Journal of Mathematical Extension
Vol. 20, No. 5 (2026) (1) 1-39
ISSN: 1735-8299
URL: http://doi.org/10.30495/JME.2026.3615
Original Research Paper

A GKM-Auxiliary Framework for the (2+1)-Dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Equation with Applications in Fluid Dynamics and Nonlinear Optics

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Abstract. This study presents a novel GKM-Auxiliary framework combining the Generalized Kudryashov Method (GKM) and the Extended Auxiliary Equation Method (EAEM) to construct a wide range of exact traveling-wave and multi-soliton solutions for the (2+1)-dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation. Based on a traveling-wave ansatz together with the homogeneous balance principle, solutions of closed form were then obtained by reducing the governing partial differential equation into an ordinary differential equation. The unified treatment yields a wide variety of solutions, including bright, kink, and dark solitons, compactons, periodic solutions, and multi-layered solitons. These solutions are expressed explicitly in terms of hyperbolic, trigonometric, exponential, and rational functions. The wave structures obtained are thoroughly confirmed by graphical simulations in Maple. In particular, 2D curves, 3D surface plots, and contour diagrams are used to visualize the propagation dynamics and spatial localization properties of solitons across various

Received: July 2026; Accepted: September 2026
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parameter regimes. These visualizations establish the conceptual importance of the proposed framework and provide direct insights into amplitude enhancement, wave propagation, and energy localization. On the other hand, the GKM-Auxiliary approach captures nonlinear wave phenomena such as multi-soliton interactions, peakons, cuspons, and rogue waves, which traditional approaches cannot capture due to their very limited solution families. This forms a flexible and powerful part of a toolkit for the study of high-order nonlinear partial differential equations, which may prove useful for (2+1)-dimensional hydrodynamics, plasma oscillations, and optical soliton theory.

AMS Subject Classification: 35Q51; 35C08; 35Q35
Keywords and Phrases: Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation; traveling wave solutions; generalized Kudryashov method; extended auxiliary equation method; solitons; solitary waves

1 Introduction

Nonlinear evolution equations (NLEEs) are fundamental tools in the study of natural sciences and engineering systems, serving as models for complex phenomena. These are used to describe crucial nonlinear phenomena associated with wave propagation, pattern formation, diffusion, and interaction dynamics in various physical systems. Unlike linear models, NLEEs can account for amplitude-dependent dispersion and self-interaction effects, which are crucial in describing nonlinear media. As a consequence, NLPDEs have found important applications in many other areas, including fluid dynamics, plasma physics, nonlinear optics, the theory of metals and dielectrics, condensed matter physics, the one-dimensional quantum-mechanical model, band theory, and problems in functional spaces related to biophysics and chemical kinetics.

Over the past few decades, increasing attention has been devoted to developing and applying mathematical techniques to obtain exact analytic solutions for NLPDEs [26]. The presence of such solutions is not only theoretically important but also serves as a benchmark for testing numerical approximations and simulations. They provide insight into why certain nonlinear phenomena, such as energy localization, soliton collisions, and modulational instability, occur. Scientists and engineers have investigated numerous equations, including the extended two-dimensional Zakharov-Kuznetsov equation [16], the Whitham-

Broer-Kaup equation [32], the Nizhnik-Novikov-Veselov equations [34], and the Kadomtsev-Petviashvili equation [38]. A few examples of such integrable equations [1] are the nonlinear fifth-order integrable equations [17], the Wazwaz-Benjamin-Bona-Mahony equations in three dimensions [22], the Biswas-Milovic equation [19], etc. Exact solutions are of great importance in nonlinear science and applied mathematics, and are a major topic.

The (2+1)-dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation [3], and other related models, such as the potential KdV or surface gravity wave equations, are particularly significant in nonlinear wave dynamics. The model combines the dispersive properties of the Benjamin-Bona-Mahony (BBM) model with those of the Kadomtsev-Petviashvili (KP) model, making it well-suited to describing the behavior of two-dimensional nonlinear waves in both time and space. The KP-BBM equation naturally arises in the modeling of weakly nonlinear dispersive waves, for which both bundling and transverse effects are significant, such as shallow-water waves, ion-acoustic waves in plasma, and weakly dissipative propagation in nonlinear optical beams. The broad range of applications of this equation, along with the naturalness of most general multidimensional wave processes in reality, has made it the subject of extensive analytical and numerical interest.

Solitons, which are a remarkable class of phenomena in nonlinear dynamics, correspond to precisely localized packets of waves, which do not change their shape and velocity as they propagate or collide. The discovery of the soliton led to a great increase in our understanding. Bright, dark, kinks, anti-kinks, and multi-layer solitons appear depending on the system's nonlinear and dispersive equilibrium. The stability of the soliton under various interactions, including collisions, reflections, and transformations, has made it a key concept across a range of fields, including optical fiber, hydrodynamics, the study of Bose-Einstein states, and plasma physics. Similarly, for systems with cubic symmetry, solitary waves and harmonic Lax pairs offer insight into the underlying nature of natural phenomena, including symmetry, special conservation laws, and integrability.

A variety of analytical techniques have been proposed for constructing solitons and other exact solutions of NLEEs. These classical tech-

niques, such as the inverse scattering transform [11], Hirota's bilinear method [15], and Bäcklund transformations [25]. They are highly effective and powerful, but are limited to integrable systems. This has motivated the emergence of direct algebraic methods [37], such as functional methods [8], the tanh method [30], the tanh-coth method [31], the Jacobi elliptic function method [36], and various Ansatz-based methods [35]. Recently, some extended approaches have been proposed, like the Improved Generalized Riccati Equation Mapping Method, the Extended Trial Equation Method [10], the Generalized Auxiliary Equation Method [2], and the Variational iteration method [12], including the extended Tanh method [5], the Painlevé expansion method [29], the Auxiliary equation method [23], and more. Improved generalized Riccati equation mapping method [21], the (G'/G)-expansion method [28], the Lie symmetry approach [6], the modified extended Fan sub-equation method [33], the classical symmetry method [13], the symmetry method [9], and the improved Kudryashov method [14], which provide frequently restricted families of solutions.

There are a few existing analytics techniques, but a significant research gap remains. For the nonlinear behaviors of weakly dispersive waves in fluid dynamics and nonlinear optics, the $(2 + 1)$-dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation is one of the simplest models of nonlinear PDEs; however, existing available algorithms are oftentimes not enough to probe all its rich complicated non-linearities over the whole parameter space. We find that most methods are either expressively efficient but analytically restrained or extremely flexible but numerically limited. As a result, there is still no comprehensive approach that covers the breadth of the analysis while remaining practically actionable.

To fill this gap, the main novelty of our study is a structured GKM-Auxiliary framework that jointly applies GKM and the Extended Auxiliary Equation Method (EAEM). Our aim, compared to recent independent applications of these methods to KP-type equations [24] or BBM-type equations [27] is to combine the GKM and the EAEM within a framework that remains novel for the (2+1)-dimensional model [4]. This combines two complementary approaches and generalizes the solution space, which provides a wider class of solitary structures than

previously reported.

This approach uses a traveling-wave transformation to convert the governing PDE into an ODE and determines the appropriate degree for the polynomial solution ansatz based on a homogeneous balance. After this, both methods are implemented to produce episodes of exact traveling-wave solutions. The 2D, 3D, and contour plots visually illustrate the effects of physical parameters on soliton height, wavelength, and interaction.

The core objectives of the present study are:

- To provide a joint analytic approach that can critically meld both GKM and EAEM, resulting in revealing novel and complicated soliton phenomenology for the (2+1)-dimensional KP-BBM equation.
- To illustrate the complementary nature of GKM and EAEM and that by combining the two, one can obtain a more comprehensive solution set than by applying only one method, especially for modeling higher-dimensional nonlinear phenomena.
- To offer frequent and graphical descriptions of the received solutions, and to point out their applicability in modeling multi-dimensional nonlinear wave problems.

The article is organized as follows. Section 1 comprises the introduction. In Section 2, we present the mathematical frameworks of the GKM (Generalized Kudryashov Method) and the EAEM (Extended Auxiliary Equation Method), and discuss their theoretical compliance. The application of the resulting proposed framework to the (2+1)-dimensional KP-BBM equation, along with some new exact solutions, is presented in Section 3. Section 4 presents the numerical and graphical results obtained and discusses their physical implications. We conclude Section 4 with important remarks on new results and potential future developments in the context of hybrid analytical and numerical techniques for nonlinear wave theory.

