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Original Research Paper

Modeling Strategic Partnerships in Two-Stage Network Data Envelopment Analysis Framework under Variable Returns to Scale

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Abstract. This study proposes a novel methodological framework for analyzing strategic alliance formation within a two-stage network Data Envelopment Analysis (DEA) structure under Variable Returns to Scale (VRS). In competitive production systems, independent efficiency evaluation may overlook potential gains achievable through cooperative resource reallocation. To address this limitation, we develop an integrated

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model that explicitly incorporates bilateral alliances between decision-making units (DMUs) while preserving their internal network topology. The proposed framework allows partnering DMUs to reassign surplus inputs through an endogenous optimization mechanism, thereby improving efficiency without violating the structural dependencies between stages. Unlike traditional DEA models that treat production units as isolated entities, the present approach embeds inter-unit interaction into the efficiency assessment process. The model is formulated in multiplier form, enabling a rigorous decomposition of efficiency improvements into stage-specific contributions and providing dual-variable interpretations of alliance-driven performance gains. From a methodological perspective, the model extends network DEA theory by simultaneously accommodating VRS technology and cooperative input reallocation across distinct production systems. The empirical applicability of the framework is illustrated using data from cement manufacturing plants. The results confirm that strategic cooperation can yield substantial reductions in input redundancy, enhance both intermediate and final-stage performance, and improve overall technical efficiency. The proposed approach contributes to the literature by bridging alliance theory and network efficiency analysis and offers a structured decision-support tool for designing and evaluating cooperative strategies in multi-stage production environments.

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Keywords and Phrases: Data envelopment analysis; Two-stage network; Business partnerships; Inverse DEA; Strategic alliances; Variable returns to scale.

1 Introduction

The evaluation of productive efficiency in multi-input–multi-output environments is intrinsically linked to the characterization of the underlying production technology. DEA provides a rigorous non-parametric approach for constructing an empirical production possibility set directly from observed data. Grounded in axiomatic properties such as convexity, free disposability, and alternative returns-to-scale specifications, DEA generates a piecewise linear frontier that represents the attainable boundary of performance and yields relative technical efficiency measures without requiring an explicit functional specification (Charnes et al. [8]; Banker et al. [6]; Cooper et al. [7]).
Within its classical envelopment and multiplier formulations, DEA es-

tablishes a dual relationship between the construction of the production frontier and the endogenous determination of input–output weights. This duality-based structure enables the consistent evaluation of DMUs while avoiding exogenous price information or restrictive parametric assumptions. Owing to these methodological strengths, DEA has been extensively implemented in diverse operational contexts, including financial services, healthcare systems, and supply chain management (Cooper et al. [7]).

Nevertheless, conventional DEA models conceptualize each DMU as a single aggregated production entity, implicitly abstracting from its internal organizational architecture. Such a "black-box" representation may be insufficient for systems characterized by sequential or interconnected sub processes. In response to this limitation, the Network DEA framework was developed to explicitly incorporate intermediate products and structural linkages into the definition of the feasible technology set (Yang et al. [32–34]). By embedding internal flow balance relationships within the optimization structure, Network DEA permits stage-wise efficiency decomposition while maintaining consistency with the global production frontier.

Over recent decades, these methodological advancements have strengthened DEA's theoretical foundations and expanded its applicability to increasingly complex production environments. The integration of internal process structures, combined with flexible returns-to-scale assumptions, has reinforced DEA's role as a cornerstone methodology in performance analysis and operational research (Charnes et al. [8]; Banker et al. [6]; Cooper et al. [7]).

Recent developments in efficiency analysis have seen the rise of network DEA models, which overcome the limitations of treating DMUs as monolithic "black boxes." These models explicitly account for internal structures and interconnections among subprocesses, allowing for a more granular assessment of performance at the divisional level and enabling managers to implement stage-specific improvements (Chen et al. [9–11]; Kao [17, 18]; Cook et al. [13]; Guo et al. [16]; Kao and Hwang [19]). Unlike conventional cooperative DEA models (e.g., Liang et al. [22, 23]) that focus on partner selection or resource allocation among pre-existing efficient units, and in contrast to two-stage network DEA studies (e.g.,

Kao and Hwang [19]) that evaluate internal structures without altering resource distribution, our approach explicitly models the endogenous redistribution of inputs and outputs between partners to actively improve their efficiency scores. Furthermore, while existing network DEA models typically treat the network structure as fixed, our proposed framework allows partners to strategically reallocate resources to achieve mutual efficiency gains, thereby offering a novel decision-support tool for partnership optimization.

By representing a DMU as a series of sequentially linked sub processes, network DEA permits simultaneous evaluation of both total system efficiency and stage-level performance. Among the various architectures explored in the literature, serial two-stage configurations are particularly prominent due to their conceptual clarity and practical relevance. Such two-stage series models serve as a foundational framework, reflecting the sequential nature of many real-world production processes.

Radial two-stage DEA models under constant returns to scale (CRS) have been widely studied, with well-established theoretical properties, including the equivalence between multiplier and envelopment forms (Kao and Hwang [19]; Li et al. [21]). However, CRS often fails to capture the heterogeneous scale characteristics observed in practical production systems, making the variable returns to scale (VRS) assumption more appropriate in many contexts (Banker et al. [6]). Introducing VRS, however, increases the methodological complexity of two-stage network models. Prior research has identified critical challenges, including ambiguous frontier projections and non-equivalence between primal and dual representations, which can reduce interpretability and undermine the theoretical soundness of efficiency scores (Chen et al. [10]; Liang et al. [22, 23]). These issues typically arise from divergent scale effects between stages, creating interactions that distort the alignment between measured efficiency and the true production frontier (Lim and Zhu [24, 25]).

Recent studies have further extended the application of network DEA in practice. Zhu et al. [39] developed a data-driven methodology to assess the sustainability of China's industrial sectors under total energy consumption constraints. Zhang et al. [38] introduced efficiency evaluation models for two-stage parallel–series systems with fixed-sum outputs,

integrating stochastic multicriteria acceptability analysis (SMAA) with DEA. Together, these contributions illustrate both the methodological sophistication and the practical relevance of network DEA models for analyzing multi-stage production systems under realistic technological assumptions. Zhao and Cheng [35] obtained estimation of optimal resource allocation for Universities' industry-education integration by a novel inverse network DEA model with shared inputs. Zhanget al. [37] evaluated green economic efficiency in Chinese cities and proposed a three-stage network SBM and machine learning approach. Zhao et al. [36] developed a new approach for estimating optimal resource allocation of agricultural sustainable systems based on inverse network.

