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Original Research Paper

P-value and Posterior Probability for the Interval Hypothesis Testing

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Abstract. This article examines the behavior of frequentist p-values and Bayesian posterior probabilities for testing interval hypotheses in the Pareto distribution. We consider the hypothesis $H_{0\epsilon} : |\theta - \theta_0| \leq \epsilon$ against $H_{1\epsilon} : |\theta - \theta_0| > \epsilon$, where θ is the shape parameter. Our analysis reveals a fundamental divergence between classical and Bayesian approaches: while the p-value for the interval hypothesis converges to the point-null p-value as $\epsilon \to 0$, the Bayesian posterior probability exhibits no such convergence. This discrepancy persists across various prior distributions, including non-informative, exponential, and conjugate priors, and remains significant even for small values of ϵ. We demonstrate that common approximation methods for reconciling these measures fail for the Pareto distribution, emphasizing the distribution-dependent nature of this problem. The results highlight important limitations in using point-null approximations for interval hypothesis testing, particularly within the Bayesian framework. Our findings contribute to the ongoing discussion about reconciling frequentist and Bayesian evidence

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and provide guidance for practitioners working with power-law distributions in various application domains.

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1 Introduction

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a population with probability density function $f(x|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}$. For testing the hypothesis $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$, Bayesian statisticians typically use the posterior probability of the null hypothesis or the Bayes factor as evidence, while frequentists employ the p-value as the observed level of significance. An important question is whether these two schools of thought can reach consistent inferences in statistical testing. This topic has a long history and extensive literature.

Lindley [**?**] and Jeffreys [**?**] demonstrated in point hypothesis testing against two-sided alternatives that with a fixed p-value, the posterior probability of the null hypothesis increases with sample size. Consequently, the null hypothesis may exhibit high posterior probability while being rejected based on p-value. Since using specific prior distributions in the methods of Lindley and Jeffreys may introduce bias toward the null hypothesis, Edwards et al. [**?**] compared the p-value with the lower bound of the posterior probability when the prior distribution belongs to the family of all reasonable distributions. They showed that for the normal distribution, the lower bound of the posterior probability exceeds the p-value.

Building upon these results, further research employing reasonable classes of prior distributions for testing simple hypotheses against two-sided alternatives was conducted by Berger and Sellke [**?**], Berger and Delampady [**?**], and Chinipardaz [**?**]. Their results for normal, binomial, and Poisson distributions indicated that the p-value for point hypothesis testing against two-sided alternatives remains smaller than the lower bound of the posterior probability. Additional work in multivariate distributions was carried out by Berger and Delampady [**?**] and Chinipardaz and Abtahi [**?**]. Comparisons of p-values and posterior probabilities have

also been extended to various statistical distributions involving nuisance parameters.

The studies by Yin [**?**, **?**], and other researchers like Chinipardaz and Mehri, [**?**] have demonstrated that the discrepancy between p-values and posterior probabilities may persist in the presence of nuisance parameters. Sellke et al. [**?**] proposed calibrating p-values to achieve agreement between the two methods, utilizing the idea that the posterior probability under the alternative hypothesis might belong to the family of beta distributions.

The issue of testing point-null or one-sided hypotheses against one-sided alternatives has also been examined extensively. Robert [**?**] showed that for testing one-sided hypotheses against one-sided alternatives in normal distributions, p-values and posterior probabilities can be equivalent. In this context, Casella and Berger [**?**] compared the p-value with the lower bound of the posterior probability for one-sided tests in normal distributions, demonstrating that unlike two-sided tests, the p-value may exceed the posterior probability in one-sided scenarios. Similar results were obtained by various authors in Poisson and exponential distributions [**?**, **?**].

Criticism has been raised regarding the conflict between classical and Bayesian methods based on point hypothesis testing against two-sided alternatives, where the null hypothesis is a single point and the alternative is an interval. This has led researchers to examine interval null hypotheses as an alternative framework. In this regard, Berger and Delampady [**?**] showed for normal distributions that the posterior probability of a point hypothesis does not differ significantly from that of an interval hypothesis. Moreover, the interval probability value can approximate the point p-value.

Following Berger and Delampady [**?**] and Gómez-Villegas and Sanz [**?**] demonstrated for normal distributions that it is possible to define the prior distribution of the null hypothesis as $\int_{|\theta-\theta_0|\leq\epsilon} \pi(\theta)d\theta$ and equate the lower bound of the posterior probability with the p-value. In a recent study, Zolfaghari et al. [**?**] showed that the p-value in interval hypothesis testing can serve as a limit for the p-value in point hypothesis testing, though such a relationship does not hold for the posterior probability of interval hypotheses [**?**, **?**].

This paper aims to examine hypothesis testing for the shape parameter of the Pareto distribution using different prior distributions, considering both cases where the scale parameter is known and where it appears as a nuisance parameter. The motivation for selecting this distribution stems from its wide-ranging applications across various fields and its utility in analyzing phenomena that follow power-law distributions. Additionally, the Pareto distribution possesses two parameters: when the shape parameter is known, it belongs to the non-regular family of distributions, and when the scale parameter is known, it belongs to the exponential family. Therefore, the methodology developed here can be adapted to other distributions.

The paper is organized into six sections. Section 2 establishes the general framework for p-values and posterior probabilities in interval hypothesis testing. Section 3 examines the p-value for interval null hypotheses in Pareto distributions and compares it with point null hypotheses. Section 4 analyzes the sensitivity of posterior probability with respect to interval length, comparing it to point null hypotheses. To provide comprehensive results, this analysis is conducted for four different prior distributions. Section 5 investigates potential approximations for the interval length to reconcile posterior probabilities between interval and point null hypotheses. Finally, Section 6 summarizes the results and presents concluding remarks.

2 Set up the Problem

2.1 p-value

Let $X_1, \ldots, X_n \sim f_\theta(x)$ and suppose we wish to test a point-null hypothesis against a two-sided alternative:

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0 \tag{1}$$

where θ_0 is a fixed value. If $T(\mathbf{X})$ is an appropriate test statistic where large values provide evidence against H_0, then the p-value for testing

(1) is given by:

$$\begin{aligned} p(t) &= 2\min\{P_{\theta_0}(T \geq t), P_{\theta_0}(T \leq t)\} \\ &= 2\begin{cases} F_{\theta_0}(t) & t \leq F_{\theta_0}^{-1}(\frac{1}{2}) \\ 1 - F_{\theta_0}(t) & t \geq F_{\theta_0}^{-1}(\frac{1}{2}). \end{cases} \end{aligned} \tag{2}$$

where t is the observed value of T.

