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Original Research Paper

Approach to Bidimensional Balancing Matrix

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Abstract. This article presents a new approach to the types of bidimensional balancing numbers using matrices 2×2 with bidimensional balancing numbers as components. It highlights the definition of the bidimensional balancing matrix, explores its key properties, and provides some examples to illustrate.

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1 Introduction

Number sequences have been a central topic of mathematical investigation, attracting considerable interest from researchers. The concept

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of balancing numbers was introduced in seminal work [6] by Finkelstein and was known as numerical centers. This sequence was also studied by Behera and Panda [1] in 1999 during their exploration of a specific Diophantine equation. They aimed to identify a natural number n that satisfies the equation

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some natural number r. The sequence of balancing numbers is indicated by $\{B_n\}_{n\geq 0}$, the sequence $A001109$ in OEIS [17]. The first ten balancing numbers are

$$\{B_n\}_{n\geq 0} = \{0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, ...\} .$$

Behera and Panda [1] determined the following recurrence relation:

$$B_{n+2} = 6B_{n+1} - B_n, \tag{1}$$

for $n \geq 2$, with initial values $B_0 = 0$ and $B_1 = 1$. They formulated the Binet formula, the generating function, and various identities associated with balancing numbers by solving (1). As Ray demonstrated in [15], a key property of balancing numbers is that a number n is a balancing number precisely when $8n^2 + 1$ is a perfect square. We can cite other recent papers here, for example, [2, 8, 10, 11, 12, 18, 14, 19], in which extensions or sequences related to balancing numbers are considered.

This work, beyond the Introduction, is structured as follows: In Section 2, we analyze the balancing numbers and bidimensional balancing numbers, which present some identities related to these sequences that will be useful for the subsequent sections. In Section 3, we provide an overview of the bidimensional balancing matrix, highlighting its key properties. Examples are included to illustrate bidimensional matrices with bidimensional balancing numbers. Finally, some conclusions are drawn.

This foundational framework serves as the basis for the study of other sequences and their extensions. This sequence is of significant interest in mathematical research and often reveals unique patterns and properties.

2 Background and Auxiliary Results

Next, we present some results relating to the integer numbers, some in particular about the balancing sequence of numbers, which will be useful in the article.

Firstly, recall that given two integers a and b, the greatest common divisor between them is indicated by $gcd(a, b)$. Let a and b be any two integers; a and b are rarely primes or co-primes when $gcd(a, b) = 1$; additional details and proof can be found, for example, in [7] and [9].

The following results show that the balancing numbers enjoy some beautiful properties.

Lemma 2.1. *(Theorem 2.13, [13]) Let $\{B_n\}_{n\geq 0}$ be the balancing sequence. If m and n are natural numbers, then $gcd(B_m, B_n) = B_{gcd(m,n)}$.*

The next result follows directly from Lemma 2.1.

Corollary 2.2. *For all integers $n \geq 1$, the integers B_{n+1} and B_n are co-prime, where $\{B_n\}_{n\geq 0}$ is the balancing sequence.*

Lemma 2.3. *(Theorem 2.3.2, [15]) Let $\{B_n\}_{n\geq 0}$ be the balancing sequence. If $n \geq 1$ are natural numbers, then $B_n^2 - B_{n-1}B_{n+1} = 1$.*

The next statement follows directly from Lemma 2.3.

Corollary 2.4. *For all integers $n \geq 1$, the integers B_n^2 and $B_{n-1}B_{n+1}$ are co-prime, where $\{B_n\}_{n\geq 0}$ is the balancing sequence.*

The next result follows directly by applying the recurrence (1), and the proof can be performed by mathematical induction.

Lemma 2.5. *For all integers $n \geq 1$, $B_n - B_{n-1}$ is an odd number, where $\{B_n\}_{n\geq 0}$ is the balancing sequence.*

A consequence of Lemma 2.5 is that B_n and B_{n-1} are not simultaneously even or odd.