The schematic diagram in Figure 1 provides a detailed overview of the proposed approach. This diagram shows that we start with the nonlinear PDE, apply the wave transform to obtain a nonlinear ODE, use

the balancing principle to determine M and N, and then plug these values into the proposed ansatz. Then this ansatz is reintroduced into the derivatives of the proposed schemes, and the resulting system of equations is solved using software tools. This system of equations is numerically solved to obtain the parameter set. Here, we differ in the following: the GKM (Generalized Kudryashov Method) is solved using this set of parameters in the objective function, whereas the EAEM (Extended Auxiliary Equation Method) uses the obtained set of parameters across 16 families of the auxiliary method. Finally, we plot a variety of solutions against parameter values in 2D and 3D, as well as in contour plots. The GKM provides a limited variety of solutions, whereas the EAEM has the advantage of its families, which generate numerous traveling-wave solutions, some of which may be novel and unreported in the literature.

The Kadomtsev-Petviashvili equation, or KP equation, is a fundamental mathematical and physical model that elucidates the dynamics of nonlinear wave phenomena [18]. We shall examine the KP-BBM equation in this context [20].

The (2+1)-dimensional KP-BBM equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial^2 u}{\partial x^2}+\alpha\frac{\partial^2}{\partial x^2}(u^2)+\beta\frac{\partial}{\partial t}\left(\frac{\partial^3 u}{\partial x^3}\right)+\gamma\frac{\partial^2 u}{\partial y^2}=0, \qquad (1)$$

where x and y are spatial while t is the temporal independent variable, and $U(x, y, t)$ is the dependent variable. The arbitrary constants in this equation are α, β, and γ. $U(x, y, t)$ is observed to be defined for all values of x, y, and $t > 0$. The weak nonlinear dispersive waves in plasma, as well as the weak long water waves in modulation, are both examined using the Kadomtsev-Petviashvili (KP) equation [24], particularly the equation in the context of nonlinear dispersion [20]. The BBM equation was developed as a long-wave propagation process [27].

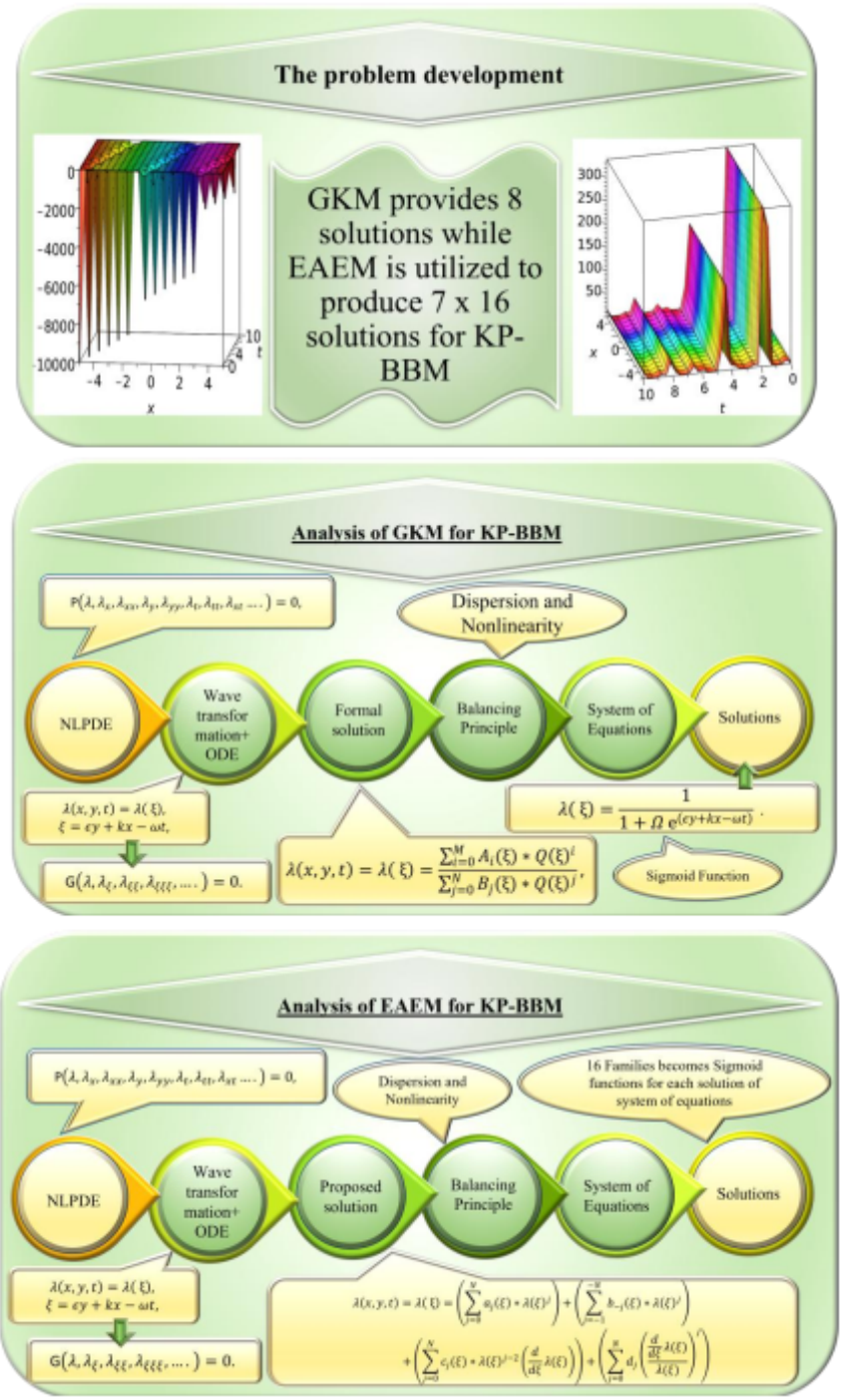


Figure 1: Schematic Diagram of Proposed Approach

2 Methodology Of The Proposed Approaches

In this study, the GKM and the EAEM are proposed to explore new types of solitary waves. The methodology of both Approaches is given below as

2.1 The GKM methodology

The general form of NLPDE in (2+1) dimensional form for $\lambda(x, y, t)$ is written as

$$H(\lambda, \lambda_x, \lambda_{xx}, \lambda_y, \lambda_{yy}, \lambda_t, \lambda_{tt}, \lambda_{xt}, \ldots) = 0. \tag{2}$$

Here H is a polynomial in a dependent function λ and independent variables x, y, t and its nonlinear partial derivatives of different orders. Consider the transformation

$$\lambda(x, y, t) = \lambda(\xi), \tag{3}$$

where

$$\xi = \epsilon y + kx - \omega t. \tag{4}$$

Thus, apply (3) to convert $\lambda(\xi)$ into the ODE

$$G(\lambda, \lambda_\xi, \lambda_{\xi\xi}, \lambda_{\xi\xi\xi}, \ldots) = 0. \tag{5}$$

The balancing principle [7] yields the value of N. Now, propose a solution of the form

$$\lambda(x, y, t) = \lambda(\xi) = \frac{\sum_{k=0}^{M} A_k(\xi)\, Q(\xi)^k}{\sum_{l=0}^{N} B_l(\xi)\, Q(\xi)^l}, \qquad l = 0 \ldots N, \quad k = 0 \ldots M, \tag{6}$$

where $A_k(\xi)$, $B_l(\xi)$, $Q(\xi)^k$ and $Q(\xi)^l$ are to be calculated, and $\lambda(\xi)$ satisfies the sigmoid function

$$\lambda(\xi) = \frac{1}{1 + \Omega\, e^{(\epsilon y + kx - \omega t)}}. \tag{7}$$

Eq. (6) meets the criteria of the following nonlinear differential equation. The derivatives of the auxiliary function $Q(\xi)$ are

$$\begin{aligned}
\frac{d}{d\xi}Q(\xi) &= Q(\xi)^2 - Q(\xi),\\
\frac{d^2}{d\xi^2}Q(\xi) &= 2Q(\xi)^3 - 3Q(\xi)^2 + Q(\xi),\\
\frac{d^3}{d\xi^3}Q(\xi) &= 6Q(\xi)^4 - 12Q(\xi)^3 + 7Q(\xi)^2 - Q(\xi),\\
\frac{d^4}{d\xi^4}Q(\xi) &= 24Q(\xi)^5 - 60Q(\xi)^4 + 50Q(\xi)^3 - 15Q(\xi)^2 + Q(\xi),\\
\frac{d^5}{d\xi^5}Q(\xi) &= 120Q(\xi)^6 - 360Q(\xi)^5 + 390Q(\xi)^4 - 180Q(\xi)^3 + 31Q(\xi)^2 - Q(\xi),\\
\frac{d^6}{d\xi^6}Q(\xi) &= 720Q(\xi)^7 - 2520Q(\xi)^6\\
&\quad + 240Q(\xi)^5 + 1020Q(\xi)^4\\
&\quad + 540Q(\xi)^3 - Q(\xi). \qquad (8)
\end{aligned}$$

Put the derivatives of eq. (8) into the ODE obtained in eq. (5), then equate the coefficients of powers of $\lambda(\xi)$ in the ansatz of eq. (6) to zero; we obtain algebraic equations in $A_i(\xi)$, $B_j(\xi)$ and $Q(\xi)^i$, $Q(\xi)^j$ which are then solved using suitable parameters with software tools such as Maple and Mathematica.

