) = N^2(K-1),$$

$$\sum_{\ell=1}^{N}\sum_{m=1}^{N}\left(\sum_{i=1}^{K-2}|j_i - j_{i+1}|\right) = N^2\sum_{i=1}^{K-2}|j_i - j_{i+1}|,$$

$$\sum_{\ell=1}^{N}\sum_{m=1}^{N}|\ell - j_1| = N\sum_{\ell=1}^{N}|\ell - j_1|, \qquad \sum_{\ell=1}^{N}\sum_{m=1}^{N}|j_{K-1} - m| = N\sum_{m=1}^{N}|j_{K-1} - m|.$$

Thus, the closed form becomes

$$RR_{\mathbf{j}}W(P_K \times P_N) = N^2(K-1) + N^2\sum_{i=1}^{K-2}|j_i - j_{i+1}| + N\left(\sum_{\ell=1}^{N}|\ell - j_1| + \sum_{m=1}^{N}|j_{K-1} - m|\right).$$

□

Corollary 3.3. *For the centered rung sequence $\mathbf{j}^* = (m, m, \ldots, m)$ with $m = \lceil N/2 \rceil$, the minimum value of the Restricted Rung Wiener index is*

$$RR_{\mathbf{j}^*}W(P_K \times P_N) = N^2(K-1) + \begin{cases} \dfrac{N(N^2-1)}{2} & \text{if } N \text{ is odd,} \\ \dfrac{N^3}{2} & \text{if } N \text{ is even.} \end{cases}$$

Proof. By Theorem 3.2, the RRW index decomposes into separable terms. The transition term $N^2\sum_{i=1}^{K-2}|j_i - j_{i+1}|$ is non-negative and vanishes for the constant sequence $\mathbf{j}^* = (m, \ldots, m)$. The remaining access cost is

$$N\left(\sum_{\ell=1}^{N}|\ell - m| + \sum_{k=1}^{N}|m - k|\right) = 2N\sum_{\ell=1}^{N}|\ell - m|.$$

The sum of absolute deviations from the median m equals

$$\sum_{\ell=1}^{N}|\ell - m| = \begin{cases} \dfrac{N^2-1}{4}, & N \text{ odd,} \\ \dfrac{N^2}{4}, & N \text{ even.} \end{cases}$$

Multiplying by $2N$ and adding the base term $N^2(K-1)$ gives the stated formula. □

Corollary 3.4. *For $K = 2$, Theorem 3.2 reduces to the ladder graph formula given in Theorem 2.2.*

Proof. For $K = 2$, the transition term $\sum_{i=1}^{K-2} |j_i - j_{i+1}|$ vanishes. Moreover $j_{K-1} = j_1 = j$, since there is only one rung. The formula reduces to $RR_jW(P_2 \times P_N) = N^2 + 2N \sum_{k=1}^{N} |k-j|$, identical to Theorem 2.2 with $n = N$. □

Definition 3.5. For a grid graph $G = P_K \times P_N$ with $K \geq 2$ rails and $N \geq 1$ vertices per rail, the sum of distances between all pairs of vertices (r, s) where r is in the first rail (V_1) and s is in the last rail (V_K) is denoted by $W_0(P_K \times P_N)$. This unrestricted sum is defined as:

$$W_0(P_K \times P_N) = \sum_{r \in V_1} \sum_{s \in V_K} d(r, s),$$

where $d(r, s)$ represents the standard shortest path distance between vertices r and s in the grid graph.