Lemma 2.6. *For all integers $n \geq 1$, the integers $(B_{n+1} - B_n)$ and $(B_n - B_{n-1})$ are co-prime, where $\{B_n\}_{n\geq 0}$ is the balancing sequence.*

Proof. By (1) we have

$$B_{n+1} - B_n = 5B_n - B_{n-1}\,.$$

Note that

$$5B_n - B_{n-1} = (B_n - B_{n-1}) + 4B_n\,.$$

Applying Euclid's Algorithm and taking into account that $B_n - B_{n-1}$ is an odd number (Lemma 2.5), we consequently obtain

$$gcd(B_{n+1} - B_n,\, B_n - B_{n-1}) = gcd(B_n - B_{n-1},\, 4B_n) = 1\,,$$

which proves the claim. □

In [3] and [4], the bidimensional extensions of the balancing numbers are shown, along with a detailed examination of their properties and characteristics. The bidimensional balancing sequence $\{B_{(m,n)}\}_{m,n\geq 0}$ is given by the following recurrence relations

$$B_{(m,n)} := B_{(m-1,n)} + B_{(m-2,n)}, \quad \text{for all } n \text{ and } m \geq 2\,, \tag{2}$$

$$B_{(m,n)} := B_{(m,n-1)} + B_{(m,n-2)}, \quad \text{for all } m \text{ and } n \geq 2\,, \tag{3}$$

with initial terms $B_{(0,0)} = 0$, $B_{(1,0)} = 1$, $B_{(0,1)} = i$, $B_{(1,1)} = 1 + i$, where i is the imaginary unit with $i^2 = -1$.

For additional information on this bidimensional sequence, see [3] and other references.

The result that follows reveals a connection between bidimensional and unidimensional balancing numbers.

Lemma 2.7. *(Theorem 3.3, [3]) Let $\{B_{(m,n)}\}_{m,n\geq 0}$ be the bidimensional balancing numbers. The general term is given by:*

$$B_{(m,n)} = B_m(B_n - B_{n-1}) + (B_m - B_{m-1})B_n i \tag{4}$$

for non-negative integers m, n, and where $\{B_j\}_{j\geq 0}$ is the balancing sequence.

In the Equation (4), we set $B_{-1} = 1$. This value can be determined using the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ by setting $n = 0$.

Table 1 presents the initial values of the bidimensional balancing sequence $\{B_{(m,n)}\}_{m,n\geq 0}$, corresponding to $0 \leq m, n \leq 4$.

$B_{(m,n)}$	m=0	m=1	m=2	m=3	m=4
n=0	0	1	6	35	204
n=1	i	1+i	6+5i	35+29i	204+169i
n=2	6i	5+6i	30+30i	175+174i	1020+1014i
n=3	35i	29+35i	174+175i	1015+1015i	5916+5915i
n=4	204i	169+204i	1014+1020i	5915+5916i	34476+34476i

Table 1: Bidimensional balancing numbers

From Lemma 2.7, it directly follows that:

Corollary 2.8. *Let $\{B_{(m,n)}\}_{m,n\geq 0}$ be the bidimensional balancing sequence, then $B_{(m,n)} \neq 0$ is a complex number.*

The combination of Lemma 2.5 and Corollary 2.8 implies that the complex number $B_{(m,n)}$ possesses an imaginary part distinct from zero for all $n > 1$.

Recall that, for any real numbers a and b, a complex number is defined as $z = a + ib$. If $a = 0$ and $b \neq 0$, the number z is referred to as a purely imaginary complex number; if $b = 0$, z is a real number. Applying this to bidimensional balancing numbers, $B_{(m,0)}$ is always real; however, $B_{(0,n)}$ is purely imaginary. Furthermore, the equality $B_{(m,0)} = B_{(0,n)} = 0$ is true if and only if $m = n = 0$.

Interchanging the real and complex parts is also exhibited by the complex numbers $B(m, n)$ and $B(n, m)$, as the following result reveals.

