Journal of Mathematical Extension
Vol. 9, No. 4, (2015), 99-105
ISSN: 1735-8299
URL: http://www.ijmex.com

The Existence of Gorenstein Injective Envelopes

Z. Heidarian
Sciences and Research Branch
Islamic Azad University

Abstract. Let R be a commutative Noetherian ring. We prove that every R-module N of finite Gorenstein injective dimension has a Gorenstein injective envelope $\psi : N \to G$ where injective dimension of Ker ψ is finite and ψ is injective.

AMS Subject Classification: 13D02; 13D05; 13C99
Keywords and Phrases: Envelope, preenvelope, gorenstein injective

1. Introduction

In general, there is little hope to describe all modules over a given ring. In order to study the structure of a module, we may try to approximate the module using the so-called $\mathcal{F}$-cover and $\mathcal{F}$-envelope where $\mathcal{F}$ is a class of R-modules. The enveloping and covering classes play a fundamental role in relative homological algebra. They are used to construct resolutions, minimal resolutions and to compute relative derived functors. A fundamental question in module theory is the existence of such covers and envelopes. We consider the existence of Gorenstein injective envelopes and totally reflexive covers.

Throughout this paper, R is a Noetherian commutative ring with an identity. We recall from [5] that, a module G is Gorenstein injective if

Received: April 2015; Accepted: July 2015

there is an exact sequence

$$... \to E_2 \to E_1 \to E_0 \to E^0 \to E^1 \to E^2 \to ...,$$

of injective modules such that $G = Ker(E^0 \to E^1)$ and $Hom_R(E, -)$ leaves the sequence exact when E is injective module.

The Gorenstein injective dimension of modules are defined in the obvious way.

Let $\mathcal{F}$ be a class of modules. Then an $\mathcal{F}$-preenvelope of a module M is a homomorphism $\varphi : M \to F$ with $F \in \mathcal{F}$ such that $Hom_R(F, G) \to Hom_R(M, G) \to 0$ is exact for all $G \in \mathcal{F}$. If, moreover, any endomorphism $f : F \to F$ such that $f\varphi = \varphi$ is an automorphism, then $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope.

We study the existence of Gorenstein injective envelopes. In [7], Enochs, Jenda and Xu showed that over a Gorenstein ring every module has Gorenstein injective envelope and then they proved that over a Cohen-Macaulay ring admitting a dualizing module all modules of finite Gorenstein injective dimension have Gorenstein injective envelope (see [6]). Recently Enochs, Estrada and Iacob proved that if the ring is commutative Noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat, then every complex has a Gorenstein injective envelope. In particular, this is the case when the ring is commutative Noetherian with a dualizing complex (cf.[4]).

On the other hand, Nikkhak Babaei and Divaani-Aazar showed that every Artinian R-module of finite Gorenstein injective dimension possesses a Gorenstein injective envelope which is especial and Artinian (cf.[11]).

We prove that every R-module of finite Gorenstein injective dimension has a Gorenstein injective envelope.

In the end, we prove some results for R-modules of finite Gorenstein injective dimension.

2. The Existence of Gorenstein Injective Envelopes

We recall some notations and definitions from [15]. If $\mathcal{X}$ is a class of R-

R-modules, then ${}^{\perp}\mathcal{X}$ is the class of R-modules M such that $Ext^1_R(M,X) = 0$ for all $X \in \mathcal{X}$. One can easily check that, the class ${}^{\perp}\mathcal{X}$ is closed under direct limits and direct summands.
A $\mathcal{X}$-preenvelope $\varphi : M \to X$ is called a special $\mathcal{X}$-preenvelope if φ is injective and $Coker\varphi \in^{\perp} \mathcal{X}$.

Definition 2.1. (See [15, Definition 2.2.1]) *Let $\mathcal{L}$ be a class of R-modules and let M be an R-module. An extension $0 \to M \to G \to L \to 0$ with $L \in \mathcal{L}$ is called a generator for $\mathcal{E}xt(\mathcal{L}, M)$ if for any extension $0 \to M \to \overline{G} \to \overline{L} \to 0$ with $\overline{L} \in \mathcal{L}$, there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \overline{G} & \to & \overline{L} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & G & \to & L & \to & 0, \end{array}$$

Furthermore, such a generator is said to be minimal provided that any commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & G & \to & L & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & G & \to & L & \to & 0, \end{array}$$

always implies that $L \to L$ is an automorphism (so that $G \to G$ is too).