Proposition 3.6. *For the grid graph $G = P_K \times P_N$ with $K \geq 2$ rails and $N \geq 1$ vertices per rail*

$$W_0(P_K \times P_N) = N^2(K-1) + \frac{N(N^2-1)}{3}.$$

Proof. The distance between vertices $(1, i)$ and (K, j) in the grid is $d((1, i), (K, j)) = (K-1) + |j - i|$. Thus,

$$W_0 = \sum_{i=1}^{N} \sum_{j=1}^{N} \big[(K-1) + |j-i|\big] = N^2(K-1) + \sum_{i=1}^{N} \sum_{j=1}^{N} |j-i|.$$

The double sum equals $2 \sum_{1 \leq i < j \leq N} (j - i)$, which simplifies to $\frac{N(N^2-1)}{3}$ (twice the Wiener index of P_N). Substituting yields the formula. □

Remark 3.7. The unconstrained sum W_0 serves as a baseline for inter-rail connectivity. By Theorem 3.2 and Proposition 3.6, the overhead Δ

(extending Note 2.6) is

$$\begin{aligned}\Delta &= RR_{\mathbf{j}}W(G) - W_0 \\ &= N^2 \sum_{i=1}^{K-2} |j_i - j_{i+1}| + N\left(\sum_{\ell=1}^{N} |\ell - j_1| + \sum_{m=1}^{N} |j_{K-1} - m|\right) - \frac{N(N^2-1)}{3}.\end{aligned}$$

Corollary 3.8. *The minimum overhead relative to the unrestricted baseline, attained at* $\mathbf{j} = \mathbf{j}^*$, *is*

$$\Delta_{\min} = RR_{\mathbf{j}^*}W(P_K \times P_N) - W_0(P_K \times P_N) = \begin{cases} \dfrac{N(N^2-1)}{6}, & N \text{ odd}, \\ \dfrac{N(N^2+2)}{6}, & N \text{ even}. \end{cases}$$

Proof. The result follows directly from Corollary 3.3 and Proposition 3.6, by subtracting $W_0(P_K \times P_N) = N^2(K-1) + \frac{N(N^2-1)}{3}$ from $RR_{\mathbf{j}^*}W(P_K \times P_N)$ in each case. $\square$

Theorem 3.9. *For the grid graph* $P_K \times P_N$ *with optimal rung sequence* $\mathbf{j}^* = (m, \ldots, m)$, *where* $m = \lceil N/2 \rceil$, $\mathbf{j} = (m + d_1, \ldots, m + d_{K-1})$, $\mathbf{d} = (d_1, d_2, ..., d_{K-1})$, *with* $d_i \in \mathbb{Z}$ *and* $1 \le m + d_i \le N$. *Then the overhead* $\Delta_{\mathbf{d}}$ *(extending Note 2.6) is*

$$\Delta_{\mathbf{d}} = RR_{\mathbf{j}}W - RR_{\mathbf{j}^*}W = \begin{cases} N^2 \displaystyle\sum_{i=1}^{K-2} |d_i - d_{i+1}| + N(d_1^2 + d_{K-1}^2), & N \text{ odd}, \\ N^2 \displaystyle\sum_{i=1}^{K-2} |d_i - d_{i+1}| + N\left(f(d_1) + f(d_{K-1})\right), & N \text{ even}, \end{cases}$$

where for N *even,* $f(d) = \begin{cases} d(d-1), & d \ge 0, \\ |d|(|d|+1), & d < 0. \end{cases}$

Proof. From Theorem 3.2, the RRW index is

$$RR_{\mathbf{j}}W = N^2(K-1) + N^2 \sum_{i=1}^{K-2} |j_i - j_{i+1}| + N\left(S_{j_1} + S_{j_{K-1}}\right),$$

where $S_j = \sum_{k=1}^{N} |k - j|$. For the optimal sequence $\mathbf{j}^* = (m, \ldots, m)$,

$$RR_{\mathbf{j}^*}W = N^2(K-1) + N(2S_m).$$

Substitute $j_i = m + d_i$, then

$$|j_i - j_{i+1}| = |d_i - d_{i+1}|, \quad S_{j_1} = S_{m+d_1}, \quad S_{j_{K-1}} = S_{m+d_{K-1}}.$$