Proposition 2.9. *For all non-negative integers m and n such that $m \neq n$, if $B_{(m,n)} = a + bi$, we have $B_{(n,m)} = b + ai$ such that a and b are integers, where $\{B_{(m,n)}\}_{m,n\geq 0}$ is the bidimensional balancing sequence and i is the imaginary unit.*

Proof. By (4), we know that

$$B_{(m,n)} = B_m(B_n - B_{n-1}) + (B_m - B_{m-1})B_n i\,.$$

Making $a = B_m(B_n - B_{n-1})$ and $b = (B_m - B_{m-1})B_n$, we get

$$B_{(m,n)} = a + bi \ .$$

Again, by (4), we obtain

$$\begin{aligned} B_{(n,m)} &= B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i \\ &= b + ai \ , \end{aligned}$$

as required. □

Ray [16] introduced the Q-matrix, denoted as Q_B, which is defined as

$$Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}.$$

This matrix is known as a generating matrix for the balancing sequence. Furthermore, Ray [16] established that for any positive integer n, the n-th power of this matrix can be expressed as

$$Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix} . \tag{5}$$

By combining (1) with (5), a simple calculation allows us to obtain the next result.

Proposition 2.10. *For all positive integers n. Let Q_B^n be the n-balancing Q-matrix given in (5). Then, for all $n \geq 2$, the identity holds*

$$Q_B^{n+1} = 6Q_B^n - Q_B^{n-1} \ ,$$

where $Q_B^1 = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$.

The matrix discussed above, along with related works, has motivated further exploration into new bidimensional balancing matrices. The work of Costa and Catarino in [5] introduced and studied the properties of the bidimensional Fibonacci matrix sequence.

3 Bidimensional Balancing Matrix

This section presents the bidimensional balancing matrix, highlighting its main properties. Based on the foundations established in the previous section, we define the bidimensional balancing matrix as follows:

Definition 3.1. For all non-negative integers m and n, let us denote the sequence of bidimensional *balancing* matrices $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$. The general term is given by

$$\mathcal{B}_{(m,n)} = \begin{bmatrix} B_{(m+2,n+2)} & -B_{(m+1,n+1)} \\ B_{(m+1,n+1)} & -B_{(m,n)} \end{bmatrix}, \tag{6}$$

where $\{B_{(m,n)}\}_{m,n\geq 0}$ denotes the sequence of bidimensional balancing numbers.

In the case where both m and n are set to 0, we obtain

$$\mathcal{B}_{(0,0)} = \begin{bmatrix} B_{(2,2)} & -B_{(1,1)} \\ B_{(1,1)} & -B_{(0,0)} \end{bmatrix} = \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix}. \tag{7}$$

When $m = 1$ and $n = 0$, we get

$$\mathcal{B}_{(1,0)} = \begin{bmatrix} B_{(3,2)} & -B_{(2,1)} \\ B_{(2,1)} & -B_{(1,0)} \end{bmatrix} = \begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix}. \tag{8}$$

For $m = 0$ and $n = 1$, we have

$$\mathcal{B}_{(0,1)} = \begin{bmatrix} B_{(2,3)} & -B_{(1,2)} \\ B_{(1,2)} & -B_{(0,1)} \end{bmatrix} = \begin{bmatrix} 174+175i & -(5+6i) \\ 5+6i & -i \end{bmatrix}. \tag{9}$$

Finally, when $m = n = 1$, we obtain

$$\mathcal{B}_{(1,1)} = \begin{bmatrix} B_{(3,3)} & -B_{(2,2)} \\ B_{(2,2)} & -B_{(1,1)} \end{bmatrix} = \begin{bmatrix} 1015+1015i & -(30+30i) \\ 30+30i & -(1+i) \end{bmatrix}. \tag{10}$$

Considering the set of all matrices $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$, the result below presents the bidimensional recurrence relations satisfied by them.

Proposition 3.2. *Let $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$ be the sequence of bidimensional balancing matrices. We have the following bidimensional recurrence relations:*

$$\mathcal{B}_{(m+1,n)} = 6\mathcal{B}_{(m,n)} - \mathcal{B}_{(m-1,n)} \quad \text{for } m \geq 1\,, \tag{11}$$
$$\mathcal{B}_{(m,n+1)} = 6\mathcal{B}_{(m,n)} - \mathcal{B}_{(m,n-1)} \quad \text{for } n \geq 1,\,, \tag{12}$$

with initial terms $\mathcal{B}_{(0,0)}$, $\mathcal{B}_{(1,0)}$, $\mathcal{B}_{(0,1)}$, and $\mathcal{B}_{(1,1)}$, respectively given in (7), (8), (9), and (10).