Now, consider testing the interval hypothesis:

$$H_{0\epsilon} : |\theta - \theta_0| \leq \epsilon \quad \text{against} \quad H_{1\epsilon} : |\theta - \theta_0| > \epsilon.$$

Since $\Theta_0 = [\theta_0 - \epsilon, \theta_0] \cup [\theta_0, \theta_0 + \epsilon]$, we define an appropriate p-value as follows. For testing $H_{0\epsilon}^{(1)}$ against $\theta > \theta_0$, the p-value is $\sup_{\theta \in [\theta_0 - \epsilon, \theta_0]} P(T \geq t)$, while for testing $H_{0\epsilon}^{(2)}$ against $\theta < \theta_0$, the p-value is $\sup_{\theta \in [\theta_0, \theta_0 + \epsilon]} P(T \leq t)$. Combining these two cases, the p-value for testing $H_{0\epsilon}$ against $H_{1\epsilon}$ is:

$$\begin{aligned} p_\epsilon(t) &= 2 \sup_{|\theta - \theta_0| \leq \epsilon} \left\{ \min\{P_\theta(T \geq t), P_\theta(T \leq t)\} \right\} \\ &= 2 \sup_{|\theta - \theta_0| \leq \epsilon} \begin{cases} F_\theta(t) & t \leq F_{\theta_0}^{-1}(\frac{1}{2}) \\ 1 - F_\theta(t) & t \geq F_{\theta_0}^{-1}(\frac{1}{2}). \end{cases} \end{aligned} \tag{3}$$

As a function of ϵ, $p_\epsilon(t)$ is nondecreasing in ϵ.

This formulation represents a more general case of (2) and is equivalent to:

$$p_\epsilon(t) = \sup_{|\theta - \theta_0| \leq \epsilon} (1 - F_\theta(|t - \theta_0|))$$

when T is unimodal and symmetric around θ_0. Intuitively, the supremum over $|\theta - \theta_0| \leq \epsilon$ guarantees a conservative p-value for the composite null hypothesis, ensuring that the Type I error rate is controlled uniformly over all $\theta \in \{\theta : |\theta - \theta_0| \leq \epsilon\}$, which extends the classical point-null p-value definition to the interval setting (see [**?**] for some special cases).

For stochastically increasing distributions [**?**], (3) can be simplified to:

$$p_\epsilon(t) = 2\begin{cases} F_{\theta+\epsilon}(t) & t \leq F_{\theta_0}^{-1}(\frac{1}{2}) \\ 1 - F_{\theta-\epsilon}(t) & t \geq F_{\theta_0}^{-1}(\frac{1}{2}). \end{cases}$$

2.2 Bayesian analysis

Let π_0 (with $\pi_1 = 1-\pi_0$) be the prior probability that $H_{0\epsilon}$ (respectively $H_{1\epsilon}$) is true, i.e., $\pi_0 = P(|\theta-\theta_0| \leq \epsilon)$ and $\pi_1 = P(|\theta-\theta_0| > \epsilon)$. In the Bayesian testing framework, the prior distribution for θ is:

$$g(\theta) = I_{|\theta-\theta_0|\leq\epsilon}(\theta)g_0(\theta) + I_{|\theta-\theta_0|>\epsilon}(\theta)g_1(\theta)$$

where $I_A(u)$ is the indicator function equal to one if $u \in A$ and zero otherwise. The functions $g_i(\theta)$, $i = 0, 1$ represent how θ is distributed over $H_{i\epsilon}$. The posterior probability for $H_{0\epsilon}$ is then:

$$P(H_{0\epsilon}|\mathbf{x}) = \left[1 + \frac{1-\pi_0}{\pi_0}\frac{1}{B_\epsilon(\mathbf{x})}\right]^{-1} \tag{4}$$

where $B_\epsilon(\mathbf{x})$ is the Bayes factor for $H_{0\epsilon}$ relative to $H_{1\epsilon}$, given by:

$$B_\epsilon(\mathbf{x}) = \frac{\int_{|\theta-\theta_0|\leq\epsilon} f(\mathbf{x}|\theta)g_0(\theta)d\theta}{\int_{|\theta-\theta_0|>\epsilon} f(\mathbf{x}|\theta)g_1(\theta)d\theta}$$

For the point-null hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, the prior distribution $g(\theta)$ is a discrete-continuous mixture given by:

$$g(\theta) = \pi_0 I_{\{\theta=\theta_0\}}(\theta) + (1-\pi_0)I_{\{\theta\neq\theta_0\}}(\theta)g_1(\theta)$$

which has a point mass at θ_0, with $g_1(\theta)$ representing the prior distribution of θ spread over the alternative parameter space. Therefore,

$$P(H_0|\mathbf{x}) = \left[1 + \frac{1-\pi_0}{\pi_0}\frac{1}{B(\mathbf{x})}\right]^{-1}$$

where $B(\mathbf{x}) = \frac{f(\mathbf{x}|\theta_0)}{\int_{-\infty}^{\infty} f(\mathbf{x}|\theta)g_1(\theta)d\theta}$.

3 P-value for Pareto Distribution

Let $X_1, \ldots, X_n \sim f(x|\theta,\beta) = \frac{\theta\beta^\theta}{x^{\theta+1}}, \quad x \geq \beta$, where $\theta > 0$ and $\beta > 0$ are shape and scale parameters, respectively. In this section, we first assume that the scale parameter β is known. This assumption allows us to derive the exact distribution of the test statistic and to obtain

closed-form expressions for the p-value. The case where β is unknown and must be estimated is treated separately in Subsection 3.1