Hence

$$\Delta_{\mathbf{d}} = N^2 \sum_{i=1}^{K-2} |d_i - d_{i+1}| + N\left[(S_{m+d_1} - S_m) + (S_{m+d_{K-1}} - S_m)\right].$$

As shown in the proof of Theorem 2.8, the deviation of $S_j = \sum_{k=1}^{N} |k-j|$ from its minimum satisfies

$$S_{m+d_i} - S_m = \begin{cases} d_i^2, & N \text{ odd}, \\ \begin{cases} d_i(d_i - 1), & d_i \geq 0, \\ |d_i|(|d_i| + 1), & d_i < 0, \end{cases} & N \text{ even}. \end{cases}$$

which yields the stated formula. □

Example 3.10. Consider the grid graph $P_3 \times P_5$ with rung sequence $\mathbf{j} = (2, 4)$. By Theorem 3.2, the components of the routing cost are computed as follows.
The base term equals $N^2(K-1) = 25 \times 2 = 50$. The transition contribution is $|2-4| = 2$, giving $N^2 \cdot 2 = 50$.
For the access term we have $\sum_{\ell=1}^{5} |\ell - 2| = 7$ and $\sum_{m=1}^{5} |m-4| = 7$, hence $N(7+7) = 70$. Thus $RR_{(2,4)}W(P_3 \times P_5) = 50 + 50 + 70 = 170$.
The baseline weight between the first and last rails is

$$W_0(P_3 \times P_5) = 50 + \frac{5(25-1)}{3} = 90.$$

This equals the sum of distances from all vertices on rail 1 to all vertices on rail 3 and serves as a lower bound for routing costs. For the optimal sequence $\mathbf{j}^* = (3,3)$ we obtain $RR_{(3,3)}W = 50 + 0 + 60 = 110$.
Hence the overhead relative to the optimal routing is $\Delta_{\mathbf{d}} = 170 - 110 = 60$, while the overhead relative to the baseline is $\Delta = RR_{\mathbf{j}}W - W_0 = 170 - 90 = 80$.

Table 1 validates Theorem 3.9 numerically for $P_4 \times P_5$; all computed values match the formula exactly.

Table 1: Validation of Theorem 3.9 for $P_4 \times P_5$. The displacement vector $\mathbf{d} = (d_1, d_2, d_3)$ indicates deviations from the optimal central rung sequence $(3,3,3)$. Column $\Delta_{\mathbf{d}}$ shows the overhead relative to the optimal rung-restricted path, and Δ shows the total overhead relative to the unrestricted baseline W_0 (see Note 2.6).

Type	$\mathbf{j}$	$\mathbf{d}$	$\Delta_{\mathbf{d}}$	Δ
Optimal	$(3,3,3)$	$(0,0,0)$	0	20
Const. shift	$(2,2,2)$	$(-1,-1,-1)$	10	30
Const. shift	$(1,1,1)$	$(-2,-2,-2)$	40	60
Single layer	$(3,4,3)$	$(0,1,0)$	50	70
Single layer	$(3,3,4)$	$(0,0,1)$	30	50
Two layers	$(2,3,4)$	$(-1,0,1)$	60	80
Worst case	$(1,5,1)$	$(-2,2,-2)$	240	260
Worst case	$(5,1,5)$	$(2,-2,2)$	240	260

4 Restricted Rung Wiener Index for Circular Ladder Graphs $CL_n = P_2 \times C_n$ (Prism Graphs)

Having established the RRW index for linear structures, we now examine circular variants.

