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Congestion in DEA under Weight Restriction Using Common Weights

A. Noura
Sistan Baluchestan University

F. Hosseinzadeh Lotfi
Science and Research branch, Islamic Azad University

A. Hajihosseini*
Zahedan Branch, Islamic Azad University

Abstract. In this article, we will propose a new approach for evaluating congestion in decision making units (DMUs) by weight restriction using common weights based on comparison of inputs. The main advantage of the proposed method is to solve one linear programming for all DMUs. Therefore, this method greatly reduces the computational costs. The proposed model for measuring congestion will be shown by examples.

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Keywords and Phrases: Data envelopment analysis, Congestion, Weight Restriction, Common Weights

1. Introduction

Charnes et al. [4] introduced data envelopment analysis to assess the performances of a group of DMUs that utilize multiple inputs to produce multiple outputs. Congestion in DEA is said to occur when an increase

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*Corresponding Author

in one or more inputs can be associated with a decrease in one or more outputs, without improving any other inputs or outputs [7]. First, the research on congestion began by Fare and Svensson[11]. Then, it was completed in 1983 and 1985 by Fare and Grosskopf [9, 10]. They presented a model according to the concept of data envelopment analysis. Another approach was presented by Cooper et al. [8]. Brockett et al. [2] and Cooper et al. [7] developed a new DEA-based approach to measure input congestion. While a significant literature exists on the subject, the two latter methodologies are considered to be fundamental congestion consideration. Other notable methods for measuring congestion are Noura et al.'s methods [16, 12].

A lot of studies have focused on common set of weights, but the idea of common weights in DEA was first introduced by Cook et al. [5] and Roll et al. [18] in the context of applying DEA to evaluate highway maintenance units. Meanwhile, the imposition of weight restrictions has been recognized as one of the important factors so as applying DEA to actual situations and several models are developed for this purpose. These include the Assurance Region (AR) model by Thompson et al. [19] and the Con-ratio Approach by Charnes et al. [3].

In this paper, first we introduce the weight restriction model. Then, we propose a new model for measuring congestion in DEA with weight restriction by using common weights which is based on Noura et al.'s methodology [12]. Also numerical examples show precision of the proposed model.

2. Weight Restriction Model

In this section, a more general model (is called the "weight restriction" model) is introduced. Assume, we have n DMUs that are evaluated in terms of m inputs and s outputs. Let x_{ij} and y_{rj} be input and output values of DMUj for i=1, $\cdots$, m and r=1, $\cdots$,s. Spot BCC model[1], the efficiencies of the n DMUs using weight restrictions are measured by the following model(model1).

$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m} v_i x_{ip} + v_0, \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rp} = 1, \\
& \sum_{i=1}^{m} v_i x_{ij} + v_0 - \sum_{r=1}^{s} u_r y_{rj} \le 0, \qquad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i p_{ik} \le 0, \qquad k = 1, \ldots, 2m-2, \\
& \sum_{r=1}^{s} u_r q_{rt} \le 0, \qquad t = 1, \ldots, 2s-2, \\
& u_r \ge \epsilon, \qquad r = 1, \ldots, s, \\
& v_i \ge \epsilon, \qquad i = 1, \ldots, m.
\end{aligned} \tag{1}
$$

Where $P_{m*2m-2} = (p_{ik})$ and $Q_{s*2s-2} = (q_{rt})$ are matrices that are associated with weight restrictions as described below[19]. For example, if ratio of weights for initial and ith of input and initial and rth of output be as follows:

$$l_{1i} \le \frac{v_i}{v_1} \le u_{1i}, \qquad l_{1i} v_1 \le v_i \le u_{1i} v_1, \qquad \text{i=2, 3, \ldots, m,}$$

$$L_{1r} \le \frac{u_r}{u_1} \le U_{1r}, \qquad L_{1r} u_1 \le u_r \le U_{1r} u_1, \qquad \text{r=2, 3, \ldots, s.}$$

Where l_{1i} and u_{1i} are lower and upper bound of $\frac{v_i}{v_1}$, and L_{1r} and U_{1r} are lower and upper bound of $\frac{u_r}{u_1}$. In this case the matrices P and Q are defined as follows:

$$P = \begin{bmatrix} l_{12} & -u_{12} & l_{13} & -u_{13} & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix},$$

$$Q = \begin{bmatrix} L_{12} & -U_{12} & L_{13} & -U_{13} & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix}.$$

3. Congestion with Weight Restriction Using Common Weights

Based on Noura et al. methodology [17], we propose the following multi objective linear programming (MOLP) with weight restriction using common set of weights (model2).

$$\begin{aligned} Min \quad & v^t x_j + v_0 - u^t y_j, \qquad\qquad j = 1, \ldots, n, \\ s.t. \quad & v^t x_j + v_0 - u^t y_j \geq 0, \qquad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i p_{ik} \leq 0, \qquad k = 1, \ldots, 2m-2, \\ & \sum_{r=1}^{s} u_r q_{rt} \leq 0, \qquad t = 1, \ldots, 2s-2, \\ & u^t \geq 1_s \epsilon, \\ & v^t \geq 1_m \epsilon. \end{aligned} \tag{2}$$

Where the decision variables are the weight vectors $v^t = (v_1, \cdots, v_m)$, $u^t = (u_1, \cdots, u_s)$ and $x_j^t = (x_{1j}, \cdots, x_{mj})$, $y_j^t = (y_{1j}, \cdots, y_{sj})$ are the input and output vectors for $DMU_j (j = 1, \cdots, n)$. Equal weights method is applied to solve the above MOLP. It also assumes that all weights are equal to one. Therefore we obtain model3.