Theorem 2.2. (See [15, Theorem 2.2.2]) *Assume that the class of R-modules $\mathcal{L}$ is closed under direct limits. Then for an R-module M, $\mathcal{E}xt(\mathcal{L}, M)$ has a generator, then there must be a minimal generator.*
Over Noetherian rings, the existence of Gorenstein injective preenvelope for all modules was proved by Enochs and L'opez-Ramos in [8]. Krause proved the next stronger result:

Theorem 2.3. (See [10, Theorem 7.12(1)]) *Every R-module has a special Gorenstein injective preenvelope.*

Theorem 2.4. *Let R be commutative Noetherian ring. Then every R-module N of finite Gorenstein injective dimension has a Gorenstein injective envelope $\psi : N \to G$ where injective dimension of $Ker\ \psi$ is finite and ψ is injective.*

Proof. We denote the class of Gorenstein injective R-modules by $\mathcal{GI}$. Let N be an R-module. Then, by Theorem 2.3, there is an exact sequence

$$0 \to N \to H \to C \to 0,$$

such that H is Gorenstein injective and $C \in^{\perp} \mathcal{GI}$. Then for any exact sequence $0 \to N \to H' \to C' \to 0$ with $C' \in^{\perp} \mathcal{GI}$, one has the exact sequence

$$Hom_R(H', H) \to Hom_R(N, H) \to Ext^1_R(C', H) = 0.$$

So,

$$\begin{array}{ccccc} 0 & \to & N & \to & H' \\ & & \| & & \\ 0 & \to & N & \to & H, \end{array}$$

can be completed to a commutative diagram. This shows that, for any exact sequence $0 \to N \to H' \to C' \to 0$ with $C' \in^{\perp} \mathcal{GI}$, there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & N & \to & H' & \to & C' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & H & \to & C & \to & 0, \end{array}$$

This means that $0 \to N \to H \to C \to 0$ generated $\mathcal{E}xt(^{\perp}\mathcal{GI}, N)$. Then since $^{\perp}\mathcal{GI}$ is closed under direct limits, by Theorem 2.2, $\mathcal{E}xt(^{\perp}\mathcal{GI}, N)$ has a minimal generator $0 \to N \xrightarrow{\psi} G \to L \to 0$, say. Therefore, we have the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \to & N & \to & G & \to & L & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & H & \to & C & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & G & \to & L & \to & 0, \end{array}$$

Since $0 \to N \to G \to L \to 0$ is a minimal generator, therefore $G \to G$ and $L \to L$ are isomorphisms. Hence G is a direct summand of H, so it is Gorenstein injective and L is a direct summand of C and will belong to ${}^{\perp}\mathcal{GI}$. So, $\psi : N \to G$ is a special Gorenstein injective preenvelope. Therefore by Definition 2.1, $\psi : N \to G$ is a Gorenstein injective envelope of N. □

In this part, $(R, \mathfrak{m})$ is a Noetherian local ring. Our main result in this direction is Theorem 2.7. It removes the assumption about Cohen-Macaulay ring from [13, Theorem 3.10]. In the following we review some definitions.

Definition 2.5. *The Krull dimension $Kdim_R(M)$ of an Artinian module M, is introduced by Roberts in [12] and is defined inductively as follows.*
When $M = 0$, we put $Kdim_R(M) = -1$. Then by induction, for any integer $\alpha \geqslant 0$, we put $Kdim_R(M) = \alpha$ if (i) $Kdim_R(M) < \alpha$ is false, and (ii) for any ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M there exists an integer n such that $Kdim_R(M_{i+1}/M_i) < \alpha$, for all $i > n$.
Following [9], we recall that if R is local ring with maximal ideal $\mathfrak{m}$, the width of a module M is defined as:

$$width_R M = inf\{i \in \mathbb{Z} | Tor_i^R(k, M) \neq 0\}.$$

Definition 2.6. (See[14]) *Assume that $(R, \mathfrak{m})$ is a local ring and M is a non-zero Artinian R-module. The module M is called a co-Cohen-Macaulay module if,*

$$width_R M = Kdim_R M.$$

Theorem 2.7. Let $(R, \mathfrak{m})$ be a local ring and M be a Cohen-Macaulay R-module of dimension n. If $H^n_{\mathfrak{m}}(M)$ is of finite Gorenstein injective dimension, then $Gid_R H^n_{\mathfrak{m}}(M) = depth R - n$. In particular, if $\mathrm{Gid}_R M$ is finite, then $\mathrm{Gid}_R H^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$.