2.2 The EAEM methodology

Consider the generic form of NLPDE as given below. Where, in the dependent function λ and the independent variables x, y, t, P is a polynomial with different orders of nonlinear partial derivatives.

$$P(\lambda, \lambda_x, \lambda_{xx}, \lambda_t, \lambda_{tt}, \lambda_{xt}, \ldots) = 0. \qquad (9)$$

Step 1. First, a (2+1) traveling wave transformation with any arbitrary constant ξ is introduced.

$$\lambda(x, y, t) = \lambda(\xi), \qquad (10)$$

where

$$\xi = \epsilon y + kx - \omega t. \qquad (11)$$

The equation (9) is transformed into an ordinary differential equation by using this transformation (10) and (11) according to the traveling wave solution $\lambda(\xi)$

$$G(\lambda, \lambda_\xi, \lambda_{\xi\xi}, \lambda_{\xi\xi\xi}, \ldots) = 0. \tag{12}$$

N is then calculated by the balancing principle [7].

Step 2. Propose a solution of the following form:

$$\begin{aligned}\lambda(x,y,t) = \lambda(\xi) = &\left(\sum_{j=0}^{N} a_j(\xi)\,\lambda(\xi)^j\right) + \left(\sum_{j=-1}^{-N} b_{-j}(\xi)\,\lambda(\xi)^j\right) \\ &+ \left(\sum_{j=0}^{N} c_j(\xi)\,\lambda(\xi)^{j-2}\Big(\frac{d}{d\xi}\lambda(\xi)\Big)\right) + \left(\sum_{j=0}^{N} d_j\Big(\frac{\frac{d}{d\xi}\lambda(\xi)}{\lambda(\xi)}\Big)^j\right),\end{aligned} \tag{13}$$

where $j = 0 \ldots N$. In this case, $\lambda(\xi)$ is the solution when (13) and (10) are incorporated into the ODE (12), where a_j, b_j, c_j, d_j and λ are functions of (x, y, t). There are 16 families of EAEM as found in the literature (here $\nabla = \eta_2^2 - 4\eta_1\eta_3$):

$$\lambda(\xi) \text{ is given by the following solution families:} \tag{14}$$

- **Family 1** ($\eta_1 > 0$):

$$\lambda(\xi) = \frac{-\eta_1\eta_2\,\mathrm{sech}^2\big(\frac{\sqrt{\eta_1}}{2}\xi\big)}{\eta_2^2 - \eta_1\eta_3\big(1 + \tanh(\frac{\sqrt{\eta_1}}{2}\xi)\big)^2}$$

- **Family 2** ($\eta_1 > 0$):

$$\lambda(\xi) = \frac{\eta_1\eta_2\,\mathrm{csch}^2\big(\frac{\sqrt{\eta_1}}{2}\xi\big)}{\eta_2^2 - \eta_1\eta_3\big(1 + \coth(\frac{\sqrt{\eta_1}}{2}\xi)\big)^2}$$

- **Family 3** ($\eta_1 > 0$, $\nabla > 0$):

$$\lambda(\xi) = \frac{2\eta_1\,\mathrm{sech}^2(\sqrt{\eta_1}\,\xi)}{\sqrt{\nabla} - \eta_2\,\mathrm{sech}(\sqrt{\eta_1}\,\xi)}$$

- **Family 4** ($\eta_1 > 0$, $\nabla < 0$):

$$\lambda(\xi) = \frac{2\eta_1 \operatorname{csch}^2(\sqrt{\eta_1}\,\xi)}{\sqrt{-\nabla} - \eta_2 \operatorname{csch}(\sqrt{\eta_1}\,\xi)}$$

- **Family 5** ($\eta_1 < 0$, $\nabla > 0$): analogous form with csch replaced by trigonometric functions.

- **Family 6** ($\eta_1 < 0$, $\nabla > 0$): form involving csc.

- **Family 7** ($\eta_1 > 0$, $\eta_3 > 0$):

$$\lambda(\xi) = \frac{-\eta_1 \operatorname{sech}^2\left(\frac{\sqrt{\eta_1}}{2}\xi\right)}{\eta_2 + 2\sqrt{\eta_1\eta_3}\,\tanh\left(\frac{\sqrt{\eta_1}}{2}\xi\right)}$$

- **Family 8** ($\eta_1 < 0$, $\eta_3 > 0$):

$$\lambda(\xi) = \frac{-\eta_1 \sec^2\left(\frac{\sqrt{-\eta_1}}{2}\xi\right)}{\eta_2 + 2\sqrt{-\eta_1\eta_3}\,\tan\left(\frac{\sqrt{-\eta_1}}{2}\xi\right)}$$

- **Family 9** ($\eta_1 > 0$, $\eta_3 > 0$): form with csch^2 and coth.

- **Family 10** ($\eta_1 < 0$, $\eta_3 > 0$): form with $\csc^2$ and cot.

- **Families 11–16**: degenerate cases ($\nabla = 0$ or selected coefficients zero) that produce peakons, cuspons, compactons and rational (rogue-wave type) solutions.

Step 3. We derive a set of solutions for Eq. (1) by inserting Eq. (13) and Eq. (10) into Eq. (12) and resolving the overdetermined systems that emerge for the coefficients.

Step 4. Using software tools such as Maple, etc., the answers found are analyzed for each family and the set of solutions.

Step 5. Using software tools, the results are finally visualized as 2-Dimensional, 3-Dimensional, and contour plots.

3 Applications Of Proposed Approaches To The KP-BBM

In this section, the proposed Approaches are applied to the KP-BBM equation as follows.

3.1 Applications of GKM to KP-BBM

We first apply the traveling wave transformation (4) to eq. (1)

$$\xi = \epsilon y + kx - \omega t. \tag{15}$$

The ODE obtained is given below

$$(-\beta k^3\omega + \epsilon^2\gamma + k^2 - k\omega)U + \alpha k^2 U^2 - \beta k^3\omega\, U'' = 0. \tag{16}$$

For U^2 and U'', the homogeneous balance principle gives us $M = 2$ and $N = 0$. Put M and N's values in the following ansatz,

$$\lambda(\xi) = \frac{A_0 + A_1 Q(\xi) + A_2 Q(\xi)^2}{B_0}, \tag{17}$$

where A_i are the unknowns to determine. Putting eq. (15) and eq. (17) into eq. (16) and setting the coefficient of powers of Q to zero using Maple to acquire a set of values as given below in eq. (18).

$$\begin{aligned}
&\epsilon = 0, \quad A_2 = A_2, \quad B_0 = B_0, \quad k = 0,\\
&\qquad A_1 = A_1, \quad \omega = \omega, \quad A_0 = A_0,\\
&\epsilon = \epsilon, \quad A_2 = A_2, \quad k = 0,\\
&\qquad \omega = \omega, \quad A_0 = A_0, \quad B_0 = 0, \quad A_1 = A_1,\\
&\epsilon = \epsilon, \quad k = k, \quad \omega = \omega,\\
&\qquad A_0 = A_0, \quad A_1 = 0, \quad A_2 = 0, \quad B_0 = B_0,\\
&\epsilon = \epsilon, \quad k = k, \quad \omega = \omega,\\
&\qquad A_0 = -B_0\frac{-\beta k^3\omega + \epsilon^2\gamma + k^2 - k\omega}{2\alpha k^2},\\
&\qquad A_1 = -\frac{6\beta k\omega B_0}{\alpha},\\
&\qquad A_2 = \frac{6\beta k\omega B_0}{\alpha}, \quad B_0 = B_0.
\end{aligned} \tag{18}$$

Solving the above set of values (18) one by one by putting them in (16) using Maple, we get four solutions, one of which gives a constant solution, which is consequently omitted. The remaining three are given below

$$\begin{aligned}
\Omega_1 &= \frac{1}{B_0}\Bigg(-\frac{B_0(-\beta k^3\omega+\epsilon^2\gamma+k^2-k\omega)}{2k\alpha} \\
&\quad -\frac{6\beta k\omega B_0}{\alpha\big(1+\Omega\exp(\epsilon y+kx-\omega t)\big)} \\
&\quad +\frac{6\beta k\omega B_0}{\alpha\big(1+\Omega\exp(\epsilon y+kx-\omega t)\big)^2}\Bigg), \\
\Omega_2 &= \frac{1}{B_0}\Bigg(-\frac{B_0(-\beta k^3\omega+\epsilon^2\gamma+k^2-k\omega)}{2k\alpha} \\
&\quad +\frac{6\beta k\omega B_0}{\alpha\big(1+\Omega\exp(\epsilon y+kx-\omega t)\big)}\Bigg), \\
\Omega_3 &= \frac{1}{B_0}\Bigg(-\frac{B_0(-\beta k^3\omega+\epsilon^2\gamma+k^2-k\omega)}{2k\alpha} \\
&\quad -\frac{6\beta k\omega B_0}{\alpha\big(1+\Omega\exp(\epsilon y+kx-\omega t)\big)} \\
&\quad +\frac{6\beta k\omega B_0}{\alpha\big(1+\Omega\exp(\epsilon y+kx-\omega t)\big)^2}\Bigg).
\end{aligned} \tag{19}$$

The KP-BBM equation is examined using the generalized Kudryashov technique. Software tools are used to formulate algebraic equations and determine their solutions, thereby identifying solution types such as compactons, kink solitons, and anti-kink solitons. The dynamic behavior of these solutions, as outlined in equation (19), along with the corresponding parameters, will be illustrated through 2D, 3D, and contour plots in the forthcoming results and discussion sections.