Proof. Using Definition 3.1 and expression (2) and (3) in (11), we have

$$\begin{aligned}
& 6\mathcal{B}_{(m,n)} - \mathcal{B}_{(m-1,n)} \\
= \; & 6\begin{bmatrix} B_{(m+2,n+2)} & -B_{(m+1,n+1)} \\ B_{(m+1,n+1)} & -B_{(m,n)} \end{bmatrix} - \begin{bmatrix} B_{(m+1,n+2)} & -B_{(m,n+1)} \\ B_{(m,n+1)} & -B_{(m-1,n)} \end{bmatrix} \\
= \; & \begin{bmatrix} 6B_{(m+2,n+2)} - B_{(m+1,n+2)} & -(6B_{(m+1,n+1)} - B_{(m,n+1)}) \\ 6B_{(m+1,n+1)} - B_{(m,n+1)} & -(6B_{(m,n)} - B_{(m-1,n)}) \end{bmatrix} \\
= \; & \begin{bmatrix} B_{(m+3,n+2)} & -B_{(m+2,n+1)} \\ B_{(m+2,n+1)} & -B_{(m+1,n)} \end{bmatrix} \\
= \; & \mathcal{B}_{(m+1,n)}\,,
\end{aligned}$$

this confirms the resulting value of the first branch.

Similarly, the argument for the second branch, presented in Equation (12), can be concluded. □

Proposition 3.2 holds for the following reason: the entries of $\mathcal{B}_{(m,n)}$ remain invariant regardless of the computational path we choose. In a particular case, we determine the value of the $\mathcal{B}_{(2,2)}$. We notice that

$$\begin{cases} 6\mathcal{B}_{(1,2)} - \mathcal{B}_{(0,2)}, & \text{path (11)}; \\ 6\mathcal{B}_{(2,1)} - \mathcal{B}_{(2,0)}, & \text{path (12)}. \end{cases}$$

Firstly, path (11) will be considered. We have

$$\begin{aligned}
\mathcal{B}_{(1,2)} &= 6\mathcal{B}_{(1,1)} - \mathcal{B}_{(1,0)} \\
&= 6\begin{bmatrix} 1015+1015i & -(30+30i) \\ 30+30i & -(1+i) \end{bmatrix} - \begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix} \\
&= \begin{bmatrix} 6(1015+1015i) & -6(30+30i) \\ 6(30+30i) & -6(1+i) \end{bmatrix} - \begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix} \\
&= \begin{bmatrix} 6(1015+1015i)-(175+174i) & -6(30+30i)+6+5i \\ 6(30+30i)-(6+5i) & -6(1+i)+1 \end{bmatrix} \\
&= \begin{bmatrix} 6090-175+6090i-174i & -180+6-180i+5i \\ 180-6+180i-5i & -6+1-6i \end{bmatrix} \\
&= \begin{bmatrix} 5915+5916i & -(174+175i) \\ 174+175i & -(5+6i) \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_{(0,2)} &= 6\mathcal{B}_{(0,1)} - \mathcal{B}_{(0,0)} \\
&= 6\begin{bmatrix} 174+175i & -(5+6i) \\ 5+6i & -i \end{bmatrix} - \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 6(174+175i)-(30+30i) & -6(5+6i)+1+i \\ 6(5+6i)-(1+i) & -6i+0 \end{bmatrix} \\
&= \begin{bmatrix} 1014+1020i & -(29+35i) \\ 29+35i & -6i \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{B}_{(2,2)} &= 6\mathcal{B}_{(1,2)} - \mathcal{B}_{(0,2)} \\
&= 6\begin{bmatrix} 5915+5916i & -(174+175i) \\ 174+175i & -(5+6i) \end{bmatrix} - \begin{bmatrix} 1014+1020i & -(29+35i) \\ 29+35i & -6i \end{bmatrix} \\
&= \begin{bmatrix} 6(5915+5916i)-(1014+1020i) & -6(174+175i)+29+35i \\ 6(174+175i)-(1+i) & -6(5+6i)+6i \end{bmatrix} \\
&= \begin{bmatrix} 34476+34476i & -(1015+1015i) \\ 1015+1015i & -(30+30i) \end{bmatrix}.
\end{aligned}$$