Suppose that we wish to test $H_0 : |\theta-\theta_0| \leq \epsilon$ against $H_1 : |\theta-\theta_0| > \epsilon$. When β is known, $T(\mathbf{X},\beta) = \sum_{i=1}^{n} \ln \frac{X_i}{\beta}$ is an appropriate statistic which is a function of the sufficient statistic $\sum_{i=1}^{n} \ln X_i$. It is clear that $2\theta T(\mathbf{X},\beta) \sim \chi^2_{2n}$ where χ^2_k has a chi-square distribution with k degrees of freedom. Let $t = t(\mathbf{x},\beta)$; then under H_0 we have $2(\theta_0+\epsilon)T(\mathbf{X},\beta) \sim \chi^2_{2n}$. Therefore, the $p_\epsilon(t) = p\text{-value}_\epsilon$ is:

$$
\begin{aligned}
p_\epsilon(t) &= 2 \sup_{|\theta-\theta_0|\leq\epsilon} \{\min\{P_\theta(T(\mathbf{X},\beta) \geq t(\mathbf{x},\beta)), P_\theta(T(\mathbf{X},\beta) \leq t(\mathbf{x},\beta))\}\} \\
&= 2 \sup_{|\theta-\theta_0|\leq\epsilon} \begin{cases} P_\theta(\chi^2_{2n} > 2\theta t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) > \frac{n}{\theta} \\ P_\theta(\chi^2_{2n} < 2\theta t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) \leq \frac{n}{\theta} \end{cases} \\
&= 2 \begin{cases} P(\chi^2_{2n} > 2(\theta_0-\epsilon) t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) > \frac{n}{\theta_0-\epsilon} \\ P(\chi^2_{2n} < 2(\theta_0+\epsilon) t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) \leq \frac{n}{\theta_0+\epsilon} \end{cases} \\
&= 2 \begin{cases} 1 - F_{\chi^2_{2n}}[2(\theta_0-\epsilon) t(\mathbf{x},\beta)] & t(\mathbf{x},\beta) > \frac{n}{\theta_0-\epsilon} \\ F_{\chi^2_{2n}}[2(\theta_0+\epsilon) t(\mathbf{x},\beta)] & t(\mathbf{x},\beta) \leq \frac{n}{\theta_0+\epsilon} \end{cases}
\end{aligned}
$$

where $t(\mathbf{x},\beta) = \sum_{i=1}^{n} \ln \frac{x_i}{\beta}$ is the observed value of $T(\mathbf{x},\beta)$ and $F_{\chi^2_\nu}(u)$ is the cumulative distribution function at point u. Now, for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, the p-value is given by:

$$
\begin{aligned}
p(t) &= 2\min\{P_{\theta_0}(T(\mathbf{X},\beta) \geq t(\mathbf{x},\beta)), P_{\theta_0}(T(\mathbf{X},\beta) \leq t(\mathbf{x},\beta))\} \\
&= 2 \begin{cases} P_{\theta_0}(\chi^2_{2n} > 2\theta_0 t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) > \frac{n}{\theta_0} \\ P_{\theta_0}(\chi^2_{2n} \leq 2\theta_0 t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) \leq \frac{n}{\theta_0} \end{cases} \\
&= 2 \begin{cases} 1 - F_{\chi^2_{2n}}(2\theta_0 t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) > \frac{n}{\theta_0} \\ F_{\chi^2_{2n}}(2\theta_0 t(\mathbf{x},\beta)) & t(\mathbf{x},\beta) \leq \frac{n}{\theta_0} \end{cases}
\end{aligned}
$$

3.1 β is unknown

In this subsection, we assume that β is unknown, which is a reasonable assumption in practice. When β is unknown, the p-value becomes a function of β. Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics and x_i be the observed value of X_i. The recommended technique in classical analysis is the plug-in method, where β is substituted with its maximum likelihood

estimator, i.e., $\hat{\beta} = t_1(\mathbf{x}) = \sum_{i=1}^{n} \ln(\frac{x_i}{x_{(1)}})$, where $x_{(1)}$ is the first order statistic. The test statistic is $T_1(\mathbf{X}) = \sum_{i=1}^{n} \ln \frac{X_i}{X_{(1)}}$, which follows a chi-square distribution with $2n-2$ degrees of freedom. Letting $t_1 = T_1(\mathbf{x})$, we have:

$$\begin{aligned} p_\epsilon(t_1) &= 2 \sup_{|\theta-\theta_0|\leq\epsilon} \begin{cases} P_\theta(\chi^2_{2n-2} > 2\theta t_1(\mathbf{x})) & t_1(\mathbf{x}) > \frac{n-1}{\theta} \\ P_\theta(\chi^2_{2n-2} < 2\theta t_1(\mathbf{x})) & t_1(\mathbf{x}) \leq \frac{n-1}{\theta} \end{cases} \\ &= \begin{cases} 1 - F_{\chi^2_{2n-2}}((2\theta_0 - \epsilon)t_1(\mathbf{x})) & t_1(\mathbf{x}) > \frac{n-1}{\theta_0-\epsilon} \\ F_{\chi^2_{2n-2}}((2\theta_0 + \epsilon)t_1(\mathbf{x})) & t_1(\mathbf{x}) \leq \frac{n-1}{\theta_0+\epsilon}. \end{cases} \end{aligned}$$

For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, however,

$$p(t_1) = \begin{cases} 1 - F_{\chi^2_{2n-2}}(2\theta_0 t_1(\mathbf{x})) & t_1(\mathbf{x}) > \frac{n-1}{\theta_0} \\ F_{\chi^2_{2n-2}}(2\theta_0 t_1(\mathbf{x})) & t_1(\mathbf{x}) \leq \frac{n-1}{\theta_0}. \end{cases}$$

Comparison between $p_\epsilon(t_1)$ and $p(t_1)$ shows that the p-value for H_0 is the limiting value of the p-value for $H_{0\epsilon}$, i.e., $\lim_{\epsilon\to 0} p_\epsilon(t_1) = p(t_1)$, since:

$$\begin{aligned} p_\epsilon(t_1) &= p(t_1) + \int_{2\theta_0 t(\mathbf{x},\beta)}^{2(\theta_0+\epsilon)t(\mathbf{x},\beta)} f_{\chi^2_{2n}}(u)du > p(t_1) \\ \text{or} \quad p_\epsilon(t_1) &= p(t_1) + \int_{(2\theta_0-\epsilon)t(\mathbf{x},\beta)}^{2\theta_0 t(\mathbf{x},\beta)} f_{\chi^2_{2n}}(u)du > p(t_1). \end{aligned}$$

This discussion confirms that when β is known, $\lim_{\epsilon\to 0} p_\epsilon(t) = p(t)$.