3.2 Applications of EAEM to KP-BBM

We apply the given transformation to eq. (1):

$$\xi = \epsilon y + kx - \omega t. \tag{20}$$

In the ODE obtained as given below,

$$k\omega\, u(\xi) + \beta k^3 \omega\, u''(\xi) - k^2 u(\xi) - \gamma\epsilon^2 u(\xi) - \alpha k^2 u(\xi)^2 = 0. \tag{21}$$

The balancing principle for u^2 and u'' yields $N = 2$. Substitute this value into the following ansatz.

$$S = a_0 + a_1\psi(\xi) + a_2\psi(\xi)^2 + c_2\Big(\tfrac{d}{d\xi}\psi(\xi)\Big) + d_1\frac{\frac{d}{d\xi}\psi(\xi)}{\psi(\xi)} + d_2\frac{\big(\frac{d}{d\xi}\psi(\xi)\big)^2}{\psi(\xi)^2}. \tag{22}$$

By incorporating the eq. (20) and eq. (22) in eq. (21), we obtain the following set of solutions as given below,

$$\begin{aligned}
&[a_0 = a_0,\ d_2 = d_2,\ a_1 = a_1,\ a_2 = a_2,\\
&\qquad c_2 = c_2,\ d_1 = d_1,\ k = 0,\ \omega = \omega,\ \epsilon = 0],\\
&[a_0 = a_0,\ a_1 = -d_2\eta_2,\ a_2 = -d_2\eta_3,\\
&\qquad c_2 = 0,\ d_1 = 0,\ d_2 = d_2,\ k = k,\ \omega = \omega,\ \epsilon = \epsilon],\\
&[a_0 = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2\alpha k^2},\\
&a_1 = -\frac{\eta_2(-3\beta k\omega + 2\alpha d_2)}{2\alpha},\\
&a_2 = \frac{\eta_3(3\beta k\omega - \alpha d_2)}{\alpha}, \quad c_2 = \frac{3\sqrt{\eta_3}\,\omega k\beta}{\alpha},\\
&d_1 = 0,\ d_2 = d_2,\ k = k,\ \omega = \omega,\ \epsilon = \epsilon],\\
&.
\end{aligned} \tag{23}$$

The above set of values (23) is then checked for the sixteen families (14) one by one, from which some unique, newly obtained, and important ones are obtained from the equation (24) and are included in the article,

due to length constraints.

$$
\begin{aligned}
\Omega_4 &= a_0 - \frac{a_1\eta_1}{2\eta_3} + \frac{a_2\eta_1^2}{4\eta_3^2},\\
\Omega_5 &= a_0 + \frac{2a_1\eta_1 \sec(\sqrt{-\eta_1}\omega t)}{\sqrt{-4\eta_1\eta_3+\eta_2^2} - \eta_2 \sec(\sqrt{-\eta_1}\omega t)}\\
&\quad + \frac{4a_2\eta_1^2 \sec^2(\sqrt{-\eta_1}\omega t)}{\left(\sqrt{-4\eta_1\eta_3+\eta_2^2} - \eta_2 \sec(\sqrt{-\eta_1}\omega t)\right)^2},\\
\Omega_6 &= a_0 - \frac{a_1\eta_1 \sec(\frac{1}{2}\sqrt{-\eta_1}\omega t)}{\eta_2 - 2\sqrt{-\eta_1\eta_3}\tan(\frac{1}{2}\sqrt{-\eta_1}\omega t)}\\
&\quad + \frac{a_2\eta_1^2 \sec^2(\frac{1}{2}\sqrt{-\eta_1}\omega t)}{\left(\eta_2 - 2\sqrt{-\eta_1\eta_3}\tan(\frac{1}{2}\sqrt{-\eta_1}\omega t)\right)^2},\\
\Omega_7 &= a_0 + \frac{4a_1\eta_1 \exp(-\sqrt{\eta_1}\omega t)}{1-4\eta_1\eta_3\exp(-2\sqrt{\eta_1}\omega t)}\\
&\quad + \frac{16a_2\eta_1^2 \exp(-2\sqrt{\eta_1}\omega t)}{(1-4\eta_1\eta_3\exp(-2\sqrt{\eta_1}\omega t))^2},\\
\Omega_8 &= a_0 + d_2\eta_1(1+\tanh\zeta_1) - \frac{d_2\eta_3\eta_1^2(1+\tanh\zeta_1)^2}{\eta_2^2},\\
\Omega_9 &= a_0 + \frac{a_1\eta_1\eta_2}{\omega^2t^2\eta_2^2-\eta_1\eta_3} + \frac{a_2\eta_1^2\eta_2^2}{(\omega^2t^2\eta_2^2-\eta_1\eta_3)^2},\\
\Omega_{10} &= a_0 + \frac{4a_1\eta_1 \exp(-\sqrt{\eta_1}\omega t)}{1-4\eta_1\eta_3\exp(-2\sqrt{\eta_1}\omega t)}\\
&\quad + \frac{16a_2\eta_1^2 \exp(-2\sqrt{\eta_1}\omega t)}{(1-4\eta_1\eta_3\exp(-2\sqrt{\eta_1}\omega t))^2},\\
\Omega_{11} &= a_0 - \frac{a_1}{\sqrt{\eta_3}\,\omega t} + \frac{a_2}{\eta_3\omega^2t^2},\\
\Omega_{12} &= a_0 + d_2\eta_1\zeta_4 - \frac{d_2\eta_3\eta_1^2\zeta_4^2}{\eta_2^2},\\
\Omega_{13} &= a_0 - \frac{2d_2\eta_2\eta_1\sec^2\zeta_2}{\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\sec\zeta_2}\\
&\quad - \frac{4d_2\eta_3\eta_1^2\sec^4\zeta_2}{\left(\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\sec\zeta_2\right)^2},\\
\Omega_{14} &= a_0 - \frac{4d_2\eta_2\eta_1\exp\zeta_1}{(\exp\zeta_1-\eta_2)^2-4\eta_1\eta_3}\\
&\quad - \frac{16d_2\eta_3\eta_1^2\exp(2\zeta_1)}{\left((\exp\zeta_1-\eta_2)^2-4\eta_1\eta_3\right)^2}.
\end{aligned}
\tag{24}
$$

$$\Omega_{15} = a_0 - \frac{4d_2\eta_2\eta_1 \exp\zeta_1}{1-4\eta_1\eta_3\exp(2\zeta_1)} - \frac{16d_2\eta_3\eta_1^2\exp(2\zeta_1)}{(1-4\eta_1\eta_3\exp(2\zeta_1))^2},$$

$$\Omega_{16} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} + \frac{\zeta_5\eta_1\zeta_4}{2\alpha} + \frac{\eta_3(3\beta k\omega - \alpha d_2)\eta_1^2\zeta_4^2}{\alpha\eta_2^2},$$

$$\Omega_{17} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{\zeta_5\eta_1\sec^2\zeta_2}{\alpha\big(\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\sec\zeta_2\big)} + \frac{4\zeta_6\eta_1^2\sec^4\zeta_2}{\alpha\big(\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\sec\zeta_2\big)^2},$$

$$\Omega_{18} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{\zeta_5\eta_1\csc^2\zeta_2}{\alpha\big(\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\csc\zeta_2\big)} + \frac{4\zeta_6\eta_1^2\csc^4\zeta_2}{\alpha\big(\sqrt{-4\eta_1\eta_3+\eta_2^2}-\eta_2\csc\zeta_2\big)^2},$$