Taking path (12) into account, we obtain

$$\begin{aligned}
\mathcal{B}_{(2,1)} &= 6\mathcal{B}_{(1,1)} - \mathcal{B}_{(0,1)} \\
&= 6\begin{bmatrix} 1015+1015i & -(30+30i) \\ 30+30i & -(1+i) \end{bmatrix} - \begin{bmatrix} 174+175i & -(5+6i) \\ 5+6i & -i \end{bmatrix} \\
&= \begin{bmatrix} 6(1015+1015i)-(174+175i) & -6(30+30i)+5+6i \\ 6(30+30i)-(5+6i) & -6(1+i)+i \end{bmatrix} \\
&= \begin{bmatrix} 5916+5915i & -(175+174i) \\ 175+174i & -(6+5i) \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_{(2,0)} &= 6\mathcal{B}_{(1,0)} - \mathcal{B}_{(0,0)} \\
&= 6\begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix} - \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 6(175+174i)-(30+30i) & -6(6+5i)+1+i \\ 6(6+5i)-(1+i) & -6-0 \end{bmatrix} \\
&= \begin{bmatrix} 1020+1014i & -(35+29i) \\ 35+29i & -6 \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{B}_{(2,2)} &= 6\mathcal{B}_{(2,1)} - \mathcal{B}_{(2,0)} \\
&= 6\begin{bmatrix} 5916+5915i & -(175+174i) \\ 175+174i & -(6+5i) \end{bmatrix} - \begin{bmatrix} 1020+1014i & -(35+29i) \\ 35+29i & -6 \end{bmatrix} \\
&= \begin{bmatrix} 6(5916+5915i)-(1020+1014i) & -6(175+174i)+35+29i \\ 6(175+174i)-(1+i) & -6(6+5i)+6 \end{bmatrix} \\
&= \begin{bmatrix} 34476+34476i & -(1015+1015i) \\ 1015+1015i & -(30+30i) \end{bmatrix},
\end{aligned}$$

which matches with the value that we found for $\mathcal{B}_{(2,2)}$, where we have used the path (11).

Next, we highlight several properties of the bidimensional balancing matrices.

Lemma 3.3. *The following properties hold for the bidimensional balancing matrices* $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$*:*

1. $\mathcal{B}_{(m,0)} = \begin{bmatrix} B_{(m+2,2)} & -B_{(m+1,1)} \\ B_{(m+1,1)} & -B_{(m,0)} \end{bmatrix}$;

2. $\mathcal{B}_{(0,n)} = \begin{bmatrix} B_{(2,n+2)} & -B_{(1,n+1)} \\ B_{(1,n+1)} & -B_{(0,n)} \end{bmatrix}$;

3. $\mathcal{B}_{(m,1)} = \begin{bmatrix} B_{(m+2,3)} & -B_{(m+1,2)} \\ B_{(m+1,2)} & -B_{(m,1)} \end{bmatrix}$;

4. $\mathcal{B}_{(1,n)} = \begin{bmatrix} B_{(3,n+2)} & -B_{(2,n+1)} \\ B_{(2,n+1)} & -B_{(1,n)} \end{bmatrix}$.

Proof. 1. Using induction on m, the proof is established.

For $m = 0$, Definition 3.1 ensures

$$\begin{aligned} \mathcal{B}_{(0,0)} &= \begin{bmatrix} B_{(0+2,2)} & -B_{(0+1,1)} \\ B_{(0+1,1)} & -B_{(0,0)} \end{bmatrix} = \begin{bmatrix} B_{(2,2)} & -B_{(1,1)} \\ B_{(1,1)} & -B_{(0,0)} \end{bmatrix} \\ &= \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix}, \end{aligned}$$

which is true. When $m = 1$, referring back to Definition 3.1, we deduce that

$$\begin{aligned} \mathcal{B}_{(1,0)} &= \begin{bmatrix} B_{(1+2,2)} & -B_{(1+1,1)} \\ B_{(1+1,1)} & -B_{(1,0)} \end{bmatrix} = \begin{bmatrix} B_{(3,2)} & -B_{(2,1)} \\ B_{(2,1)} & -B_{(1,0)} \end{bmatrix} \\ &= \begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix}, \end{aligned}$$

which is also true.