Table 1 shows the ratio of $p(t)$ to $p_\epsilon(t)$ and $p(t_1)$ to $p_\epsilon(t_1)$ (in parentheses), specifically $\frac{p\text{-value}}{p\text{-value}_\epsilon}$, as ϵ varies. The test statistic $T(\mathbf{x})$ is chosen to yield the usual p-values: 0.1, 0.05, and 0.01. Two cases are considered. First, β is known (with $\beta = 1$ presented in the table; other values yield similar results). Second, the values in parentheses correspond to the case where β is replaced by its maximum likelihood estimator. As can be seen from the table, as ϵ takes small values, $\frac{p\text{-value}}{p\text{-value}_\epsilon}$ tends to one. A ratio close to 1 indicates that the standard point-null p-value can be used as a reliable approximation for the interval p-value when the interval width ϵ is sufficiently small. As the ratio decreases, the discrepancy between the two p-values becomes more pronounced, suggesting that the interval nature of the null hypothesis cannot be ignored. This

Table 1: The ratio of p-value to p-value$_\epsilon$ for commonly used p-values when $\theta_0 = 1$. The values in parentheses correspond to the case where the scale parameter β is unknown.

		ϵ				
p-value	**n**	**0.0001**	**0.001**	**0.01**	**0.1**	**1**
0.1	5	0.9994(0.9994)	0.9943(0.9948)	0.9450(0.9497)	0.5761(0.6031)	0.05(0.05)
	10	0.9992(0.9992)	0.9924(0.9927)	0.9271(0.9302)	0.4846(0.4991)	0.05(0.05)
	25	0.9988(0.9988)	0.9886(0.9888)	0.8929(0.8948)	0.3526(0.3585)	0.05(0.05)
	35	0.9986(0.9986)	0.9967(0.9869)	0.8765(0.8780)	0.3044(0.3085)	0.05(0.05)
0.05	5	0.9993(0.9993)	0.9933(0.9938)	0.9357(0.9406)	0.5202(0.5479)	0.025(0.025)
	10	0.9991(0.9991)	0.9911(0.9915)	0.9153(0.9187)	0.4270(0.4417)	0.025(0.025)
	25	0.9986(0.9987)	0.9869(0.9871)	0.8772(0.8792)	0.2962(0.3020)	0.025(0.025)
	35	0.9984(0.9984)	0.9847(0.9849)	0.8589(0.8606)	0.2497(0.2537)	0.025(0.025)
0.01	5	0.9991(0.9991)	0.9910(0.9917)	0.9144(0.9207)	0.4154(0.4430)	0.005(0.005)
	10	0.9988(0.9988)	0.9884(0.9888)	0.8904(0.8945)	0.3248(0.3389)	0.005(0.005)
	25	0.9983(0.9983)	0.9831(0.9834)	0.8448(0.8473)	0.2045(0.2096)	0.005(0.005)
	35	0.9980(0.9980)	0.9805(0.9808)	0.8231(0.8251)	0.1643(0.1676)	0.005(0.005)

means that reporting the p-value for point-null hypotheses can be used for the p-value for small interval-null hypotheses. The next two sections are devoted to investigating this behavior for the posterior probability.

4 Posterior Probability for Pareto Distribution

Consider testing $H_{0\epsilon} : |\theta - \theta_0| \leq \epsilon$ against $H_{1\epsilon} : |\theta - \theta_0| > \epsilon$ for the Pareto distribution. Initially, assume β is known. We have:

$$B_\epsilon(\mathbf{x}) = \frac{\int_{|\theta-\theta_0|\leq\epsilon} f(\mathbf{x}|\theta,\beta)g_0(\theta)d\theta}{\int_{|\theta-\theta_0|>\epsilon} f(\mathbf{x}|\theta,\beta)g_1(\theta)d\theta}.$$

When using the invariant Jeffreys prior $g_i(\theta) = \frac{1}{\theta}$, $i = 0, 1$:

$$\begin{aligned} m(\mathbf{x}|H_{0\epsilon}) &= \int_{\theta_0-\epsilon}^{\theta_0+\epsilon} \theta^n \beta^{n\theta} \frac{1}{\prod_{i=1}^n x_i^{(\theta+1)}} \frac{1}{\theta} d\theta \\ &= \frac{1}{\prod_{i=1}^n x_i} \frac{(n-1)!}{t(\mathbf{x},\beta)^n} \left[\Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0+\epsilon) - \Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0-\epsilon)\right] \end{aligned}$$

where $\Gamma_{(n,t(\mathbf{x},\beta))}(u)$ denotes the cumulative gamma distribution function with parameters n and $t(\mathbf{x},\beta) = \sum_{i=1}^{n} \ln \frac{x_i}{\beta}$ evaluated at point u. Following from (4), we obtain

$$P(H_{0\epsilon}|\mathbf{x},\beta) = \Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0+\epsilon) - \Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0-\epsilon).$$

Now, consider testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$:

$$B(\mathbf{x}) = \frac{\theta_0^n \beta^{n\theta_0} e^{-(\theta_0+1)\log\sum_{i=1}^{n} x_i}}{\int_0^\infty \theta^n \beta^{n\theta} e^{-(\theta+1)\log\sum_{i=1}^{n} x_i} d\theta} = \frac{(\theta_0 t(\mathbf{x},\beta))^n}{(n-1)! e^{\theta_0 t(\mathbf{x},\beta)}}$$

and

$$P(H_0|\mathbf{x}) = \left[1 + \frac{(n-1)! e^{\theta_0 t(\mathbf{x},\beta)}}{(\theta_0 t(\mathbf{x},\beta))^n}\right]^{-1}.$$

Example 4.1. To obtain an analytic solution for ϵ, consider the special case $n = 1$ and $g(\theta) = \frac{1}{\theta}$, then:

$$\begin{aligned} P(H_0|x) &= \left[1 + \frac{(\frac{x}{\beta})^{\theta_0}}{\theta_0 \ln \frac{x}{\beta}}\right]^{-1}, \\ P(H_{0\epsilon}|x) &= \Gamma_{(1,\ln\frac{x}{\beta})}(\theta_0+\epsilon) - \Gamma_{(1,\ln\frac{x}{\beta})}(\theta_0-\epsilon) \\ &= e^{-(\theta_0-\epsilon)\ln\frac{x}{\beta}} - e^{-(\theta_0+\epsilon)\ln\frac{x}{\beta}} \\ &= 2(\frac{x}{\beta})^{-\theta_0} \sinh(\epsilon \ln \frac{x}{\beta}). \end{aligned}$$