Proof. By [3, Theorem 2.2], we have:

$$Gid_R H^n_{\mathfrak{m}}(M) = sup\{depth R_{\mathfrak{p}} - width_{R_{\mathfrak{p}}}(H^n_{\mathfrak{m}}(M))_{\mathfrak{p}} | \mathfrak{p} \in Spec(R)\}$$

$$= depth R - width_R H^n_{\mathfrak{m}}(M).$$

On the other hand M is Cohen-Macaulay R-module, therefore by [14, Proposition 2.6], $H^n_{\mathfrak{m}}(M)$ is co-Cohen-Macaulay R-module and hence $width_R H^n_{\mathfrak{m}}(M) = Kdim_R H^n_{\mathfrak{m}}(M) = n$. Therefore

$$Gid_R H^n_{\mathfrak{m}}(M) = depth R - n.$$

The second claim follows easily because by [3, Corollary 2.3], $Gid_R M = depth\ R$. $\square$

Acknowledgements
I am grateful to Professor M. Tousi for useful discussion, and for helpful suggestions, guidance and encouragement.

References

[1] M. Auslander and M. Bridger,Stable module theory, *Mem. Amer. Math. Soc.*, 94 (1969), 1-146.

[2] M. Auslander and R. O. Buchweitz, Maximal CohenMacaulay approximations, *Soc. Math. France, Mmoire*, 38 (1989), 5-37.

[3] L. W. Christensen and S. Sather-Wagstaff, Transfer of Gorenstein dimensions along ring homomorphisms, *J. Pure Appl. Algebra,* 214 (2010), 982-989.

[4] E. E. Enochs, S. Estrada, and A. Iacob, Gorenstein injective precover, covers and envelops, *Arxive: 1301. 5694 V1 [Math. AC],* 2013.

[5] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, de gruyter expositions in mathematics, 30 Walter de Gruyter and Co., Berlin, 2000.

[6] E. E. Enochs, O. M. G. Jenda, and J. Xu, A generalization of auslanders last theorem, *Algebr. Represent. Theory,* 2 (1999), 259-268.

[7] E. E. Enochs, O. M. G. Jenda, and J. Xu, Covers and envelopes over Gorenstein rings, *Tsukuba J. Math.,* 20 (1996), 487-503.

[8] E. E. Enochs and J. A. L'opez-Ramos, Kaplansky classes, *Rend. Sem. Mat. Univ. Padova*, 107 (2002), 67-79.

[9] H. B. Foxby, *Hyperhomological algebra and commutative rings*, notes in preparation.

[10] H. Krause, The stable derived category of a noetherian scheme, *Compos. Math.*, 141 (2005), 1128-1162.

[11] M. Nikkhah Babaei and K. Divaani-Aazar, Gorenstein injective envelops of Artinian modules, *Comm. Algebra,* to appear, arxive: 1306. 4452 v1 [math. AC], 2013.

[12] R. N. Roberts, Krull dimension of Artinian modules over quasi local commutative rings, *Quart. J. Math.(Oxford)*, 3 (26) (1975), 269-273.

[13] R. Sazeedeh, Gorenstein injectivity of section functor, *ArXive: 1008. 4485 v1 [Math. AC],* 2010.

[14] Z. Tang, Local Homology and Local Cohomology, *Algebra Colloquium*, 11 (4) (2004), 467-476.

[15] J. Z. Xu, *Flat Cover of Modules*, Lecture Notes in Math. 1634, Springer-Verlag, Berlin, 1996.

Zahra Heidarian
Department of Mathematics
Assistant Professor of Mathematics
Sciences and Research Branch
Islamic Azad University
Tehran, Iran
E-mail: zahraheidarian@gmail.com