Early methodological advances in two-stage DEA predominantly employed radial efficiency measures under the assumption of constant returns to scale. While these models provided a solid analytical foundation, their effectiveness is limited in contexts where production technologies exhibit scale heterogeneity. Incorporating VRS into two-stage network models introduces substantial complexity in defining the production possibility set (Lozano et al. [26]). Previous research has demonstrated that, under VRS, conventional formulations may fail to yield consistent frontier projections and can produce discrepancies between primal and dual representations. These issues are largely associated with the interaction of scale effects across stages and the dual function of intermediate products, which serve simultaneously as outputs of the first stage and inputs to the second.

Alongside the development of network DEA, there has been increasing attention to non-radial efficiency measures, which account directly for input excesses and output shortfalls rather than assuming proportional reductions or expansions. In this vein, slack-based efficiency measures have gained prominence, offering a more realistic evaluation of inefficiency (Tone [29]). Recent enhancements of these measures emphasize identifying the closest efficient benchmark instead of the furthest, thereby providing a more actionable assessment for performance improvement. Extensions of such maximum slack-based measures to network DEA environments have demonstrated important theoretical properties, including monotonicity and decomposability across stages.

The Slacks-Based Measure (SBM) has emerged as a particularly robust

non-radial approach, integrating all input and output slacks directly into the efficiency score and remaining unaffected by weakly efficient frontiers (Kao [17, 18]). Among various network configurations, the two-stage serial structure, where first-stage outputs feed into the second stage, has attracted considerable attention. Foundational studies by Kao and Hwang [19], Liang et al. [22], and Chen, Cook, and Zhu [9] have established the methodological basis for evaluating these systems, including overall efficiency, stage-specific efficiencies, and frontier projections for underperforming units.

Despite these advances, two-stage network DEA continues to face methodological challenges. A major concern is the loss of primal-dual equivalence, a property ensuring consistency between envelopment and multiplier models in standard DEA (Chen et al. [10]). Additionally, multiplier formulations frequently encounter non-unique optimal solutions, resulting in ambiguity in assigning stage-level efficiencies to DMUs (Kao and Hwang [19]). Another persistent issue is the practical projection of intermediate measures, simultaneously first-stage outputs and second-stage inputs, onto the efficient frontier, which remains conceptually and operationally challenging (Lim and Zhu [24]). Recent applications have explored novel solutions to these challenges, such as Yin et al.'s [31] two-stage DEA framework for hotel performance, which emphasizes internal cooperation and coordinated stage-level evaluation.

To overcome the methodological limitations discussed earlier, recent studies have introduced the notion of "production scale alignment," inspired by organizational contingency theory, which emphasizes that the effectiveness of organizational structures depends on their fit with environmental conditions (Chen and Wang [11]). Chu and Zhu [12] operationalized this concept within a two-stage network DEA framework under CRS, proposing a three-step frontier projection procedure that aligns the production scale of each stage with the adjusted level of intermediate measures. Their method employs radial projection for intermediate outputs and allows flexible weighting in the multiplier formulation while ensuring total value flow equivalence, effectively restoring primal-dual correspondence and producing a unique pair of stage-specific efficiencies for each DMU.

Despite these contributions, several research gaps remain. Notably, Chu

and Zhu's [12] model is limited to the CRS assumption, leaving open the challenge of extending production scale alignment to the more realistic VRS context. Additionally, the incorporation of exogenous inputs into the second stage or the presence of undesirable outputs within the scale-matching framework has received limited attention and warrants further investigation.

The Inverse DEA methodology, introduced by Wei et al. [30], addresses a complementary question: "How much additional input is required to sustain the current efficiency level if outputs increase?" This framework provides managers with guidance for setting new production targets while maintaining efficiency. Subsequent research has expanded its application, including VRS contexts (Ghiyasi et al. [14]), unit consolidations (Amin and Ibn Boamah [4]), and cases with undesirable outputs (Amin et al. [1, 2]).

Another prominent research trajectory has focused on modeling inter-organizational collaborations, including strategic alliances, partnerships, mergers, and acquisitions. Drawing from resource dependence and transaction cost theories, these collaborations aim to generate synergies, expand market access, and enhance competitiveness (Arslan et al. [5]). Traditional DEA-based merger and acquisition studies, often using inverse DEA models, primarily evaluate efficiency gains from fully integrated entities (Amin et al. [1, 2]). However, many firms pursue partnerships without full consolidation, seeking efficiency improvements while maintaining their distinct identities. Soltanifar et al. [28] proposed a CSW-Inverse DEA model for merger target setting, while Amin and Ibn Boamah [3] introduced DEA partnership models that optimize input and output redistribution between collaborating DMUs, encompassing concepts such as production partnerships (input sharing) and sales partnerships (output sharing). Gerami [15] explored modeling business partnerships under semi-additive technologies, and Sohrabi et al. [27] developed approaches for strategic alliances using DEA-R models. Amin and Ibn Boamah [4] further advanced models for strategic alliances and partnerships in the context of mergers and acquisitions.

Nevertheless, despite these advancements, a notable gap persists: the development of Inverse DEA methodologies tailored for multi-stage network production systems. Most existing inverse DEA applications fo-

cus on single-stage units without considering internal process structures (Ghiyasi et al. [14]). In contrast, real-world production environments—including banking, energy, and supply chains—often comprise multiple sequential or parallel subunits, and neglecting these internal stages may lead to inaccurate input requirements and suboptimal resource planning.

Despite substantial advances in network DEA and partnership DEA separately, a notable research gap persists at their intersection. Existing literature on two-stage network DEA predominantly addresses the optimization of internal processes within individual DMUs, whereas partnership-oriented DEA models focus on resource allocation between independent, aggregated DMUs. To date, no study has examined how strategic alliances can be formed and optimized among DMUs that themselves possess internal two-stage network structures. This scenario is common in real-world settings; for example, bank branches, often modeled as two-stage systems involving deposit collection and lending, may collaborate to share intermediate resources such as deposits or loan officers, and manufacturing firms with distinct production and distribution stages may engage in strategic partnerships to optimize resource usage. Compared to existing approaches, our proposed model offers several distinct advantages while acknowledging certain limitations. The primary advantages include: (i) simultaneous optimization of shared resource allocation and cooperative gains through the integration of inverse DEA with cooperative game theory; (ii) provision of net resource savings rather than mere reallocation; and (iii) practical applicability for cost reduction and performance benchmarking in two-stage production systems. However, the model has limitations, including its current focus on two-stage series structures, reliance on input-orientation, and computational challenges when scaling to large-scale multi-partner alliances. Despite these limitations, our approach provides a significant advancement over conventional DEA models that treat decision-making units independently and overlook cooperative synergies.