4.1 β is unknown

Assuming β is unknown represents a more realistic scenario. The recommended prior for this parameter is Jeffreys prior, $g(\beta|\theta) = \frac{1}{\beta}$. We

then have:

$$
\begin{aligned}
m_{g_1}(\mathbf{x}|H_{0\epsilon}) &= \int_{\theta_0-\epsilon}^{\theta_0+\epsilon}\int_0^{x_{(1)}} L(\theta,\beta|\mathbf{x})g(\beta|\theta)g(\theta)d\beta d\theta \qquad (5)\\
&= \int_{\theta_0-\epsilon}^{\theta_0+\epsilon}\int_0^{x_{(1)}} \theta^n\beta^{n\theta}\exp\{-(\theta+1)\sum_{i=1}^n \ln x_i\}\frac{1}{\beta}\frac{1}{\theta}d\beta d\theta\\
&= \frac{1}{\prod_{i=1}^n x_i}\int_{\theta_0-\epsilon}^{\theta_0+\epsilon}\theta^{n-2}\exp\{-\theta(\sum_{i=1}^n \ln\frac{x_i}{x_{(1)}})\}d\theta\\
&= \frac{1}{\prod_{i=1}^n x_i}\frac{(n-2)!}{t_1(\mathbf{x})^{n-1}}\big(\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0+\epsilon)-\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0-\epsilon)\big),
\end{aligned}
$$

where $t_1(\mathbf{x})=\sum_{i=1}^n \ln\frac{x_i}{x_{(1)}}$. For testing H_0 against H_1, we have:

$$
m_{g_0}(\mathbf{x},H_0) = \theta_0^n\beta^{n\theta_0}\exp\{-(\theta_0+1)\sum_{i=1}^n \ln x_i\}\frac{1}{\beta}d\beta, \qquad (6)
$$

and

$$
\begin{aligned}
m_{g_1}(\mathbf{x}|H_{1\epsilon}) &= \int_0^{\infty}\int_0^{x_{(1)}} L(\theta,\beta|\mathbf{x})\pi(\beta|\theta)\pi(\theta)d\beta d\theta\\
&= \frac{1}{\prod_{i=1}^n x_i}\frac{(n-2)!}{t_1(\mathbf{x})^{n-1}}. \qquad (7)
\end{aligned}
$$

Now, $P(H_{0\epsilon}|\mathbf{x})$ and $P(H_0|\mathbf{x})$ can be obtained using (5), (6) and (7). This analysis was repeated for three additional prior distributions:

1. Noninformative prior, $g_2(\theta)=k$ where k is a constant
2. Exponential distribution $g_3(\theta)=ue^{-u\theta},\ u>0$
3. The conjugate distribution $g_4(\theta)=\frac{b^a\theta^{a-1}}{(a-1)!}e^{-b\theta}$ with $\theta_0=\text{mean}(\theta)$

The posterior probabilities for point-null and interval-null hypotheses are obtained and presented in Table 2. Four prior distributions are considered in two cases: when the scale parameter β is known (first part of table) and when β is unknown (second part of table).

To assess the sensitivity of our Bayesian results to the choice of prior parameters, we conducted additional analyses for the conjugate

Table 2: The Bayes factor for four priors, $g_1(\theta) = \frac{1}{\theta}$, $g_2(\theta) = 1$, $g_3(\theta) = e^{-\theta}$ and $g_4(\theta) = \frac{b^a \theta^{a-1}}{(a-1)!} e^{-b\theta}$. Note that $t(\mathbf{x}, \beta) = \sum \ln \frac{x_i}{\beta}$ and $t_1(\mathbf{x}) = \sum \ln \frac{x_i}{x_{(1)}}$

Prior distribution	$B_\epsilon(\mathbf{x})$	$B(\mathbf{x})$
$g_1(\theta) = \frac{1}{\theta}$	$\frac{\Gamma_{(n-1,t(\mathbf{x},\beta))}(\theta_0+\epsilon)-\Gamma_{(n-1,t(\mathbf{x},\beta))}(\theta_0-\epsilon)}{1+\Gamma_{(n-1,t(\mathbf{x},\beta))}(\theta_0-\epsilon)-\Gamma_{(n-1,t(\mathbf{x},\beta))}(\theta_0+\epsilon)}$	$\frac{(t(\mathbf{x},\beta)\theta_0)^{n-1}}{(n-2)!e^{\theta_0 t(\mathbf{x},\beta)}}$
$g_2(\theta) = 1$	$\frac{\Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0+\epsilon)-\Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0-\epsilon)}{1+\Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0-\epsilon)-\Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0+\epsilon)}$	$\frac{\theta_0^{n-1}(t(\mathbf{x},\beta))^n}{e^{-\theta_0 t(\mathbf{x},\beta)}(n-1)!}$
$g_3(\theta) = ue^{-\theta u}$	$\frac{\Gamma_{(n,t(\mathbf{x},\beta)+u)}(\theta_0+\epsilon)-\Gamma_{(n,t(\mathbf{x},\beta)+u)}(\theta_0-\epsilon)}{1+\Gamma_{(n,t(\mathbf{x},\beta)+u)}(\theta_0-\epsilon)-\Gamma_{(n,t(\mathbf{x},\beta)+u)}(\theta_0+\epsilon)}$	$\frac{\theta_0^{n-1}(t(\mathbf{x},\beta)+u)^n}{ue^{\theta_0 t(\mathbf{x},\beta)}(n-1)!}$
$g_4(\theta) = \frac{b^a \theta^{a-1}}{(a-1)!} e^{-b\theta}$	$\frac{\Gamma_{(n+a-1,t(\mathbf{x},\beta)+b)}(\theta_0+\epsilon)-\Gamma_{(n+a-1,t(\mathbf{x},\beta)+b)}(\theta_0-\epsilon)}{1+\Gamma_{(n+a-1,t(\mathbf{x},\beta)+b)}(\theta_0-\epsilon)-\Gamma_{(n+a-1,t(\mathbf{x},\beta)+b)}(\theta_0+\epsilon)}$	$\frac{(a-1)!\theta_0^{n-1}(t(\mathbf{x},\beta)+b)^{n+a-1}}{b^a(n+a-2)!e^{\theta_0 t(\mathbf{x},\beta)}}$
$g_1(\theta) = \frac{1}{\theta}$	$\frac{\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0+\epsilon)-\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0-\epsilon)}{1+\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0-\epsilon)-\Gamma_{(n-1,t_1(\mathbf{x}))}(\theta_0+\epsilon)}$	$\frac{(t_1(\mathbf{x})\theta_0)^{n-1}}{(n-2)!e^{\theta_0 t_1(\mathbf{x})}}$
$g_2(\theta) = 1$	$\frac{\Gamma_{(n,t_1(\mathbf{x}))}(\theta_0+\epsilon)-\Gamma_{(n,t_1(\mathbf{x}))}(\theta_0-\epsilon)}{1+\Gamma_{(n,t_1(\mathbf{x}))}(\theta_0-\epsilon)-\Gamma_{(n,t_1(\mathbf{x}))}(\theta_0+\epsilon)}$	$\frac{\theta_0^{n-1}(t_1(\mathbf{x}))^n}{e^{\theta_0 t_1(\mathbf{x})}(n-1)!}$
$g_3(\theta) = ue^{-\theta u}$	$\frac{\Gamma_{(n,t_1(\mathbf{x})+u)}(\theta_0+\epsilon)-\Gamma_{(n,t_1(\mathbf{x})+u)}(\theta_0-\epsilon)}{1+\Gamma_{(n,t_1(\mathbf{x})+u)}(\theta_0-\epsilon)-\Gamma_{(n,t_1(\mathbf{x})+u)}(\theta_0+\epsilon)}$	$\frac{\theta_0^{n-1}(t_1(\mathbf{x})+u)^n}{ue^{\theta_0 t_1(\mathbf{x})}(n-1)!}$
$g_4(\theta) = \frac{b^a \theta^{a-1}}{(a-1)!} e^{-b\theta}$	$\frac{\Gamma_{(n+a-1,t_1(\mathbf{x})+b)}(\theta_0+\epsilon)-\Gamma_{(n+a-1,t_1(\mathbf{x})+b)}(\theta_0-\epsilon)}{1+\Gamma_{(n+a-1,t_1(\mathbf{x})+b)}(\theta_0-\epsilon)-\Gamma_{(n+a-1,t_1(\mathbf{x})+b)}(\theta_0+\epsilon)}$	$\frac{(a-1)!\theta_0^{n-1}(t_1(\mathbf{x})+b)^{n+a-1}}{b^a(n+a-2)!e^{\theta_0 t_1(\mathbf{x})}}$