$$\Omega_{19} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} + \frac{\zeta_5\eta_1\csc^2(\frac{1}{2}\zeta_2)}{2\alpha\big(\eta_2+2\sqrt{-\eta_1\eta_3}\cot(\frac{1}{2}\zeta_2)\big)} + \frac{\zeta_6\eta_1^2\csc^4(\frac{1}{2}\zeta_2)}{\alpha\big(\eta_2+2\sqrt{-\eta_1\eta_3}\cot(\frac{1}{2}\zeta_2)\big)^2},$$

$$\Omega_{20} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{2\zeta_5\eta_1\exp\zeta_1}{\alpha\big((\exp\zeta_1-\eta_2)^2-4\eta_1\eta_3\big)} + \frac{16\zeta_6\eta_1^2\exp(2\zeta_1)}{\alpha\big((\exp\zeta_1-\eta_2)^2-4\eta_1\eta_3\big)^2}.$$

$$\Omega_{21} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{2\zeta_5\eta_1 \exp\zeta_1}{\alpha(1 - 4\eta_1\eta_3 \exp(2\zeta_1))} + \frac{16\eta_3(3\beta k\omega - \alpha d_2)\eta_1^2 \exp(2\zeta_1)}{\alpha(1 - 4\eta_1\eta_3 \exp(2\zeta_1))^2},$$

$$\Omega_{22} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{\eta_2^2(-3\beta k\omega + 2\alpha d_2)\eta_1}{2\alpha\big(\eta_2^2(\epsilon y + kx - \omega t)^2 - \eta_1\eta_3\big)} + \frac{\zeta_6\eta_1^2\eta_2^2}{\alpha\big(\eta_2^2(\epsilon y + kx - \omega t)^2 - \eta_1\eta_3\big)^2},$$

$$\Omega_{23} = \frac{\beta k^3\omega\eta_1 + (-2\alpha d_2\eta_1 - 1)k^2 + k\omega - \gamma\epsilon^2}{2k^2\alpha} - \frac{\zeta_5}{2\alpha\sqrt{\eta_3}(\epsilon y + kx - \omega t)} + \frac{3\beta k\omega - \alpha d_2}{\alpha(\epsilon y + kx - \omega t)^2}.$$

Where

$$\zeta_1 = \sqrt{\eta_1}(\epsilon y + kx - \omega t), \qquad \zeta_2 = \sqrt{-\eta_1}(\epsilon y + kx - \omega t),$$
$$\zeta_3 = 1 - \coth\big(\tfrac{1}{2}\sqrt{\eta_1}\,\omega t\big), \qquad \zeta_4 = 1 + \tanh\big(\tfrac{1}{2}\zeta_1\big),$$
$$\zeta_5 = \eta_2(-3\beta k\omega + 2\alpha d_2), \qquad \zeta_6 = \eta_3(3\beta k\omega - \alpha d_2).$$

4 Results And Discussions

Using the solutions obtained from GKM and EAEM for the (2+1)-dimensional KP-BBM equation, we make observations on the propagation properties, shape features, and parametric dependence of soliton structures. The solutions obtained in this paper exhibit a wealth of nonlinear wave phenomena, including single solitons, kink and antikink waves, compactons, and multi-soliton interactions. We obtained seven exact traveling wave solutions using EAEM, which increased to

dozens when multiplied and checked across the sixteen families. However, the article's length constraints mean we can select only a few. Similarly, we obtained several exact traveling-wave solutions yielding distinct types of solitary-wave solutions. The two-dimensional (2D) and three-dimensional (3D) plots, along with contour plots, illustrate the dynamic behavior of our findings. These simulations have been conducted with computational tools such as Mathematica and Maple.

4.1 Analysis of the GKM

A graphical overview of the major conclusions drawn by applying the GKM is given below. Figures show Singular Bright Grey soliton, Kink soliton, Smooth Bell-Shaped soliton, and Parabolic Type soliton for different parameter sets.

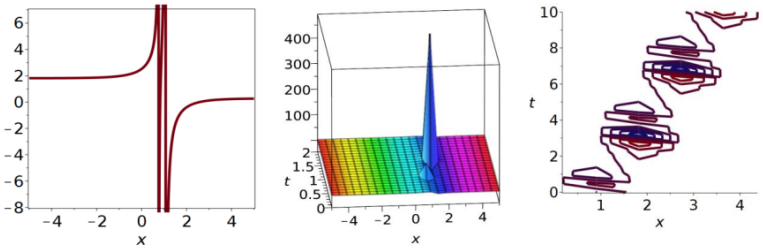


Figure 2: 2-D, Contour and 3-D Plots of Singular Bright Grey soliton

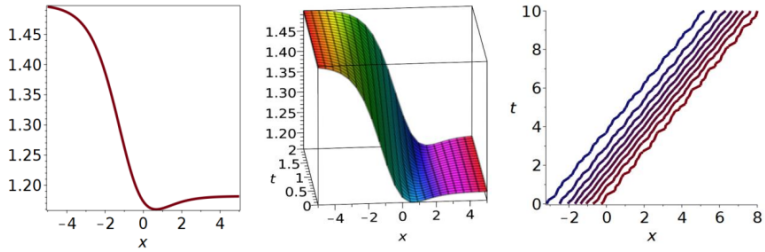


Figure 3: 2-D, Contour and 3-D Plots of Kink soliton

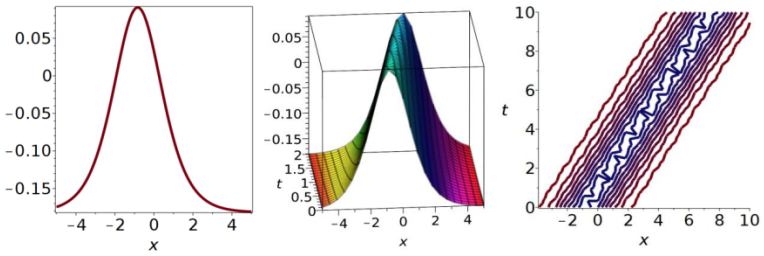


Figure 4: 2-D, Contour and 3-D Plots of Smooth Bell-Shaped soliton

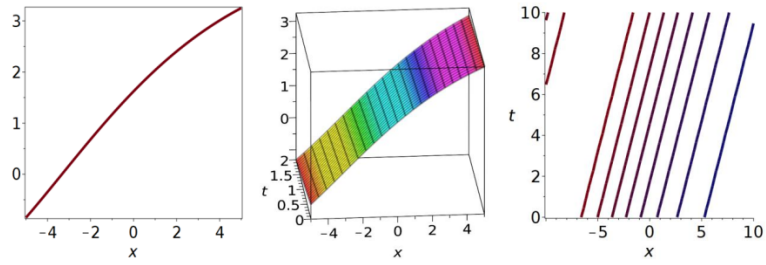


Figure 5: 2-D, Contour and 3-D Plots of Parabolic Type soliton

In this section, we discuss the effect of parameters on soliton dynamics. The impacts of variations in the traveling-wave parameters ϵ, k, ω, the auxiliary-equation parameters η_1, η_2, η_3, and the KP-BBM coefficients α, β, γ can significantly change soliton features. In particular, increasing values of k or ω narrow the width of the soliton and increase its propagation speed, while larger ϵ enhances the transverse velocity. The qualitative type of the solution is controlled by the sign of η_1: $\eta_1 > 0$ yields hyperbolic (localized) solitons, $\eta_1 < 0$ produces trigonometric (periodic) waves, and the degenerate case $\eta_1 = 0$ (or vanishing discriminants) generates rational or rogue-type solutions. Amplitude and steepness are primarily governed by η_2 and η_3, respectively, whereas the relative strength of nonlinearity, dispersion, and transverse modulation is determined by the three physical coefficients α, β, and γ.

To summarize these effects, Table 1 lists the principal parameters together with their influence on the wave profile and the corresponding figures in which each regime is illustrated.

Table 1: Influence of the main parameters on soliton morphology and the figures that display the corresponding regimes.

Parameter	**Principal effect on the soliton**	**Figures**
k, ω	Increase $\Rightarrow$ narrower width, higher speed	Figs. 2–5
ϵ	Controls transverse velocity and y-direction modulation	Figs. 2–5
$\eta_1 > 0$	Hyperbolic / localized structures (bright, dark, kink)	Figs. 2, 3, 4
$\eta_1 < 0$	Trigonometric / periodic waves	Fig. 5
η_2, η_3	Amplitude and steepness	All GKM figures
α, β, γ	Balance of nonlinearity, dispersion and transverse effects	All GKM figures

The three non-trivial analytical solutions Ω_1–Ω_3 obtained by the GKM are given explicitly in Eq. (19). For the representative parameter set

$$\begin{aligned} &\epsilon = 1.8, \quad A_0 = 0.3, \quad A_1 = 0.4, \quad A_2 = 0.2, \\ &B_0 = 1.1, \quad k = 0.2, \quad \omega = 0.1, \quad \Omega = 1.2, \\ &\alpha = 1.2, \quad \beta = 1.3, \quad \gamma = 1.2, \end{aligned}$$

these solutions produce the distinct nonlinear structures shown in Figures 2–5.