The present study aims to address this gap by developing an Inverse DEA framework for two-stage network DMUs under VRS. The primary contribution is the formulation of a model that simultaneously accounts for internal two-stage structures and inter-DMU partnerships. Specifi-

cally, the proposed approach integrates production scale alignment, following the logic of Chu and Zhu [12], with input/output redistribution mechanisms as introduced by Amin and Ibn Boamah [3]. This allows the determination of optimal resource reallocation, including initial inputs, intermediate measures, and final outputs—between partner DMUs to enhance both overall and stage-level efficiencies, while preserving the technological constraints of each two-stage system.

The remainder of the paper is organized as follows. Section 2 introduces the two-stage network DEA model under VRS for evaluating DMU performance. Section 3 presents the proposed framework for modeling business partnerships between DMUs with two-stage network structures. Section 4 demonstrates the application of the methodology to the cement production industry in southern Iran, illustrating the resource-sharing and efficiency-improvement process. Finally, Section 5 summarizes the findings and discusses managerial and policy implications.

In addition, supplementary sections provide the theoretical underpinnings: preliminary notations and definitions are introduced first, followed by a brief review of conventional DEA models for cost and revenue efficiency. Subsequently, the paper presents fuzzy and inverse DEA extensions, examines their properties, and illustrates their practical applicability through numerical examples. These examples highlight the capacity of the proposed framework to calculate stage-wise and overall efficiency while accounting for both internal DMU structures and inter-unit resource sharing.

2 Two-Stage Network DEA

Standard DEA models treat each DMU as a black box, which fails to capture the internal process interdependencies that commonly exist in real production systems. Network DEA addresses this limitation by explicitly modeling the flow of intermediate products between interconnected stages, thereby enabling more comprehensive and informative efficiency evaluations.

Among various network configurations, the two-stage serial structure is one of the most widely studied. In this framework, the first stage transforms external inputs into intermediate products, which sub-

sequently serve as inputs to the second stage for generating the final outputs. Early work by Kao and Hwang [19] defined the overall system efficiency as the product of the efficiencies of the two stages under the assumption of constant returns to scale (CRS). However, extending this multiplicative formulation to variable returns to scale (VRS) presents theoretical difficulties. To overcome this limitation, Lee [20] proposed a loss-based additive decomposition of overall system inefficiency into the inefficiencies of the individual stages. This approach is applicable under both CRS and VRS technologies, ensuring theoretical consistency and practical interpretability while preserving the linkage between stages through shared multipliers. Consequently, the two-stage VRS network DEA model has become an effective and widely adopted framework for evaluating the performance of complex production systems with interconnected processes. Consider a network consisting of two sequential processes, as illustrated in Figure 1. Let there be n decision-making units (DMUs), each operating according to the two-stage network structure shown in Figure 1.

Suppose that there are n DMUs to be evaluated, and each DMU_o uses

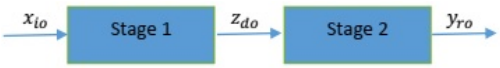


Figure 1: Two-stage network structure.

m inputs x_{io}, $(i = 1, \ldots, m)$, to produce D intermediate measures z_{do}, $(d = 1, \ldots, D)$, in the first stage. Also, each DMU_o uses D intermediate measures z_{do}, $(d = 1, \ldots, D)$, to produce s outputs y_{ro}, $(r = 1, \ldots, s)$, in the second stage.

The vector of external inputs is defined as

$$X_j = (x_{1j}, \ldots, x_{mj}),$$

and the intermediate vector is defined as

$$Z_j = (z_{1j}, \ldots, z_{Dj}).$$

In the second stage, the vector $Z_j = (z_{1j}, \ldots, z_{Dj})$ is used to produce the vector of external outputs

$$Y_j = (y_{1j}, \ldots, y_{sj}),$$

where $j = 1, \ldots, n$. Assume that all data are nonnegative.

The efficiency scores of DMU_j, $j = 1, \ldots, n$, for the first stage, second stage, and overall system are defined as follows (Lee [20]):

$$E_j^1 = \frac{\sum_{d=1}^{D} w_d z_{dj} - u_o^1}{\sum_{i=1}^{m} v_i x_{ij}}, \qquad E_j^2 = \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^2}{\sum_{d=1}^{D} w_d z_{dj}},$$
$$E_j = \frac{\sum_{r=1}^{s} u_r y_{rj} - u_o^1 - u_o^2}{\sum_{i=1}^{m} v_i x_{ij}}. \tag{1}$$

We suppose that the efficiency score of DMU_o is determined by maximizing the above objective function under the conditions that, for all DMUs, the efficiency scores of the two processes and the overall system are less than or equal to one. Therefore, we impose the following constraints:

$$E_j^1 \leq 1, \qquad E_j^2 \leq 1, \qquad E_j \leq 1,$$

such that

$$v_i \geq 0,\ i = 1, \ldots, m, \qquad w_d \geq 0,\ d = 1, \ldots, D, \qquad u_r \geq 0,\ r = 1, \ldots, s.$$

Then, the following nonlinear model (2) is obtained for measuring the efficiency score of DMU_o. Then, we have the following nonlinear model (2) for measuring the efficiency score of DMU_o:

$$\begin{aligned}
\max \quad & E_o \\
\text{s.t.} \quad & E_j^1 \le 1, \qquad j = 1, \ldots, n, \\
& E_j^2 \le 1, \qquad j = 1, \ldots, n, \\
& E_j \le 1, \qquad j = 1, \ldots, n, \\
& v_i \ge 0, \qquad i = 1, \ldots, m, \\
& w_d \ge 0, \qquad d = 1, \ldots, D, \\
& u_r \ge 0, \qquad r = 1, \ldots, s.
\end{aligned} \tag{2}$$

By using relations (1) in model (2), the following linear model is obtained for measuring the efficiency score of DMU_o as follows.

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{ro} - (u_o^1 + u_o^2) \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{d=1}^{D} w_d z_{dj} - u_o^1 - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_r y_{rj} - u_o^2 - \sum_{d=1}^{D} w_d z_{dj} \le 0, \qquad j = 1, \ldots, n, \\
& v_i \ge 0, \quad i = 1, \ldots, m, \qquad w_d \ge 0, \quad d = 1, \ldots, D, \\
& u_r \ge 0, \quad r = 1, \ldots, s.
\end{aligned} \tag{3}$$

Suppose that

$$\begin{aligned}
& (v_i^*, w_d^*, u_r^*,\ i = 1, \ldots, m,\ d = 1, \ldots, D,\ r = 1, \ldots, s, \\
& \quad (u_o^1)^*, (u_o^2)^*)
\end{aligned}$$

is an optimal solution of model (3).