prior $g_4(\theta)$ by varying the hyperparameters a and b while keeping the prior mean $E(\theta) = a/b = \theta_0$ fixed. We also explored cases where the prior variance was increased or decreased by a factor of 2. The results, not reported here for brevity, indicate that while the absolute values of the posterior probabilities $P(H_{0\epsilon}|\mathbf{x})$ and $P(H_0|\mathbf{x})$ are somewhat affected by these changes, the qualitative conclusion remains unchanged: the posterior probability for the interval hypothesis does not converge to that of the point-null hypothesis as $\epsilon \to 0$, and the discrepancy persists across all prior specifications examined. This robustness reinforces the main message of the paper and suggests that the observed divergence is not an artifact of a particular prior choice but rather a fundamental feature of the Pareto distribution in this testing context.

5 Closeness of Posterior Probabilities of H_0 and $H_{0\epsilon}$

The closeness of $P(H_0|\mathbf{x})$ and $P(H_{0\epsilon}|\mathbf{x})$ can be measured by their ratio, $\frac{P(H_{0\epsilon}|\mathbf{x})}{P(H_0|\mathbf{x})}$. Suppose that β is known and $g_i(\theta) = \frac{1}{\theta}$ for $i = 0, 1$. As long as $\frac{P(H_0|\mathbf{x})}{P(H_{0\epsilon}|\mathbf{x})}$ is bounded between d_1 and d_2,

$$d_1\left[1+\frac{(n-1)!e^{\theta_0 t(\mathbf{x},\beta)}}{(\theta_0 t(\mathbf{x},\beta))^n}\right] \leq \Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0+\epsilon) - \Gamma_{(n,t(\mathbf{x},\beta))}(\theta_0-\epsilon)$$
$$\leq d_2\left[1+\frac{(n-1)!e^{\theta_0 t(\mathbf{x})}}{(\theta_0 t(\mathbf{x}))^n}\right].$$

As an example, consider Example 4.1. From $d_1 \leq \frac{P(H_0|x)}{P(H_{0\epsilon}|x)} \leq d_2$, we obtain

$$\sinh^{-1}\left[2d_2(1+\frac{x}{\beta})^{\theta_0}\ln(\frac{x}{\beta})^{-\theta_0}\right]^{-1} \leq \epsilon \leq \sinh^{-1}\left[2d_1(1+\frac{x}{\beta})^{\theta_0}\ln(\frac{x}{\beta})^{-\theta_0}\right]^{-1}.$$

Suppose that $d_1 = 0.9$ and $d_2 = 1.1$, i.e., the error is approximately no more than 0.1. Table 3 shows the values of ϵ when the ratio of the posterior probability of $H_{0\epsilon}$ to the posterior probability of H_0 is bounded between 0.9 and 1.1. The results are given for observations under which the p-value is 0.01, 0.05, and 0.1. Four prior distributions are considered: $g_1(\theta) = \frac{1}{\theta}$, $g_2(\theta) = 1$, $g_3(\theta) = e^{-\theta}$, and $g_4(\theta) = \frac{b^a\theta^{a-1}}{(a-1)!}e^{-b\theta}$. For $g_4(\theta)$, we consider $E(\theta) = \frac{a}{b} = \theta_0$. It is common in Bayesian analysis to reduce the parameters of the prior distribution by focusing the mean on the H_0 hypothesis.

We follow this method when the nuisance parameter β is unknown. Since β is a scale parameter, the recommended prior distribution $g(\beta) = \frac{1}{\beta}$ is used. The reader can also examine other reasonable priors. The results are given in Table 4. As can be seen from the two tables, the interval for ϵ becomes smaller as n increases. Also, comparing the results with those in Table 1 shows that, in contrast to the p-value, the posterior probabilities for $H_{0\epsilon}$ cannot be approximated by the posterior probability of H_0 when ϵ is small. To illustrate this, consider a p-value equal to 0.01 and $n = 5$ for $\beta = 1$. To have a difference of no more

Table 3: The interval for ϵ when the posterior probability ratio comparing hypothesis H_0 to $H_{0\epsilon}$ is between 0.9 and 1.1 ($0.9 \leq \frac{P(H_0|\mathbf{x})}{P(H_{0\epsilon}|\mathbf{x})} \leq 1.1$). For all cases, $\beta = 1$, $\theta_0 = 4$, and for $g_4(\theta)$, $a = 3$ and $b = a/\theta_0 = 3/4$ so that $E(\theta) = \theta_0 = 4$.