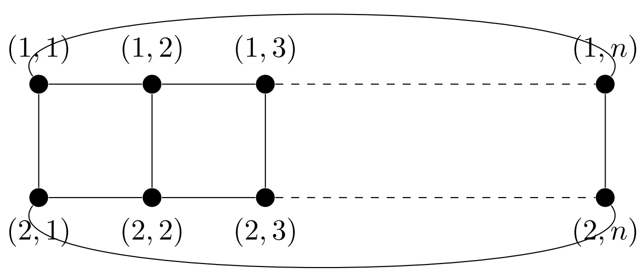

Figure 3: The Circular Ladder Graphs CL_n

Definition 4.1. For the prism graph CL_n (Figure 3) the distance between two positions $k, j \in \{1, \ldots, n\}$ on an n-cycle is

$$d_c(k, j) = \min\{|k - j|, n - |k - j|\},$$

representing the shortest path length between positions k and j along the cycle. The Restricted Rung Wiener index through rung j is

$$RR_jW(CL_n) = \sum_{k=1}^{n}\sum_{m=1}^{n} d_j((1,k),(2,m)),$$

where the restricted distance from vertex $(1,k)$ to vertex $(2,m)$ via rung j is

$$d_j((1,k),(2,m)) = d_c(k,j) + 1 + d_c(j,m).$$

Theorem 4.2. *For the Circular Ladder Graph CL_n and any rung $j \in \{1,\dots,n\}$, the quantity $RR_jW(CL_n)$ is independent of j by rotational symmetry. Moreover,*

$$RR_jW(CL_n) = \begin{cases} n^2 + \dfrac{n^3}{2} & \text{if } n \text{ is even,} \\ \\ n^2 + \dfrac{n(n^2-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Definition 4.1

$$RR_jW(CL_n) = \sum_{k=1}^{n}\sum_{m=1}^{n} d_j((1,k),(2,m)), \quad \text{where } d_j((1,k),(2,m)) = d_c(k,j)+1+d_c(j,m).$$

Thus,

$$RR_jW(CL_n) = n\sum_{k=1}^{n} d_c(k,j) + n^2 + n\sum_{m=1}^{n} d_c(j,m) = n^2 + 2n\sum_{k=1}^{n} d_c(k,j),$$

since $d_c(j,m) = d_c(m,j)$.

The quantity $\sum_{k=1}^{n} d_c(k,j)$ is invariant under any rotation $\rho_t(v) = v + t \pmod{n}$ of C_n, which preserves all circular distances. Therefore, it suffices to evaluate $\sum_{k=1}^{n} d_c(k,1)$.

Case 1. $n = 2m$ (even). For $k \leq m$, $d_c(k,1) = k-1$; for $k > m$, $d_c(k,1) = 2m - k + 1$. Thus,

$$\sum_{k=1}^{n} d_c(k,1) = \sum_{i=0}^{m-1} i + \sum_{\ell=1}^{m} \ell = \frac{(m-1)m}{2} + \frac{m(m+1)}{2} = m^2 = \frac{n^2}{4},$$

$$RR_jW(CL_n) = n^2 + \tfrac{n^3}{2}.$$

Case 2. $n = 2m+1$ (odd). For $k \le m+1$, $d_c(k,1) = k-1$; for $k > m+1$, $d_c(k,1) = 2m+2-k$. Therefore

$$\sum_{k=1}^{n} d_c(k,1) = \sum_{i=0}^{m} i + \sum_{\ell=1}^{m} \ell = m(m+1) = \tfrac{n^2-1}{4},$$

Hence,

$$RR_jW(CL_n) = n^2 + \frac{n(n^2-1)}{2}.$$

This completes the proof. $\square$

Proposition 4.3. *For the prism graph CL_n, the sum of unrestricted distances between all pairs of vertices where one is in rail 1 and the other is in rail 2 is*

$$W_0(CL_n) = n^2 + 2 \cdot W(C_n),$$

where $W(C_n)$ is the Wiener index of the n-cycle. Explicitly,

$$W_0(CL_n) = \begin{cases} n^2 + \dfrac{n^3}{4} & \text{if } n \text{ is even,} \\[2ex] n^2 + \dfrac{n(n^2-1)}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For vertices $(1,i) \in V_1$ and $(2,j) \in V_2$, the unrestricted distance is

$$d((1,i),(2,j)) = 1 + d_c(i,j).$$

Thus,

$$W_0(CL_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} d((1,i),(2,j)) = \sum_{i=1}^{n}\sum_{j=1}^{n}(1 + d_c(i,j)) = n^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} d_c(i,j).$$