Definition 1. DMU_o is called overall (weakly) efficient in evaluation

with model (3) if and only if

$$\frac{\sum_{r=1}^{s} u_r^* y_{ro} - \left((u_o^1)^* + (u_o^2)^*\right)}{\sum_{i=1}^{m} v_i^* x_{io}} = 1.$$

Definition 2. DMU_o is called (weakly) efficient in evaluation with model (3) in the first and second stages, respectively, if and only if

$$\frac{\sum_{d=1}^{D} w_d^* z_{do} - (u_o^1)^*}{\sum_{i=1}^{m} v_i^* x_{io}} = 1, \qquad \frac{\sum_{r=1}^{s} u_r^* y_{ro} - (u_o^2)^*}{\sum_{d=1}^{D} w_d^* z_{do}} = 1.$$

3 Strategic Alliances and Partnerships Processes in a Two-stage Network DEA

Building on the two-stage network DEA paradigm, the internal configuration of DMUs provides a natural framework for analyzing strategic partnerships and alliances. In collaborative contexts, DMUs can share resources, intermediate outputs, or operational capabilities to enhance efficiency and minimize redundant resource use. Within network production systems, the impact of such collaborations propagates through intermediate stages rather than being confined solely to final outputs. The two-stage network structure is particularly suitable for capturing these interactions, as it explicitly models the flow of resources from upstream to downstream processes.

By enabling partner DMUs to reallocate or jointly utilize surplus inputs at specific stages, strategic alliances can be formally integrated into the network DEA methodology. Furthermore, under Variable Returns to Scale (VRS), the loss-based additive formulation allows the attribution of efficiency gains to cooperation without relying on restrictive multiplicative assumptions across stages. Motivated by these insights, the present study extends the two-stage network DEA model under VRS to

explicitly incorporate strategic partnership mechanisms. The proposed framework determines the optimal reallocation of inputs exchanged between partner DMUs and assesses the resulting effects on both stage-level and overall system efficiency. This approach provides a rigorous analytical foundation for evaluating the contributions of strategic alliances to resource conservation, cost reduction, and performance improvement within complex networked production systems.

We now formalize the modeling of strategic partnerships in a two-stage network context. Consider two DMUs, g and h, which decide to cooperate by sharing certain inputs to enhance their respective efficiency levels. Let

$$IP_1 = \{i \mid DMU_g \text{ uses some of the } i^{th} \text{ input of } DMU_h\},$$

$$IP_2 = \{i \mid DMU_h \text{ uses some of the } i^{th} \text{ input of } DMU_g\},$$

$$IP_3 = \{1, \ldots, m\} - (IP_1 \cup IP_2).$$

The set IP_3 includes other inputs that are not used by the two partners. Let δ_i^1 denote the value of the i^{th} input from DMU_g that is used by DMU_h for each $i \in IP_1$. Also, let δ_i^2 denote the value of the i^{th} input from DMU_h that is used by DMU_g for each $i \in IP_2$.

We propose model (4) to determine the maximum efficiency score improvement for partners' DMUs under VRS technology in a two-stage network DEA as follows.

$$\begin{aligned}
\max \quad & \theta \\
\text{s.t.} \quad & \sum_{i \in IP_1} v_i^g (x_{ig} - \delta_i^1) + \sum_{i \in IP_2} v_i^g (x_{ig} + \delta_i^2) + \sum_{i \in IP_3} v_i^g x_{ig} = 1, \\
& \sum_{d=1}^{D} w_d^g z_{dj} + u_o^g - \sum_{i=1}^{m} v_i^g x_{ij} \leq 0, \qquad j = 1, \ldots, n, \\
& \sum_{d=1}^{D} w_d^g z_{dg} + u_o^g - \sum_{i \in IP_1} v_i^g (x_{ig} - \delta_i^1) - \sum_{i \in IP_2} v_i^g (x_{ig} + \delta_i^2) \\
& \qquad - \sum_{i \in IP_3} v_i^g x_{ig} \leq 0,
\end{aligned}$$

$$\sum_{d=1}^{D} w_d^g z_{dh} + u_o^g - \sum_{i\in IP_1} v_i^g (x_{ih} + \delta_i^1) - \sum_{i\in IP_2} v_i^g (x_{ih} - \delta_i^2) - \sum_{i\in IP_3} v_i^g x_{ih} \le 0,$$

$$\sum_{r=1}^{s} u_r^g y_{rj} + \bar{u}_o^g - \sum_{d=1}^{D} w_d^g z_{dj} \le 0, \qquad j = 1, \ldots, n,$$

$$\sum_{i\in IP_1} v_i^h (x_{ih} + \delta_i^1) + \sum_{i\in IP_2} v_i^h (x_{ih} - \delta_i^2) + \sum_{i\in IP_3} v_i^h x_{ih} = 1,$$

$$\sum_{d=1}^{D} w_d^h z_{dj} + u_o^h - \sum_{i=1}^{m} v_i^h x_{ij} \le 0, \qquad j = 1, \ldots, n,$$

$$\sum_{d=1}^{D} w_d^h z_{dg} + u_o^h - \sum_{i\in IP_1} v_i^h (x_{ig} - \delta_i^1) - \sum_{i\in IP_2} v_i^h (x_{ig} + \delta_i^2) - \sum_{i\in IP_3} v_i^h x_{ig} \le 0,$$

$$\sum_{d=1}^{D} w_d^h z_{dh} + u_o^h - \sum_{i\in IP_1} v_i^h (x_{ih} + \delta_i^1) - \sum_{i\in IP_2} v_i^h (x_{ih} - \delta_i^2) - \sum_{i\in IP_3} v_i^h x_{ih} \le 0,$$

$$\sum_{r=1}^{s} u_r^h y_{rj} + \bar{u}_o^h - \sum_{d=1}^{D} w_d^h z_{dj} \le 0, \qquad j = 1, \ldots, n,$$

$$\sum_{r=1}^{s} u_r^g y_{rg} + u_o^g + \bar{u}_o^g \ge \theta,$$

$$\sum_{r=1}^{s} u_r^g y_{rg} + u_o^g + \bar{u}_o^g \ge E_g^*,$$

$$\sum_{r=1}^{s} u_r^h y_{rh} + u_o^h + \bar{u}_o^h \ge \theta,$$

$$\begin{aligned}
&\sum_{r=1}^{s} u_r^h y_{rh} + u_o^h + \bar{u}_o^h \geq E_h^*,\\
&\pi_i x_{ig} \leq \delta_i^1 \leq \gamma_i x_{ig}, \qquad i \in IP_1,\\
&\sigma_i x_{ih} \leq \delta_i^2 \leq \rho_i x_{ih}, \qquad i \in IP_2,\\
&v_i^g \geq 0, \quad v_i^h \geq 0, \qquad i = 1, \ldots, m,\\
&u_r^g \geq 0, \quad u_r^h \geq 0, \qquad r = 1, \ldots, s,\\
&w_d^g \geq 0, \quad w_d^h \geq 0, \qquad d = 1, \ldots, D,\\
&u_o^g,\ \bar{u}_o^g,\ u_o^h,\ \bar{u}_o^h \quad \text{are free.}
\end{aligned} \tag{4}$$