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n}(v^t x_j + v_0 - u^t y_j), \\
s.t. \quad & v^t x_j + v_0 - u^t y_j \geq 0, \qquad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i p_{ik} \leq 0, \qquad k = 1, \ldots, 2m-2, \\
& \sum_{r=1}^{s} u_r q_{rt} \leq 0, \qquad t = 1, \ldots, 2s-2, \\
& u^t \geq 1_s \epsilon, \\
& v^t \geq 1_m \epsilon,
\end{aligned}
\tag{3}
$$

which implies model4.

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n}(v^t x_j + v_0 - u^t y_j), \\
s.t. \quad & v^t x_j + v_0 - u^t y_j - \Delta_j = 0 \qquad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i p_{ik} \leq 0 \qquad k = 1, \ldots, 2m-2, \\
& \sum_{r=1}^{s} u_r q_{rt} \leq 0 \qquad t = 1, \ldots, 2s-2, \\
& u^t \geq 1_s \epsilon, \\
& v^t \geq 1_m \epsilon, \\
& \Delta_j \geq 0, \qquad j = 1, \ldots, n.
\end{aligned}
\tag{4}
$$

From $v^t x_j + v_0 - u^t y_j - \Delta_j = 0$ we have $v^t x_j + v_0 - u^t y_j = \Delta_j$. Hence we obtain model5

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n} \Delta_j, \\
s.t. \quad & v^t x_j + v_0 - u^t y_j - \Delta_j = 0, \qquad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i p_{ik} \leq 0, \qquad k = 1, \ldots, 2m-2, \\
& \sum_{r=1}^{s} u_r q_{rt} \leq 0, \qquad t = 1, \ldots, 2s-2, \\
& u^t \geq 1_s \epsilon, \\
& v^t \geq 1_m \epsilon, \\
& \Delta_j \geq 0, \qquad j = 1, \ldots, n.
\end{aligned}
\tag{5}
$$

Common weight in above model is similar to Liu Peng's model [14] and Noura Hoseini's model [15]. Now we suppose $(u^*, v^*)^t$ will be the optimal solution of problem (5), which is called common set of weights (CSW) with weight restriction for ranking and comparing DMUs. According to achieved CSW, the efficiency score of $DMU_j (j = 1, \cdots, n)$ will be $\varphi_j^* = \frac{v^{*t} x_j + v_0^*}{u^{*t} y_j}$. If φ_j^* is equal to one then the DMU under evaluation is efficient.

Now according Noura et al. [16], the efficient set of DMUs (E) is defined as follows:

$$E = \{j : \varphi_j^* = 1\},$$

The highest value in each input among DMUs of E for all components is introduced with x_i^*.

$$x_i^* = max\{x_{ij} : j \in E\} \qquad \text{i=1,...,m.}$$

So the following revised definition is suggested to identifying congestion.

Definition 3.1. *Congestion in DMUo eventually occurs if for the optimal solution of DMUo (φ_o^*), the following condition is satisfied:*
$\varphi_o^* > 1$, *and there is at least one* $x_{io} > x_i^*, i = 1, \ldots, m$.

The amount of congestion in the ith input of DMUo is shown with s^c_{io} as follow:

$$s^c_{io} = x_{io} - x^*_i \ ,$$

The sum of all s^c_{io} is the amount of congestion in DMUo.

$$s^c_o = \sum_{i=1}^{m} s^c_{io} \ ,$$

Congestion does not present in DMUo when $x_{io} \leq x^*_i$ or $s^c_{io} = 0$ for all $i = 1, \ldots, m$.

4. Numerical Examples

Example 4.1. Table1 shows 14 public hospitals with two inputs (doctors and nurses) and two outputs (outpatient and inpatient)[6]. φ^*_j in this table is obtained from GAMS software for model5. Moreover, we get

$$0.2 \leq \frac{v_2}{v_1} \leq 5, \qquad 0.2 \leq \frac{u_2}{u_1} \leq 5,$$

With considering table 2, we have

$$E = \{DMU_3, DMU_{10}\},$$

$$x^*_1 = \max Input1 = 8554, \quad \forall \ DMU \in E,$$

Now, due to definition 3.1;

For DMU_5, $x_{1,5} = 8836 > x^*_1 \Rightarrow DMU_5$ has congestion in Input 1. and $S^c_{1,5} = 8836 - 8554 = 282$.

For DMU_{13}, $x_{1,13} = 13479 > x^*_1 \Rightarrow DMU_{13}$, has congestion in Input 1. and $S^c_{1,13} = 13479 - 8554 = 4925$.