The double sum $\sum_{i=1}^{n}\sum_{j=1}^{n} d_c(i,j)$ counts each unordered pair $\{i,j\}$ twice (once as (i,j) and once as (j,i)), plus n zero terms (when $i = j$). Therefore:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_c(i,j) = 2 \sum_{1 \le i < j \le n} d_c(i,j) = 2W(C_n),$$

where $W(C_n)$ is the Wiener index of the cycle C_n.

The Wiener index of C_n is well known (see, e.g., [8]):

$$W(C_n) = \begin{cases} \dfrac{n^3}{8} & \text{if } n \text{ is even,} \\ \dfrac{n(n^2-1)}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$W_0(CL_n) = n^2 + 2W(C_n) = \begin{cases} n^2 + \dfrac{n^3}{4} & \text{if } n \text{ is even,} \\ n^2 + \dfrac{n(n^2-1)}{4} & \text{if } n \text{ is odd.} \end{cases}$$

□

Corollary 4.4. *The overhead imposed by routing through a fixed gateway in CL_n is*

$$\Delta(CL_n) = RR_jW(CL_n) - W_0(CL_n) = \begin{cases} \dfrac{n^3}{4} & \textit{if } n \textit{ is even,} \\ \dfrac{n(n^2-1)}{4} & \textit{if } n \textit{ is odd.} \end{cases}$$

This overhead is independent of the choice of gateway j, a consequence of rotational symmetry in ring topologies.

Proof. Direct calculation from Theorem 4.2 and Proposition 4.3.

If n is even,

$$\Delta(CL_n) = \left(n^2 + \frac{n^3}{2}\right) - \left(n^2 + \frac{n^3}{4}\right) = \frac{n^3}{4}.$$

If n is odd,

$$\Delta(CL_n) = \left(n^2 + \frac{n(n^2-1)}{2}\right) - \left(n^2 + \frac{n(n^2-1)}{4}\right) = \frac{n(n^2-1)}{4}.$$

□

Table 2 illustrates the numerical agreement with the derived formulas for $RR_jW(CL_n)$, $W_0(CL_n)$, and $\Delta(CL_n)$ for $3 \leq n \leq 10$.

Table 2: Values of $RR_jW(CL_n)$, $W_0(CL_n)$, and $\Delta(CL_n)$ for $n = 3$ to 10

n	$RR_jW(CL_n)$	$W_0(CL_n)$	$\Delta(CL_n)$
3	21	15	6
4	48	32	16
5	85	55	30
6	144	90	54
7	217	133	84
8	320	192	128
9	441	261	180
10	600	350	250

Conclusion

We have introduced the Restricted Rung Wiener index, denoted by RRW, to quantify routing costs in layered networks with mandatory gateway constraints. The overhead relative to the unrestricted baseline is measured by $\Delta = RR_jW - W_0$, while the overhead relative to the central (optimal) gateway is measured by $\Delta_d = RR_jW - RR_{\mathsf{j}^*}W$. Explicit closed-form formulas were derived for ladder graphs L_n, general grid graphs $P_K \times P_N$, and prism graphs CL_n. In linear topologies (ladders and grids), the RRW index is minimized at central gateways, with the minimum overhead Δ independent of the number of layers K, while the displacement overhead Δ_d increases with distance from the optimal gateway; in circular topologies, rotational symmetry yields uniform overhead across all gateway choices. These closed-form results provide a quantitative basis for gateway placement decisions in such layered systems. Several directions remain open. For instance, it would be interesting to examine toroidal or cylindrical graphs, or to allow crossover costs that vary by gateway position.

Declaration regarding generative AI
Generative AI tools were used only to enhance language clarity and phrasing in parts of the manuscript. The author bears full responsibility for all content and originality.

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