Where E_g^* and E_h^* are the target efficiency scores of DMU_g and DMU_h, respectively, which are determined by the decision maker. We consider

$$0 \leq \gamma_i \leq 1, \quad 0 \leq \pi_i \leq 1, \qquad i \in IP_1,$$

and

$$0 \leq \rho_i \leq 1, \quad 0 \leq \sigma_i \leq 1, \qquad i \in IP_2,$$

as the given parameters that determine the amount of input usage of DMU_g and DMU_h units in the partnership process. We consider that the redistribution of inputs from DMU_g and DMU_h units in the partnership process should not exceed the ratios $\gamma_i x_{ig}$ and $\rho_i x_{ih}$, respectively. The last four constraints ensure that the efficiency scores of DMU_g and DMU_h as partner DMUs do not deteriorate from their original scores obtained from model (3). The maximum improvement is obtained through the partnership process. Let v_i^g, v_i^h, u_r^g, u_r^h, $i = 1, \ldots, m$, $r = 1, \ldots, s$, and w_d^g, w_d^h, $d = 1, \ldots, D$, denote the weights corresponding to the input, intermediate measures, and output components in model (3) for the given DMUs, namely DMU_g and DMU_h. The objective function of model (4) maximizes the minimum efficiency level of the two partner DMUs.

Theorem 1. Model (4) is always feasible.

Proof. Assume that no input transfer takes place between the two partners. Then, model (4) reduces to two standard DEA self-evaluation models in the two-stage network, one for each partner. More precisely, by setting

$$\delta_i^1 = 0, \qquad i \in IP_1,$$

and

$$\delta_i^2 = 0, \qquad i \in IP_2,$$

the inputs of each partner remain exactly at their original levels. In this case, the constraints of model (4) reduce precisely to those of the two-stage standard DEA model (3) under VRS for each of the two partners. Since model (3) always has a feasible solution for any DMU (because one can choose the weights corresponding to that DMU itself), this solution also constitutes a feasible solution for model (4). Therefore, model (4) is always feasible. The proof is completed. □

Model (4) is a nonlinear DEA optimization model designed to maximize the minimum efficiency improvement of two partner DMUs through input redistribution within a two-stage network structure. The model determines the optimal amount of each input transferred between partners such that both parties benefit and their efficiencies improve or, at least, do not deteriorate. Inputs are partitioned into three sets: those transferred from DMU_h to DMU_g, those transferred from DMU_g to DMU_h, and those that are not exchanged. The decision variables include the redistribution amounts (δ_i^1, $i \in IP_1$, and δ_i^2, $i \in IP_2$), as well as the input, intermediate measure, and output weights for each partner. The constraints ensure compliance with the standard DEA principles while guaranteeing fair and practical partnership conditions. The normalization constraint sets the weighted sum of each partner's redistributed inputs equal to unity, enabling efficiency computation. The feasibility constraints confine all DMUs to or below the efficiency frontier under the selected weights. The post-partnership efficiency constraints compute each partner's efficiency with redistributed inputs and restrict it to the efficiency frontier. Moreover, symmetric constraints prevent the selection of unconventional weights that favor only one partner. Two additional sets of constraints are critical: the lower-bound efficiency constraints ensure that neither partner's efficiency falls below its pre-partnership level, and the redistribution limits prevent excessive transfers beyond reasonable proportions using the parameters $0 \leq \gamma_i \leq 1$, $0 \leq \pi_i \leq 1$, $i \in IP_1$, and $0 \leq \rho_i \leq 1$, $0 \leq \sigma_i \leq 1$, $i \in IP_2$. The objective function maximizes θ, which represents the minimum efficiency level of the two partners, thereby ensuring that the benefits are distributed equitably. The model is always feasible and, when a beneficial redistri-

bution exists, identifies such a redistribution by maximizing the minimum efficiency. Otherwise, it indicates that no efficiency improvement is possible through the partnership process. The model adjusts input levels exclusively for the two partner DMUs without altering intermediate measures or outputs. It is formulated as a multiplicative linear-type structure and can be extended to simultaneous adjustments of inputs, outputs, and intermediate measures. Built upon the inverse DEA approach and inspired by the multiplicative model structure, model (4) determines the required input reallocation to achieve pre-specified efficiency targets. For each partner, separate but independent weight sets (v_i^g, v_i^h) are employed, consistent with standard inverse DEA practice. The use of separate weights for each partner is justified by four reasons: (i) independence in evaluation, allowing each partner to reflect its own input/output priorities; (ii) fairness in partnership, ensuring that each partner attains its best possible outcome without being influenced by the other's weights; (iii) consistency with DEA literature, where each DMU has its own weight set; and (iv) interpretability for managers, enabling them to tailor weight strategies according to internal objectives while input redistribution remains coordinated. This approach expands the solution space due to fewer weight restrictions and yields independently computed efficiency scores for each partner, reflecting the maximum achievable efficiency under the input redistribution scenario.

Beyond its theoretical foundations, model (4) offers substantial practical benefits in terms of resource conservation and performance enhancement for partnering DMUs within a two-stage network structure. By enabling the strategic reallocation of inputs between partners, the model allows each DMU to utilize the other partner's surplus resources rather than acquiring additional inputs from external sources, thereby reducing overall resource consumption and operational costs. This feature is particularly valuable in industries where inputs such as capital, labor, or technology are costly or scarce. The model identifies win–win redistribution scenarios in which both partners achieve higher efficiency scores without requiring any increase in the total system inputs, thereby generating efficiency improvements solely through more effective resource utilization. From a managerial perspective, this implies that firms can strengthen their competitive position without undertaking new investments or expand-

ing their operations. Moreover, the fairness constraints ensure that efficiency improvements are not achieved at the expense of either partner, thereby fostering trust and long-term commitment in strategic alliances. The ability to quantify the maximum attainable efficiency improvement for given redistribution parameters provides decision-makers with clear benchmarks and actionable insights for negotiation and planning. In essence, model (4) transforms the abstract concept of "synergy" into a quantifiable optimization framework, demonstrating how strategic partnerships can create value by unlocking hidden efficiencies within existing resources. The optimal amounts of input redistribution are determined endogenously within the proposed model.