For DMU_{14}, $x_{1,14} = 21808 > x^*_1 \Rightarrow DMU_{14}$ has congestion in Input 1. and $S^c_{1,13} = 21808 - 8554 = 13254$.

Table 1: Data of 14 public hospitals, source: Tone et al. [6]

Hospital	Inpatients	Outpatients	Doctors	Nurses
H1	101225	97775	3008	20980
H2	130580	135871	3985	25643
H3	168473	133655	4324	26978
H4	100407	46243	3534	25361
H5	215616	176661	8836	40796
H6	217615	182576	5376	37562
H7	167278	98880	4982	33088
H8	193393	136701	4775	39122
H9	256575	225138	8046	42958
H10	312877	257370	8554	48955
H11	227099	165274	6147	45514
H12	321623	203989	8366	55140
H13	341743	174270	13479	68037
H14	487539	322990	21808	78302

Table 2: The Efficiencies by Model 5

DMUs	φ_j^*
DMU1	1.147
DMU2	1.072
DMU3	1.000
DMU4	1.917
DMU5	1.213
DMU6	1.073
DMU7	1.412
DMU8	1.353
DMU9	1.037
DMU10	1.000
DMU11	1.338
DMU12	1.225
DMU13	1.566
DMU14	1.171

$$x_2^* = \max Input2 = 48955, \quad \forall \ DMU \in E,$$

For DMU_{12}, $x_{2,12} = 55140 > x_2^* \Rightarrow DMU_{12}$ has congestion in Input 2. and $S_{2,12}^c = 55140 - 48955 = 6185.$

For DMU_{13}, $x_{2,13} = 68037 > x_2^* \Rightarrow DMU_{13}$ has congestion in Input 2. and $S_{2,13}^c = 68037 - 48955 = 19082.$

For DMU_{14}, $x_{2,14} = 78302 > x_2^* \Rightarrow DMU_{14}$ has congestion in Input 2. and $S_{2,13}^c = 78302 - 48955 = 29347.$

Example 4.2. This example involves six hypothetical university departments (Table 3), each DMU have two inputs and three outputs[13] as follows:

Inputs
T/S: Teaching staff
R/S: Researching staff

Outputs
U/S: Undergraduate students
M/S: Master students
P: Publications

With considering table 4, we have

$$E = \{C, E\},$$

$$x_1^* = \max Input1 = 98, \quad \forall \ DMU \in E.$$

Now, due to definition 3.1, we have

For $DMU_1(A)$, $x_{1,1} = 100 > x_1^* \Rightarrow DMU_1(A)$ has congestion in Input 1. and $S_{1,1}^c = 100 - 98 = 2.$

Table 3: The Data Set of Example 2, Source: Jahanshahloo et al. [13]

Department	U/S	M/S	P	T/S	R/S
A	1540	154	59	100	70
B	1408	186	23	120	123
C	690	59	76	50	20
D	674	73	90	67	17
E	1686	197	12	98	20
F	982	63	15	76	12

Table 4: The Efficiencies of example 2 by Model 5.

DMUs	φ_j^*
A	1.046
B	1.513
C	1.000
D	1.135
E	1.000
F	1.398

For $DMU_2(B)$, $x_{1,2} = 120 > x_1^* \Rightarrow DMU_2(B)$ has congestion in Input 1. and $S_{1,2}^c = 120 - 98 = 22$.

$$x_2^* = \max Input2 = 20, \quad \forall\ DMU \in E.$$

For $DMU_1(A)$, $x_{2,1} = 70 > x_2^* \Rightarrow DMU_1(A)$ has congestion in Input 2. and $S_{2,1}^c = 70 - 20 = 50$.

For $DMU_2(B)$, $x_{2,2} = 123 > x_2^* \Rightarrow DMU_2(B)$ has congestion in Input 2. and $S_{2,2}^c = 123 - 20 = 103$.

5. Conclusion

In this paper, we measured congestion in DEA with weight restriction using common weights. So far, many methods have been presented for measuring congestion in DEA, but the measuring of congestion under weight restriction by using common set of weights is a new approach. The main advantage of the proposed method is to solve one linear programming for all DMUs. Therefore, this method greatly reduces the computational costs. Moreover, we compared the results of examples solved by pervious methods with the proposed approach via numerical examples. Due to imposing weight restriction and common weights in DEA models, the numerical examples are similar to those of the last approaches for measuring congestion.

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Abbasali Noura
Department of Mathematics
Associate Professor of Mathematics
Sistan Baluchestan University
Zahedan, Iran
E-mail: *abbasali_noura@yahoo.com*

Farhad Hosseinzadeh Lotfi
Department of Mathematics
Professor of Mathematics
Science and Research branch
Islamic Azad University
Tehran, Iran
E-mail: *farhad@hosseinzadeh.ir*

Alireza Hajihosseini
Department of Mathematics
Assistant Professor of Mathematics
Zahedan Branch
Islamic Azad University
Zahedan, Iran
E-mail: *hajihosseiny54@yahoo.com*