p-value	**n**	Intervals for ε			
		$g_1(\theta) = \frac{1}{\theta}$	$g_2(\theta) = 1$	$g_3(\theta) = e^{-\theta}$	$g_4(\theta) = \Gamma(a, b)$
0.01	5	[0.590, 0.650]	[0.282, 0.322]	[1.331, 1.417]	[0.254, 0.283]
	10	[0.507, 0.555]	[0.253, 0.287]	[1.109, 1.184]	[0.205, 0.225]
	25	[0.403, 0.440]	[0.214, 0.239]	[0.849, 0.912]	[0.141, 0.152]
	40	[0.355, 0.386]	[0.193, 0.215]	[0.731, 0.790]	[0.112, 0.12]
	100	[0.272, 0.294]	[0.156, 0.171]	[0.534, 0.587]	[0.068, 0.071]
0.05	5	[0.698, 0.785]	[0.310, 0.361]	[1.643, 1.789]	[0.273, 0.307]
	10	[0.597, 0.670]	[0.280, 0.323]	[1.352, 1.495]	[0.219, 0.242]
	25	[0.473, 0.530]	[0.239, 1.273]	[0.999, 1.141]	[0.149, 0.162]
	40	[0.416, 0.465]	[0.218, 0.247]	[0.840, 0.984]	[0.118, 0.127]
	100	[0.316, 0.354]	[0.177, 0.199]	[0.581, 0.745]	[0.070, 0.074]
0.1	5	[0.749, 0.856]	[0.322, 0.379]	[1.778, 1.977]	[0.281, 0.318]
	10	[0.639, 0.729]	[0.293, 0.341]	[1.444, 1.651]	[0.225, 0.250]
	25	[0.504, 0.576]	[0.250, 0.289]	[1.038, 1.269]	[0.152, 0.166]
	40	[0.441, 0.505]	[0.228, 0.262]	[0.860, 1.122]	[0.120, 0.129]
	100	[0.332, 0.385]	[0.186, 0.212]	[0.582, 0.999]	[0.071, 0.075]

than 10%, ϵ must lie in $(0.59; 0.65)$. This means that $P(H_{0\epsilon}|\mathbf{x})$ is not close to $P(H_0|\mathbf{x})$ when $|\epsilon| \leq 0.59$. Consequently, $P(H_0|\mathbf{x})$ cannot be an approximation to $P(H_{0\epsilon}|\mathbf{x})$ unless $|\theta - \theta_0| > 0.59$.

5.1 An approximation for ϵ

The complexity in obtaining $P(H_{0\epsilon}|\mathbf{x})$ presents a challenge for researchers. Some authors assumed that $f(x|\theta)$ is sufficiently smooth that there is only a small change in the small interval $(\theta_0 - \epsilon, \theta_0 + \epsilon)$ [**?**, **?**]. Therefore,

Table 4: The interval for ϵ when the posterior probability ratio comparing hypothesis H_0 to $H_{0\epsilon}$ is between 0.9 and 1.1 ($0.9 \leq \frac{P(H_0|\mathbf{x})}{P(H_{0\epsilon}|\mathbf{x})} \leq 1.1$). For all cases, $\theta_0 = 2$, $a = 3$, $b = a/\theta_0 = 3/2$, and β is unknown with prior distribution $g(\beta|H_0) = g(\beta|H_1) = g(\beta) = 1/\beta$.

p-value	**n**	Intervals for ε			
		$g_1(\theta) = \frac{1}{\theta}$	$g_2(\theta) = 1$	$g_3(\theta) = e^{-\theta}$	$g_4(\theta) = \Gamma(a, b)$
0.1	5	[0.393, 0.449]	[0.386, 0.394]	[0.368, 0.381]	[0.357, 0.709]
	10	[0.328, 0.374]	[0.293, 0.334]	[0.264, 0.297]	[0.245, 0.574]
	25	[0.255, 0.291]	[0.214, 0.243]	[0.194, 0.267]	[0.159, 0.367]
	40	[0.222, 0.254]	[0.184, 0.209]	[0.173, 0.202]	[0.158, 0.272]
	100	[0.167, 0.193]	[0.137, 0.155]	[0.237, 0.302]	[0.218, 0.272]
0.001	5	[0.366, 0.412]	[0.384, 0.431]	[0.341, 0.417]	[0.337, 0.411]
	10	[0.306, 0.343]	[0.287, 0.322]	[0.272, 0.317]	[0.264, 0.323]
	25	[0.240, 0.268]	[0.207, 0.231]	[0.194, 0.217]	[0.173, 0.196]
	40	[0.211, 0.234]	[0.177, 0.197]	[0.173, 0.196]	[0.159, 0.177]
	100	[0.159, 0.177]	[0.113, 0.146]	[0.230, 0.268]	[0.211, 0.245]
0.01	5	[0.309, 0.340]	[0.359, 0.394]	[0.329, 0.357]	[0.307, 0.353]
	10	[0.260, 0.284]	[0.263, 0.288]	[0.247, 0.271]	[0.203, 0.282]
	25	[0.204, 0.222]	[0.186, 0.203]	[0.171, 0.196]	[0.159, 0.177]
	40	[0.179, 0.194]	[0.158, 0.171]	[0.157, 0.196]	[0.159, 0.177]
	100	[0.137, 0.147]	[0.115, 0.125]	[0.203, 0.220]	[0.187, 0.203]

they used

$$\int_{\theta_0-\epsilon}^{\theta_0+\epsilon} f(\mathbf{x}|\theta)\pi(\theta)d\theta \approx 2\epsilon f(\mathbf{x}|\theta_0)\pi(\theta_0). \tag{8}$$

Then, as long as $d_1 \leq \frac{P(H_{0\epsilon}|\mathbf{x})}{P(H_0|\mathbf{x})} \leq d_2$, it leads to

$$d_1 \leq \frac{2\epsilon f(\mathbf{x}|\theta_0)\pi(\theta_0)}{[1 + \frac{m(\mathbf{x})}{f(\mathbf{x}|\theta_0)}]^{-1}} \leq d_2.$$

After some manipulation, we derive

$$\frac{d_1}{2\pi(\theta_0)[f(\mathbf{x}|\theta_0) + m(\mathbf{x})]} \leq \epsilon \leq \frac{d_2}{2\pi(\theta_0)[f(\mathbf{x}|\theta_0) + m(\mathbf{x})]}.$$

Table 5: The interval for ϵ_1 (Section 4) and ϵ_2 from (8) when the posterior probability ratio of H_0 to $H_{0\epsilon}$ is between 0.9 and 1.1 ($0.9 \leq \frac{P(H_0|\mathbf{x})}{P(H_{0\epsilon}|\mathbf{x})} \leq 1.1$). For all cases, $\beta = 1$ and $\theta_0 = 4$. For $g_4(\theta)$, $a = 3$.