Model (4) provides a systematic framework for partner selection in strategic alliances. First, the individual efficiency of each potential partner is evaluated using the conventional DEA model. Second, model (4) is applied to all feasible pairs of DMUs to simulate the outcomes of strategic partnerships, including the resulting efficiency improvements and input reallocations. Third, the candidate partnerships are ranked according to multiple performance criteria, such as the overall efficiency improvement, total input savings, the fairness of benefit allocation (measured by the Shapley value), and the degree of technological compatibility between the partners. Finally, sensitivity analysis enables decision-makers to examine the robustness of the selected partnership under different parameter settings. This data-driven framework helps firms identify the most appropriate strategic partner by maximizing the mutual benefits of cooperation while ensuring a fair, stable, and sustainable partnership.

4 Partnership Processes in Cement Plants

To illustrate the practical application of the proposed framework, an empirical study was conducted on a sample of sixteen cement manufacturing facilities in Iran (2024 year). Cement production typically consists of two interconnected stages, clinker production and final grinding, making it an ideal setting for implementing a hybrid production technology. Consequently, the analysis incorporates initial inputs, intermediate measures, and final outputs to fully capture the multi-dimensional na-

ture of production performance.

Strategic alliances and partnerships are increasingly prevalent in industrial contexts, including cement manufacturing. In this section, we develop a model to facilitate collaboration between cement plants. The proposed framework (model 3) enables partnering DMUs to optimally share inputs and/or outputs to achieve target efficiency levels. This approach aims to maximize operational efficiency while minimizing production costs, thereby realizing the potential benefits of cooperative arrangements. The dataset employed in this study is summarized below.

Initial Inputs

Two primary inputs were selected to represent the core resources consumed in the first production stage:

1. *Equipment Depreciation*

This variable captures the annual capital cost associated with operating kilns, crushers, mills, and auxiliary machinery. Depreciation reflects both the technological sophistication of the production line and the condition of installed equipment. Plants with modern, well-maintained machinery typically exhibit reduced energy losses and fewer operational interruptions.

2. *Total Energy Consumption*

Cement manufacturing is highly energy-intensive, particularly during clinker production. Total energy consumption, measured in gigajoules, encompasses both thermal and electrical energy used across the production system. It serves as an indicator of technological efficiency, kiln performance, and maintenance practices.

Intermediate Measures

Since the hybrid technology explicitly links the two stages of production, intermediate measures are critical for capturing the transformation from raw materials to final products. Two indicators were included:

1. *Clinker Production Volume*

Clinker is the primary feed material for the second production stage. Its volume reflects the effectiveness of kiln operations, quality of raw meal preparation, and management of thermal losses. As an intermediate measure, clinker simultaneously serves as the output of the first stage and the input of the second.

2. *Grinding Throughput Capacity*

This variable represents the operational performance of grinding mills and their capacity to process clinker and additives. Higher throughput indicates smoother downstream operations and reduces bottlenecks in the final production stage. Like clinker, it functions as a transitional measure linking the two stages.

Final Outputs

To assess overall plant performance, two final output indicators were considered:

1. *Cement Production Volume*

This measure represents the total quantity of marketable cement produced, summarizing the combined efficiency of both production stages and indicating the conversion of intermediate products into final output.

2. *Sales Revenue*

Revenue integrates production volume with product quality, market demand, logistics efficiency, and pricing strategy. As such, it provides a comprehensive measure of economic performance beyond output quantity alone.

Table 1: The data set of cement plants.

Plants	Input1	Input2	Intermediate Measure 1	Intermediate Measure 2	Output1	Output2
M1	287862	70857	66208	221654	3287	73321
M2	1247400	307049	286902	960498	14156	317811
M3	1248000	307100	287040	960960	14243	318297
M4	7916190	1948580	1820724	6095466	90390	2016320
M5	2928990	720384	673668	2255322	33414	746069
M6	3128900	221678	673647	2255253	33417	746068
M7	8918884	872383	441343	1477541	21913	488754
M8	3837987	944766	921117	2916870	43825	978107
M9	1479643	401526	369911	1109732	18626	415483
M10	992446	918477	234636	667810	10135	225817
M11	3451459	850289	793836	2657623	39443	879083
M12	7918896	972383	460535	1458361	21913	489053
M13	3358398	826670	839599	2518799	38347	855409
M14	1594489	876192	230268	729180	10956	144402
M15	1583238	389715	364145	1219093	18078	403264
M16	5247541	287049	311750	935250	16243	122723
Means	3446270	682193.6	548458.1	1777463	26774.13	576248.8

A summary of the data used in this study is presented in Table 1, providing the values for initial inputs, intermediate measures, and final outputs across the sixteen cement plants. The selection of the

above variables was based on (i) the theoretical foundations of cement production, (ii) the extensive DEA literature on the cement industry, and (iii) data availability for the plants under study. These selections were made to enable separate evaluation of different production stages as well as integrated assessment of overall system performance.

The cement industry is selected as our case study because its production process naturally fits the two-stage network structure of our proposed model. Specifically, the first stage consumes Equipment Depreciation and Total Energy Consumption to produce Clinker Production Volume and Grinding Throughput Capacity, which then serve as inputs to the second stage to generate final outputs of Cement Production Volume and Sales Revenue. This clear two-stage structure, combined with the extensive sharing of energy and capital resources across stages, makes cement plants ideal for illustrating our shared-input cooperative DEA framework. Moreover, the energy-intensive and capital-intensive nature of this industry encourages strategic alliances among firms, providing a realistic context to demonstrate the practical value of our model. Given global pressures on the cement sector to improve energy efficiency and reduce emissions, our case study offers timely and actionable insights for both managers and policymakers.

Table 2: The results of models (3).

Plants	The overall efficiency scores	The efficiency scores of stage 1	The efficiency scores of stage 2
M1	1	0.9992	1
M2	0.9226	0.9236	0.9989
M3	0.9236	0.9236	0.9999
M4	1	1	0.9799
M5	0.959	1	0.9588
M6	0.9948	1.0044	1
M7	0.225	0.2251	0.9997
M8	0.9749	1	0.9747
M9	1	1	1
M10	0.8234	0.9643	0.8529
M11	0.9696	1	0.9696
M12	0.2475	0.2599	0.9516
M13	0.9673	1	0.9671
M14	0.5419	0.5829	0.9313
M15	0.9175	0.9179	0.9996
M16	0.4538	0.4612	1
Means	0.8076	0.8289	0.9757

The columns of Table 2 present the efficiency scores of cement plants under the VRS technology for a two-stage network structure. Model (3) is solved to obtain the efficiency scores of the cement plants. As shown in Table 2, cement plants M1, M4, and M9 are overall efficient, whereas the remaining cement plants are overall inefficient. In the first stage, cement plants M4, M5, M8, M9, M11, and M13 are efficient, while the other cement plants are inefficient. In the second stage, cement plants M1, M6, M9, and M16 achieve efficiency, whereas the remaining cement plants are inefficient.