- Figure 2 displays the singular bright-grey soliton associated with $\Omega_1(x, y, t)$. Its abrupt peak and spatially isolated character indicate strong energy localization in a narrow region.

- Figure 3 shows the kink-type soliton of $\Omega_2(x, y, t)$. The profile connects two distinct asymptotic states and is therefore a natural model for shock waves and phase transitions in fluid dynamics—one of the principal application domains of the KP-BBM equation. The corresponding 3-D surface confirms that the localized transition propagates steadily along the x-direction while remaining robust.

- Figure 4 presents the smooth bell-shaped soliton of $\Omega_3(x, y, t)$. The wave is symmetric, peaked, and remains spatially confined throughout its evolution.

- Figure 5 illustrates a broader, parabola-type solitary wave (obtained from a limiting case of the same family). The wider spatial support is typically associated with longer-range interactions in nonlinear media.

Taken together, these plots demonstrate the flexibility of the Generalized Kudryashov Method in generating a rich variety of solitary-wave morphologies that are of direct relevance to stability analysis and wave-propagation problems arising in fluid dynamics, plasma physics, and nonlinear optics.

4.2 Analysis of the EAEM

The EAEM approach applied to the same model provides a set of many solitonic solutions in an explicit form via trigonometric, hyperbolic, exponential, and rational functions. This variety stems from the versatility of the EAEM in producing several nonlinear wave families by adjusting the auxiliary parameters.

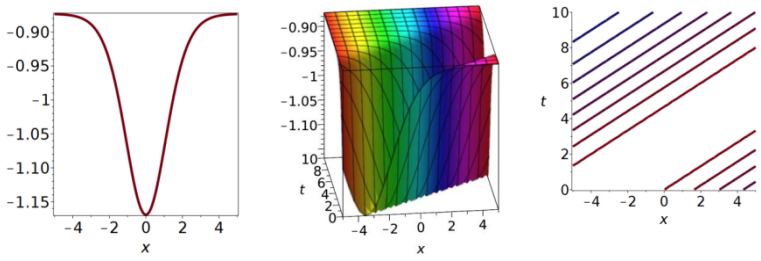


Figure 6: 2-D, contour, and 3-D Plots of Bell-shaped soliton

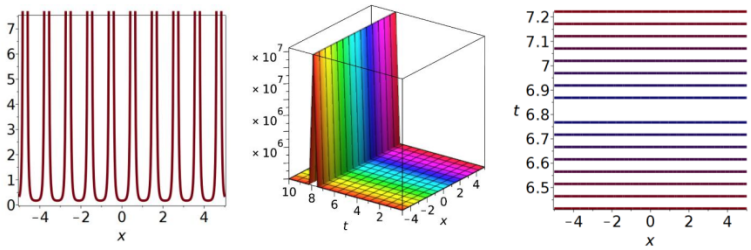


Figure 7: 2-D, contour, and 3-D Plots of Peakons shape

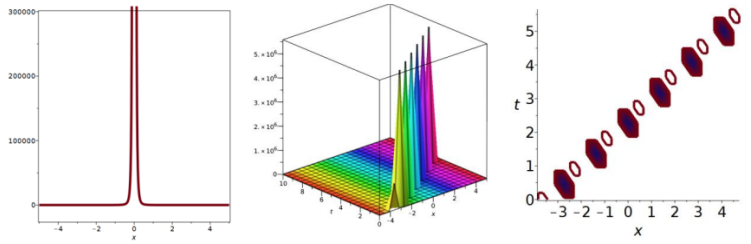


Figure 8: 2-D, contour, and 3-D Plots of Multi-Singular Bright Grey Solitons

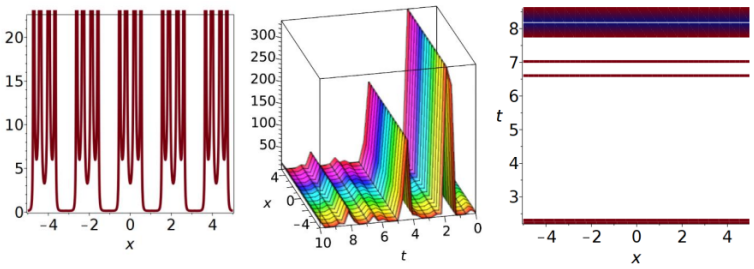


Figure 9: 2-D, contour, and 3-D Plots of Multi-Layered Singular Solitons

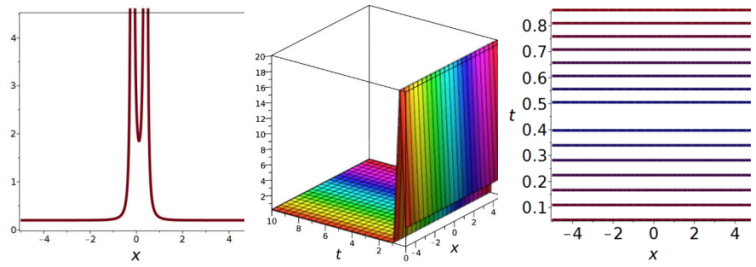


Figure 10: 2-D, contour, and 3-D Plots of the Singular soliton

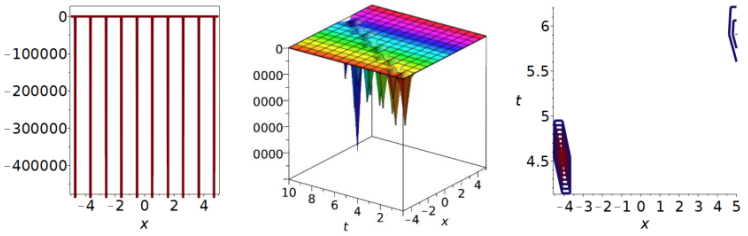


Figure 11: 2-D, contour, and 3-D Plots of multi-dark soliton

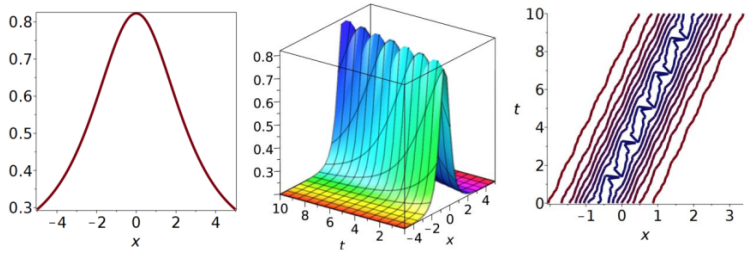


Figure 12: 2-D, contour, and 3-D Plots of Rogue-type bell-shaped Multihump Cuspon

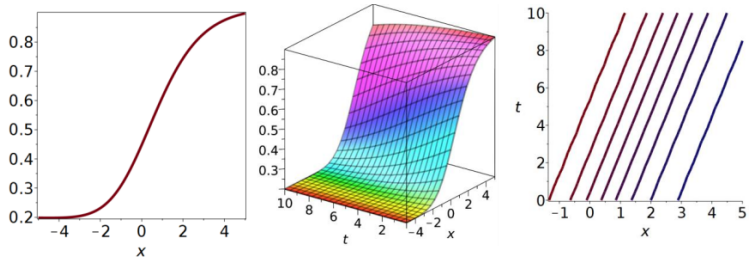


Figure 13: 2-D, contour, and 3-D Plots of Kink type soliton

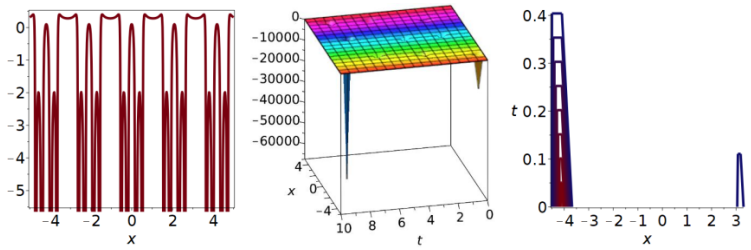


Figure 14: 2-D, contour, and 3-D Plots of Multi Dark and Grey solitons

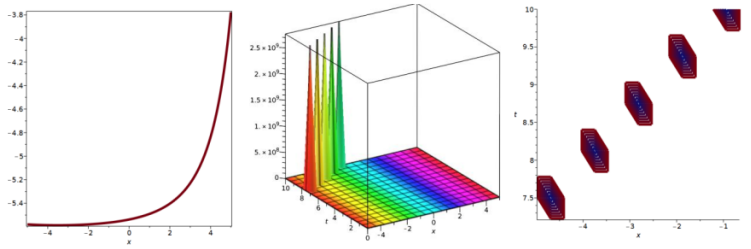


Figure 15: 2-D, contour, and 3-D Plots of Parabolic type periodic Bright solitons