ϵ	n	$g_1(\theta) = \frac{1}{\theta}$	$g_2(\theta) = 1$	$g_3(\theta) = e^{-\theta}$	$g_4(\theta) = \frac{b^a \theta^{a-1}}{(a-1)!} e^{-b\theta}$
ϵ_1	25	[0.504, 0.576]	[0.250, 0.289]	[1.038, 1.269]	[0.401, 0.423]
ϵ_2	25	[1.560, 1.590]	[1.590, 1.620]	[1.920, 1.960]	[1.930, 1.960]
ϵ_1	40	[0.441, 0.505]	[0.228, 0.262]	[0.860, 1.122]	[0.120, 0.129]
ϵ_2	40	[1.390, 1.410]	[1.510, 1.530]	[1.620, 1.640]	[1.750, 1.768]
ϵ_1	100	[0.332, 0.385]	[0.186, 0.212]	[0.582, 3.999]	[0.071, 0.075]
ϵ_2	100	[1.150, 1.160]	[1.320, 1.330]	[1.253, 1.259]	[1.420, 1.430]

This provides an approximate bound for ϵ to achieve closeness between $P(H_{0\epsilon}|\mathbf{x})$ and $P(H_0|\mathbf{x})$. As an example, consider Example 4.1 with $n = 1$ and $\pi(\theta) = \frac{1}{\theta}$. We have $m(x) = (x \ln(\frac{x}{\beta}))^{-1}$ and therefore,

$$\frac{\ln(\frac{x}{\beta})^{d_1\theta_0 x}}{2[\ln(\frac{x}{\beta})^x + \frac{\theta_0}{x}(\frac{x}{\beta})^{-\theta_0}]} \leq \epsilon \leq \frac{\ln(\frac{x}{\beta})^{d_2\theta_0 x}}{2[\ln(\frac{x}{\beta})^x + \frac{\theta_0}{x}(\frac{x}{\beta})^{-\theta_0}]}.$$

To demonstrate that this approximation is not generally good, consider again testing $H_{0\epsilon}$ against $H_{1\epsilon}$ when β is known and the p-value is 0.1. Table 5 presents ϵ obtained from Section 4 (denoted ϵ_1) and this approximation (denoted ϵ_2). The difference between ϵ_1 and ϵ_2 shows that reducing $P(H_{0\epsilon}|\mathbf{x})$ to $P(H_0|\mathbf{x})$ in this manner is not generally reliable.

The failure of the approximation (8) for the Pareto distribution can be explained analytically as follows. The approximation relies on the assumption that the likelihood function $f(\mathbf{x}|\theta)$ is sufficiently smooth and varies slowly over the interval $(\theta_0 - \epsilon, \theta_0 + \epsilon)$, so that it can be well-approximated by its value at the midpoint θ_0. However, for the Pareto distribution, the likelihood function takes the form

$$f(\mathbf{x}|\theta) = \theta^n \beta^{n\theta} \exp\left\{-(\theta+1)\sum_{i=1}^{n} \ln x_i\right\},$$

which is not smooth in θ when β is unknown, due to the presence of the term $\beta^{n\theta}$ and the fact that the support depends on β. More importantly,

when β is estimated by the plug-in method, the likelihood becomes proportional to $\theta^{n-2}e^{-\theta t_1(\mathbf{x})}$, which can be highly skewed and concentrated near the boundary of the parameter space. Consequently, the integral over the interval cannot be accurately approximated by $2\epsilon f(\mathbf{x}|\theta_0)\pi(\theta_0)$, especially when the interval width ϵ is not extremely small or when the sample size is moderate. This non-regular behavior is a distinctive feature of the Pareto distribution and explains why the approximation yields unreliable bounds for ϵ, as evidenced by the substantial differences between ϵ_1 and ϵ_2 in Table 5.

6 Summary and Conclusion

This article has provided a comprehensive analysis of interval hypothesis testing for the shape parameter of the Pareto distribution, contrasting frequentist p-values with Bayesian posterior probabilities. Our investigation yields several key insights that underscore the fundamental differences between these two statistical paradigms.

The primary finding of this study is the divergent behavior of p-values and posterior probabilities as the interval width ϵ diminishes. We have rigorously demonstrated that the p-value for the interval hypothesis $H_{0\epsilon}$ converges to the p-value for the point-null hypothesis H_0 as $\epsilon \to 0$. This result justifies the use of the standard point-null p-value as a practical approximation for testing small-interval hypotheses within the frequentist framework.

In stark contrast, the Bayesian posterior probability exhibits no such convergence. Even for small values of ϵ, a significant discrepancy persists between $P(H_{0\epsilon}|\mathbf{x})$ and $P(H_0|\mathbf{x})$. This finding, robust across various prior distributions (including the non-informative $1/\theta$, constant, exponential, and conjugate Gamma priors), reveals a fundamental irreconcilability in the Bayesian context. Our results, detailed in Tables 3 and 4, show that the bounds on ϵ required to achieve closeness between the two posterior probabilities are often substantial and highly sensitive to sample size and the chosen prior. Consequently, the posterior probability of a point-null hypothesis cannot generally serve as a reliable approximation for that of an interval hypothesis.

Furthermore, we have shown that a common smoothness approxi-

mation, which leads to a simplified bound for ϵ (Equation (8)), fails to provide accurate results for the Pareto distribution, as evidenced by the comparison in Table 5. This underscores the necessity of employing the exact methodological framework developed in this paper for reliable inference.

Our conclusions differ from those of Berger and Delampady [**?**] in the normal distribution context, highlighting that the reconcilability of these statistical measures is not universal but is instead dependent on the underlying distribution. The Pareto distribution, with its distinct properties as both a non-regular and exponential family member, provides a critical counterexample. In summary, while the frequentist p-value offers a consistent measure as the null hypothesis shifts from an interval to a point, the Bayesian posterior probability does not. This work emphasizes the importance of carefully considering the chosen statistical framework and the underlying model assumptions when testing interval-based hypotheses. For future research, it would be valuable to extend this analysis to other non-regular distributions and to develop more robust Bayesian methods for interval hypothesis testing that are less sensitive to prior specification and interval width.

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