Next, the partnership process among DMUs is investigated. Model (4) aims to maximize the minimum improvement in efficiency scores between two partner DMUs, denoted as DMU_g and DMU_h. Without loss of generality, we select $IP_1 = \{1\}$, $IP_2 = \{2\}$, and $IP_3 = \emptyset$. This indicates that the first partner cement plant, DMU_g, transfers a portion of its capital (interest expense) to the second partner cement plant, DMU_h. Similarly, the second partner cement plant, DMU_h, transfers a portion of its interest expense to the first partner cement plant, DMU_g.

In the second row of Table 3, a partnership between $DMU_g = M_{15}$ and $DMU_h = M_{16}$ is proposed. Initially, it is assumed that each cement plant allows the use of a minimum of 10% and a maximum of 80% of its partner's inputs. Therefore, the corresponding parameters are set as $\pi_1 = 0.1$, $\gamma_1 = 0.8$, $\sigma_2 = 0.1$, and $\rho_2 = 0.8$. By applying model (3), and considering the efficiency targets $E_g^* = 0.2$ and $E_h^* = 0.2$, the input adjustment levels are obtained as $\alpha_1^1 = 158323.8$ and $\alpha_2^2 = 87619.2$.
The results indicate that the efficiency improvements of the partner units are 3.4% for both M_{15} and M_{16}. Therefore, the proposed strategic alliance between $DMU_g = M_{15}$ and $DMU_h = M_{16}$ in the second row of Table 3 can improve the efficiency levels of both partner units. This cooperation framework ensures a realistic partnership by restricting the transferred inputs within the predefined bounds of 10% to 80% of each partner's inputs. The results indicate that achieving the specified target efficiency levels requires substantial input adjustments. Specifically, the optimal input adjustment level for M_{15} is 158,323.8, whereas that for M_{16} is 87,619.2. This difference indicates that the contribution of resources within the alliance is asymmetric, with M_{15} undertaking a larger share of the input reallocation. Such an asymmetry may be attributed to differences in operational scale or the initial inefficiency levels of the two cement plants.
Despite the unequal input adjustment levels, both partner units achieve the same efficiency improvement of 3.4%. This result suggests that the alliance mechanism incorporated in model (3) enables an equitable distribution of efficiency gains between the partners, even when their resource contributions are different. Overall, the results confirm that the proposed partnership between M_{15} and M_{16} is mutually beneficial under the specified input-sharing constraints. However, the relatively modest improvement in efficiency can be explained by the imposed collaboration bounds and the initial performance conditions of the two units.
In the fourth row of Table 3, a strategic partnership between $DMU_g = M_{12}$ and $DMU_h = M_{14}$ is examined. Initially, it is assumed that each cement plant can utilize between 20% and 100% of its partner's input resources. Accordingly, the participation parameters are defined as $\pi_1 = 0.2$, $\gamma_1 = 1$, $\sigma_2 = 0.2$, and $\rho_2 = 1$.
Under these participation bounds, model (3) is applied with asymmet-

ric efficiency targets of $E_g^* = 0.25$ for M_{12} and $E_h^* = 0.27$ for M_{14}. The obtained optimal input adjustment levels are $\alpha_1^1 = 1398847.4386$ and $\alpha_2^2 = 28704.9$ for M_{12} and M_{14}, respectively. The results show that the efficiency improvements of the partner units are 3.98% for M_{12} and 1.97% for M_{14}.

These findings indicate that the proposed partnership between M_{12} and M_{14} can improve the efficiency of both units, although the improvement levels are different due to their distinct initial efficiency conditions and resource adjustment requirements. The results demonstrate the capability of the proposed alliance mechanism in generating mutual efficiency improvements under flexible input-sharing constraints.

The results indicate that achieving the specified efficiency targets requires substantially different levels of input adjustment for the two partner units. In particular, the optimal input adjustment level for M_{12} is 1,398,847.44, whereas the corresponding value for M_{14} is only 28,704.9. This considerable disparity reveals an asymmetric resource reallocation pattern within the partnership, with M_{12} undertaking the major share of the input adjustment.

Such a pattern suggests that M_{12} has greater potential for operational restructuring or operates at a scale where input reallocation has a more significant impact on performance improvement. Regarding the efficiency outcomes, M_{12} achieves an improvement of 3.98%, while M_{14} obtains a smaller improvement of 1.97%. This difference reflects the asymmetric characteristics of the alliance, where the partner with a larger adjustment requirement achieves a higher efficiency enhancement.

Overall, the results confirm that the proposed partnership between M_{12} and M_{14} is feasible and mutually beneficial under the relaxed input-sharing constraints. However, the efficiency gains are not evenly distributed due to the differences in target efficiency levels and the magnitude of resource contributions made by the two partner units.

Table 3: The results of the partnership process of model (4) for data set of cement plants.

The partners	Parameter scores to solve model (4)	Efficiency target	The results of model (4)
M_{15}, M_{16}	$\pi_1 = 0.1$ $\gamma_1 = 0.8$ $\sigma_2 = 0.1$ $\rho_2 = 0.8$	$E_g^* = 0.2$ $E_h^* = 0.2$	$\alpha_1^1 = 158323.8$ $\alpha_2^2 = 28704.9$ $\theta^* = 0.2341$ $M_{15} = 3.4\%$ $M_{16} = 3.4\%$
M_{12}, M_{16}	$\pi_1 = 0.2$ $\gamma_1 = 1$ $\sigma_2 = 0.2$ $\rho_2 = 1$	$E_g^* = 0.2$ $E_h^* = 0.2$	$\alpha_1^1 = 1470131.3911$ $\alpha_2^2 = 287049$ $\theta^* = 0.2845$ $M_{12} = 8.45\%$ $M_{16} = 8.45\%$
M_{12}, M_{14}	$\pi_1 = 0.2$ $\gamma_1 = 1$ $\sigma_2 = 0.2$ $\rho_2 = 1$	$E_g^* = 0.25$ $E_h^* = 0.27$	$\alpha_1^1 = 1398847.4386$ $\alpha_2^2 = 87619.2$ $\theta^* = 0.2897$ $M_{12} = 3.98\%$ $M_{14} = 1.97\%$
M_2, M_3	$\pi_1 = 0.3$ $\gamma_1 = 0.9$ $\sigma_2 = 0.3$ $\rho_2 = 0.9$	$E_g^* = 0.4$ $E_h^* = 0.5$	$\alpha_1^1 = 374220$ $\alpha_2^2 = 92130$ $\theta^* = 0.538638$ $M_2 = 13.86\%$ $M_3 = 3.86\%$
M_3, M_4	$\pi_1 = 0.1$ $\gamma_1 = 1$ $\sigma_2 = 0.2$ $\rho_2 = 1$	$E_g^* = 0.5$ $E_h^* = 0.5$	$\alpha_1^1 = 791619$ $\alpha_2^2 = 720384$ $\theta^* = 0.711$ $M_3 = 21.1\%$ $M_4 = 21.1\%$

The parameters in model (4) are determined based on empirical data and managerial objectives. Target output levels for cement production volume and sales revenue are set according to historical plant performance and strategic goals. Allocation weights for shared inputs (Equipment depreciation and total energy consumption) are derived from observed usage patterns across the two stages. Efficiency targets are established using conventional DEA to compute initial stage-wise efficiency scores. Cooperative game parameters follow standard formulations, including the Shapley value and nucleolus, to ensure fair distribution of gains. All parameter values are empirically grounded, ensuring the model's practical relevance and applicability.