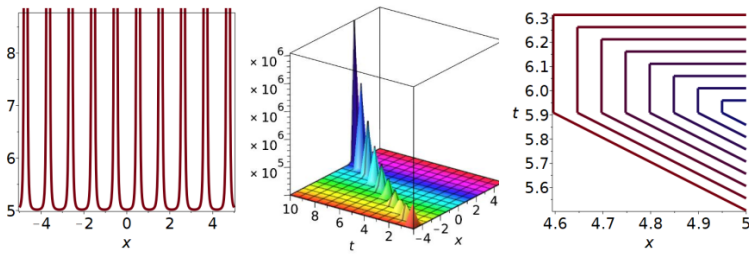


Figure 16: 2-D, contour, and 3-D Plots of Multi Bright and Grey solitons

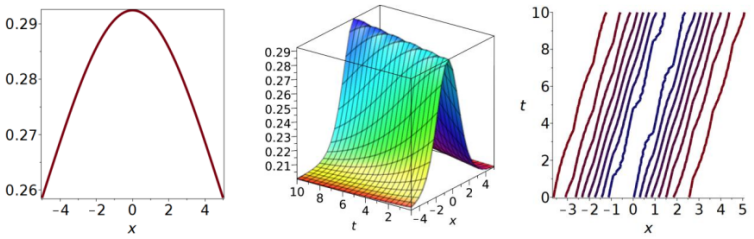


Figure 17: 2-D, contour, and 3-D Plots of Compacton

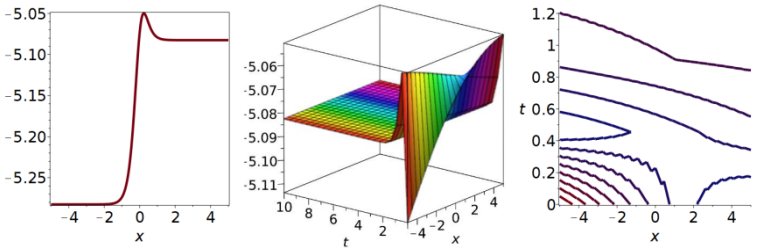


Figure 18: 2-D, contour, and 3-D Plots of Exponential soliton

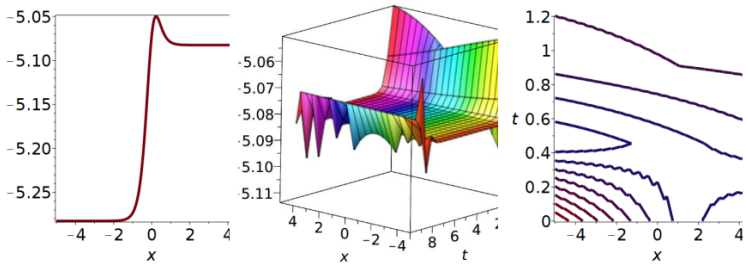


Figure 19: 2-D, contour, and 3-D Plots of a singular kink-type soliton

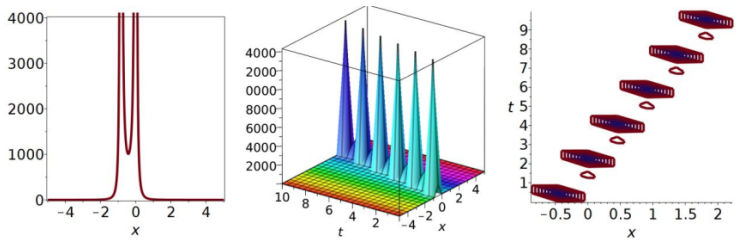


Figure 20: 2-D, contour, and 3-D Plots of Periodic Peakons and Grey Solitons

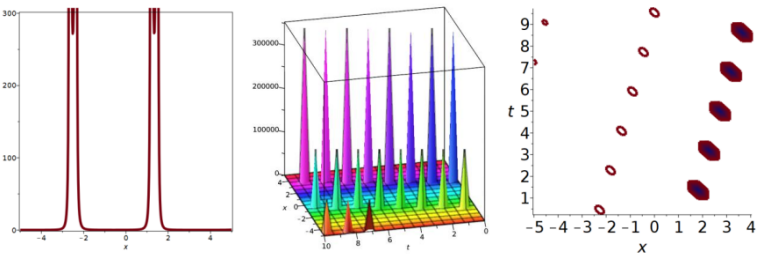


Figure 21: 2-D, contour, and 3-D Plots of Periodic Solitons

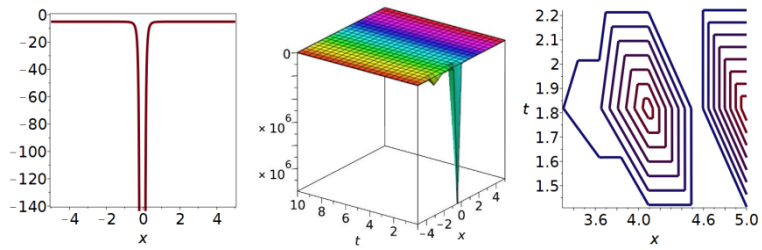


Figure 22: 2-D, contour, and 3-D Plots of Dark and Grey solitons

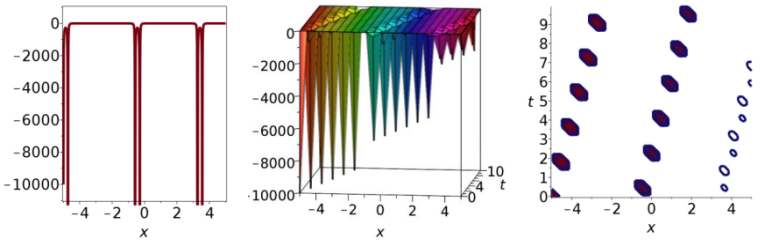


Figure 23: 2-D, contour, and 3-D Plots of Multi-Layered Periodic Dark Solitons

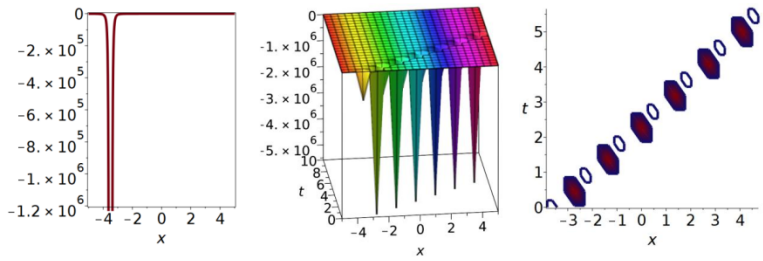


Figure 24: 2-D, contour, and 3-D Plots of Multi-Dark and Grey soliton

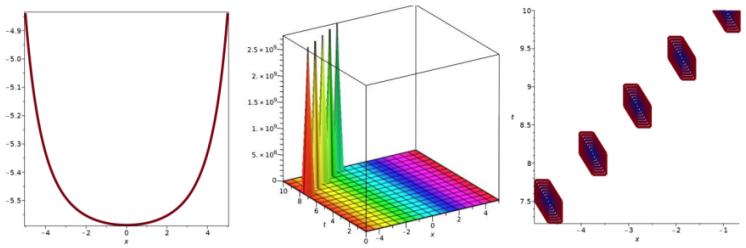


Figure 25: 2-D, contour, and 3-D Plots of Multi-Bright soliton

- Hyperbolic families relate to bright and dark solitons in terms of localized pulse or dip-like solutions propagating in opposite directions.
- The trigonometric families give rise to periodic and oscillatory waves, which can be viewed as the result of an inexact balance between nonlinearity and dispersion.
- Exponential solutions correspond to dissipative or decaying structures, which are typically associated with damped waveforms or shock fronts.
- Algebraic (and rogue-like) solitons occur for rational solutions, which tend to slow down decay and exhibit long-range behavior.

These various families show that the EAEM is a suitable framework for all local and periodic wave phenomena. Furthermore, the auxiliary parameters determine the amplitude, width, and velocity of their soliton. For example, an increase of certain parameters makes the soliton size smaller, while others change either its amplitude or its localization.

Representative plots include Bell-shaped soliton, Peakons, Multi-Singular Bright Grey Solitons, Multi-Layered Singular Solitons, Singular soliton, multi-dark soliton, Rogue-type bell-shaped Multihump Cuspon, Kink type soliton, Multi Dark and Grey solitons, Parabolic type periodic Bright solitons, Multi Bright and Grey solitons, Compacton, Exponential soliton, singular kink-type soliton, Periodic Peakons and Grey Solitons, Periodic Solitons, Dark and Grey solitons, Multi-Layered Periodic Dark Solitons, Multi-Dark and Grey soliton, and Multi-Bright soliton.