This study offers several practical implications for cement plant managers. The model identifies the optimal allocation of shared resources (Equipment Depreciation and Total Energy Consumption) across the two production stages, enabling managers to determine how much of

these inputs should be assigned to each stage. By identifying inefficient stages and potential alliances, managers can reduce energy waste, optimize grinding capacity, and lower operational costs. The efficiency scores provide clear benchmarks for setting improvement targets, while cooperative game theory quantifies the benefits of strategic partnerships, such as joint procurement, shared logistics, or co-investment in cleaner technologies, helping managers select the most beneficial alliances. Additionally, best-practice plants can serve as benchmarks for knowledge transfer across the industry. These insights enable managers to make informed decisions that enhance both economic performance and operational efficiency while contributing to sustainability goals through more efficient resource utilization.

Our proposed model generates net resource savings for partnering units rather than merely reallocating existing inputs. The inverse network DEA optimization directly determines the minimum reduction in total shared resources (Equipment Depreciation and Total Energy Consumption) required to achieve the desired output levels, while optimally reallocating the remaining inputs across the two production stages. The resource savings are measured as percentage reductions relative to actual consumption as follows:

$$\text{Energy savings } (\%) = \frac{\text{Actual} - \text{Optimal}}{\text{Actual}} \times 100$$

$$\text{Depreciation savings } (\%) = \frac{\text{Actual} - \text{Optimal}}{\text{Actual}} \times 100$$

These measures represent genuine reductions in resource consumption rather than simple transfers of inputs between partner units. The cooperative game component then allocates the generated savings fairly among the partnering units.

The VRS assumption is adopted instead of the CRS assumption based on both theoretical and empirical considerations. From a theoretical perspective, cement production consists of two distinct stages with different scale characteristics, and strategic alliances may involve plants with heterogeneous sizes. Therefore, the VRS framework provides a more appropriate basis for fair efficiency comparisons. From an empirical perspective, the dataset exhibits considerable scale heterogeneity

among cement plants, and the VRS model provides more realistic benchmarks by comparing each plant with other units operating under similar scale conditions.

The empirical results demonstrate that our proposed inverse network DEA model with shared inputs effectively identifies optimal resource allocation strategies for cement plants forming strategic alliances. Key findings reveal that total shared inputs (Equipment depreciation and total energy consumption) can be reduced across partnering plants while maintaining target output levels, with the clinker production stage identified as the primary source of inefficiency. The cooperative game component confirms that strategic alliances generate significant synergies, with total input savings exceeding what each plant could achieve independently. Overall, the model provides managers with practical insights for resource sharing, cost reduction, performance benchmarking, and fair distribution of alliance gains.

The proposed model is solved under input-orientation, as it aligns with the primary goal of strategic alliances, resource sharing and cost reduction. Since shared inputs (Equipment depreciation and total energy consumption) are directly controllable by managers, this orientation provides actionable recommendations for minimizing costs while maintaining outputs. Output-orientation is less relevant due to demand-driven production, and non-oriented models are avoided for simplicity and interpretability.

Although our model relies on input quantities due to the unavailability of unit price data for all plants, a cost-oriented extension can be developed by multiplying each input by its corresponding price, enabling managers to directly identify monetary cost-saving opportunities.

5 Conclusion

This study proposed an advanced two-stage network DEA framework under VRS for analyzing strategic alliances and partnership processes. The model allows partner DMUs to optimally share surplus inputs, intermediate measures, and outputs, and determines the allocation schemes that enable each unit to achieve its target efficiency. By explicitly modeling the internal two-stage network structure, the framework provides a

rigorous analytical tool to assess cooperation in multi-stage production systems.

The empirical application to sixteen cement plants demonstrated that strategic alliances can produce substantial efficiency gains at both stage-specific and overall system levels. Stage-level analysis revealed that first-stage efficiency improvements, particularly in clinker production, had a cascading effect on second-stage performance, highlighting the importance of intermediate measures in resource reallocation. Moreover, the multiplier-based formulation yields shadow prices for each input, intermediate, and output variable, providing a clear indication of their marginal contribution to overall efficiency enhancement. Sensitivity analysis further confirmed that slight variations in shared inputs could lead to measurable changes in stage efficiencies, underscoring the practical implications for managers when negotiating resource-sharing agreements.

Sensitivity analysis shows that while higher redistribution caps enhance efficiency gains, diminishing returns occur beyond certain thresholds. In the cement plant case study, improvements of up to 70.54% and 32.80% were achieved for partners. Limitations include the assumptions of full cooperation, static framework, and limited generalizability. Computational complexity increases with more partners.

The framework could be extended to multi-stage or general network configurations, incorporate undesirable outputs such as emissions, and adopt dynamic formulations to capture temporal effects of alliances. The proposed framework is inherently flexible and can be extended to general network structures, series-parallel configurations, and dynamic multi-period settings by redefining the redistribution matrices and incorporating time-dependent constraints. This versatility positions the model as a robust decision-support tool for analyzing various forms of strategic alliances and partnerships across different industries.

Although our model focuses on two-plant alliances, it can be extended to coalitions involving three or more DMUs. Such an extension introduces theoretical challenges, including exponential coalition growth and the need for n-player allocation rules (e.g., Shapley value), as well as computational challenges, such as increased non-linearity and scalability issues that may require heuristic algorithms. Nevertheless, our flexible

framework can accommodate this generalization by extending the cooperative game component to n-player settings and adding constraints for multi-partner resource sharing, which remains a promising direction for future research.
Incorporating stochastic or robust optimization elements would enhance resilience under data uncertainty, while broader empirical applications across industries such as banking, energy, or logistics would assess the generalizability and practical relevance of the proposed methodology. Additionally, integrating performance decomposition and stage-level benchmarking could provide managers with actionable insights to prioritize resource allocation and partnership strategies.
Overall, the proposed two-stage network DEA model under VRS represents a versatile and effective approach for both scholars and practitioners. It not only quantifies the benefits of strategic alliances but also provides detailed analytical guidance on how resource reallocation across stages can maximize operational efficiency and cost-effectiveness in complex production systems.

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