A bell-shaped soliton is a stationary, symmetric pulse that is an energy-packet solution that serves as a fundamental building block in optical fiber communication. Its non-diffusive evolution suggests its potential use as an information carrier in optical systems. Peakons provide localized pulses with various peak shapes. The method also discloses Multi-Singular Bright Grey Solitons and Multi-Layered Singular Solitons, which exhibit complex, multi-peaked structures. Dark and Grey solitons, as well as Multi-Dark and Grey Solitons, demonstrate the method's ability to capture localized dips and interactions among different soliton types. Kink-type and Singular kink-type solitons also exhibit sudden state changes. The plots confirm that the EAEM can be used to obtain many solitary-wave solutions, which are vital for understanding nonlinear wave dynamics.

Table 2: The Compared Analysis of the GKM and the EAEM

Aspect	**GKM**	**EAEM**
Basic idea	Reduction of PDE to ODE through wave transformations, assume the solution as a finite-series rational expression with an auxiliary function (Kudryashov ansatz).	Apply an auxiliary ordinary differential equation (extended set) to construct families of exact solutions; agrees with polynomial/trigonometric/exponential ansatz forms.
Usual solution forms found	Rational, singular bright or grey, kink-shaped, bell-shaped, and parabolic-type.	Extremely wide diversity: bell, peakons, multilayer solitons, compactons, and cuspons as well as periodic and multi-dark/grey.
Algebraic complexity	Moderate: solution of the algebraic coefficient systems resulting from the balance principle; feasible for lower-order ansatz.	The complexity increases with the number of auxiliary families and free parameters.
Parameter flexibility	Good for tuned parameter sets, with fewer auxiliary degrees of freedom.	High also supports many parameter variations, allowing for the generation of a wider range of wave shapes.
Convenience of achieving multi-half/poly-layer solutions	Possible but cumbersome.	The multi-layer and multi-hump structures are more readily obtained for the natural EAEM.
Symbolic computational cost	Low-to-moderate (few smaller algebraic systems).	Moderate to high (more unknown coefficients; larger systems).
Robustness (singularities and cusps)	Could generate a singular solution; needs careful handling of singular denominators.	Better performance for cuspons/peakons/compactons by selection of auxiliary ODEs.
Optimal for	Generating compact closed-form rational-type solitary waves, with clear analytical expressions.	Studying a wide solution family space (periodic, multi-hump, dark or grey families, including new shapes).
Strengths (for publication)	Simpler presentation; easier reproducibility of formulas; clean parameter dependence, compressible algebra.	Shows diversity and novelty in several visually diverse solution families, making it suitable for presenting the richness of the model.
Limitations	It also cannot capture some non-rational solution shapes or intricate multi-layer structures.	The closed-form simplicity may be masked by complexity; the algebraic systems are large and require CAS support.

4.3 GKM and EAEM compared analysis

The strengths of the two methods are discussed and compared based on theoretical considerations. The GKM is also easier to solve on a computer, and we found nice solutions for single- and kink-soliton solutions with controllable parameters. However, its rational functional structure confines it to a certain class of solution symmetries.

In turn, EAEM extends the analytical scope by incorporating additional functional families, thereby encompassing multi-soliton, periodic, and rational solutions within a unified formalism. The introduction of the auxiliary parameters allows for fine-tuning amplitude, steepness, and phase, producing more complex dynamical structures. Both approaches are inherently consistent with the KP-BBM equation when reduced to traveling-wave form. However, the collaboration EAEM is more general and becomes more relevant when modeling highly nonlinear media, particularly when non-integrable effects dominate and higher-dimensional interactions are present.

- Kink and anti-kink solitons are a solid model of shock on fluid layers.
- Bright and dark solitons correspond to energy localization in plasma or optics.
- Periodic and multi-soliton waves are related to modulated or interacting pulses.
- Rogue-like solitons are linked to rational and exponential waves.

This comparison reveals that innovation is not inherent in the methods themselves, but rather in their precise combination. The GKM is an effective method for constructing fundamental soliton solutions, and the EAEM provides the flexibility needed to investigate more complex, physically meaningful effects, such as periodic oscillations and multi-soliton interactions. They are united to create a foundation that is greater than the sum of their individual qualities. The results provide valuable insights into nonlinear wave propagation in dispersive media. The KP-BBM equation incorporates weak high-order nonlinearity and dispersion

in multidimensional waves. The soliton families obtained here are physically realizable in fluid mechanics, plasma oscillations, and nonlinear optical fibers. Specifically:

Table 3: The Summary of the comparison of the proposed scheme with the previous ones

Aspect	**Sivasundaram et al.**	**Usman et al.**	**Khater et al.**	**Our Approach**
Equation	First type KP Equation	Dispersive modified BBM	Modified version of KM	(2+1)-D KP-BBM
Proposed method	SGEM and Rational-SGEM	Solitary wave Ansatz method	Improved Kudryashov method	GKM and EAEM
Obtained solutions	Complex and Logarithmic	Exponential Hyperbolic	Rational and Exponential	Hyperbolic, trigonometric, exponential, and rational
Dimensionality	(2+1)-Dimensional	(1+1)-Dimensional	Multi-Dimensional	(2+1)-Dimensional
Visualization	2D, 3D, and contour surfaces	2D and 3D plots	–	2D curves, 3D surfaces, and contour diagrams
Novelty	First application of RSGEM to the KP Hierarchy	Combined study of NDMBBM and the KS equation	Developed version of the classical Method	First combination of GKM and EAEM for higher-dimensional KP-BBM

Table 3 summarizes a direct comparison between the proposed GKM-Auxiliary framework and the recent studies cited, further highlighting the novelty and positioning of our work. Sivasundaram et al. [24] were able to apply the SGEM and RSGEM to (2+1)-dimensional KP hierarchy equation; however, they used complex solutions and logarithmic solutions. In contrast, our framework focuses on the physically significant KP-BBM equation and yields a rich set of real solitary wave solutions. Likewise, Usman et al. [27] have found bright, dark, and rogue solitons for (1+1)-dimensional BBM-type equations. We extend this study to a (2+1)-dimensional coupled system to investigate multidimensional in-

teraction phenomena, including multilayered and compacton solutions. In addition, the developed Kudryashov method [14] is superior to the classical one, but our method combines the Generalized Kudryashov Method with the Extended Auxiliary Equation Method, which are not combined in known methods. This hybrid approach enables us to explore a more comprehensive solution space than either method could provide on its own, especially for higher-dimensional, more complex nonlinear wave systems.

Table 4: The Summary of the GKM and the EAEM

Method	**Solution Type**	**Functional Form**	**Physical Nature**
GKM	Single soliton, kink, compactons	Rational (sigmoid-type)	Localized and steady waveforms
EAEM	Multi-soliton, periodic, rational	Hyperbolic, trigonometric, exponential	Rich, nonlinear, and interacting wave structures

The GKM-EAEM approach successfully reproduces known solutions of the equation, while also revealing new families of soliton and quasi-soliton waves, thereby enriching the analytical panorama of the KP-BBM equation.

5 Conclusions

To conclude, this work established a novel framework (GKM-Auxiliary) of the Generalized Kudryashov Method (GKM) and the Extended Auxiliary Equation Method (EAEM) for the (2+1)-dimensional KP-BBM equation and successfully applied it to the present equation. The proposed framework generates a large class of exact solitary wave and multi-soliton solutions, including bright, kink, dark, compacton, periodic, multi-layered, peakon, cuspon, and rogue-type waves. This study primarily focuses on extensive graphical simulations: two-dimensional (2D) curves, three-dimensional (3D) surface plots, and contour diagrams. More particularly, Figures for GKM display singular bright-grey soliton, kink, smooth bell-shaped, and parabolic-type solitons, respectively, while figures for EAEM exhibit the entire variety of the frame-

work: a bell-shaped soliton, a peakon, a multi-singular bright-grey soliton, a multi-layered singular soliton, a rogue-type multi-hump cuspon, a kink-type soliton, a multi-dark/grey soliton, a compacton, a periodic soliton, and a multi-bright soliton, respectively. Such visualizations not only verify the morphological richness and dynamical stability of the obtained solutions, but also the physical relevance of the proposed approach. The GKM-Auxiliary framework then provides the flexibility and analytical power needed to explore complex nonlinear wave phenomena in (2+1)-dimensional hydrodynamics, plasma oscillations, and optical soliton theory. The extension of this framework to other higher-dimensional non-integrable PDEs will be pursued, and the numerical stability of the newly found soliton structures will be studied.

Acknowledgements
The authors would like to thank the anonymous referees for their valuable comments and suggestions that improved the quality of this paper.

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