Assume the property holds for all non-negative integers $k \leq m$. We will show that it remains valid for $m+1$. Proceeding, we have

$$\begin{aligned} \mathcal{B}_{(m+1,0)} &= 6\mathcal{B}_{(m,0)} - \mathcal{B}_{(m-1,0)} \\ &= 6\begin{bmatrix} B_{(m+2,2)} & -B_{(m+1,1)} \\ B_{(m+1,1)} & -B_{(m,0)} \end{bmatrix} - \begin{bmatrix} B_{(m+1,2)} & -B_{(m,1)} \\ B_{(m,1)} & -B_{(m-1,0)} \end{bmatrix} \\ &= \begin{bmatrix} 6B_{(m+2,2)} - B_{(m+1,2)} & -(6B_{(m+1,1)} - B_{(m,1)}) \\ 6B_{(m+1,1)} - B_{(m,1)} & -(6B_{(m,0)} - B_{(m-1,0)}) \end{bmatrix}, \end{aligned}$$

and using the recurrence relation indicated in Equations (2), we obtain

$$\mathcal{B}_{(m+1,0)} = \begin{bmatrix} B_{(m+3,2)} & -B_{(m+2,1)} \\ B_{(m+2,1)} & -B_{(m+1,0)} \end{bmatrix}.$$

Thus, the property 1. is confirmed to be true.

2. The proof is conducted by induction on n.

For $n = 0$, we have previously determined that

$$\mathcal{B}_{(0,0)} = \begin{bmatrix} B_{(2,0+2)} & -B_{(1,0+1)} \\ B_{(1,0+1)} & -B_{(0,0)} \end{bmatrix} = \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix}$$

being true. For $n = 1$ and using Definition 3.1, we have that

$$\begin{aligned} \mathcal{B}_{(0,1)} &= \begin{bmatrix} B_{(2,1+2)} & -B_{(1,1+1)} \\ B_{(1,1+1)} & -B_{(0,1)} \end{bmatrix} = \begin{bmatrix} B_{(2,3)} & -B_{(1,2)} \\ B_{(1,2)} & -B_{(0,1)} \end{bmatrix} \\ &= \begin{bmatrix} 174+175i & -(5+6i) \\ 5+6i & -i \end{bmatrix}, \end{aligned}$$

and also, this is valid.

Suppose the property 2. holds for all non-negative integers $k \leq n$. We will verify that it also holds for $n+1$. Thus, we have

$$\begin{aligned} \mathcal{B}_{(0,n+1)} &= 6\mathcal{B}_{(0,n)} - \mathcal{B}_{(0,n-1)} \\ &= 6\begin{bmatrix} B_{(2,n+2)} & -B_{(1,n+1)} \\ B_{(1,n+1)} & -B_{(0,n)} \end{bmatrix} - \begin{bmatrix} B_{(2,n+1)} & -B_{(1,n)} \\ B_{(1,n)} & -B_{(0,n-1)} \end{bmatrix} \\ &= \begin{bmatrix} 6B_{(2,n+2)} - B_{(2,n+1)} & -(6B_{(1,n+1)} - B_{(1,n)}) \\ 6B_{(1,n+1)} - B_{(0,n)} & -(6B_{(0,n)} - B_{(0,n-1)}) \end{bmatrix} \end{aligned}$$

and applying the recurrence relation from (3), we get

$$\mathcal{B}_{(0,n+1)} = \begin{bmatrix} B_{(2,n+3)} & -B_{(1,n+2)} \\ B_{(1,n+2)} & -B_{(0,n+1)} \end{bmatrix}.$$

Therefore, the property 2. is verified to be true.

3. The proof is obtained by induction on m. Similar to what we did in 1.

4. The proof proceeds by induction on n. Similar to what we have done in 2. □

Furthermore, the determinant of the bidimensional balancing matrix $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$, for all non-negative integers m and n, is given by

$$\begin{aligned} \det \mathcal{B}_{(m,n)} &= \begin{vmatrix} B_{(m+2,n+2)} & -B_{(m+1,n+1)} \\ B_{(m+1,n+1)} & -B_{(m,n)} \end{vmatrix} \\ &= B_{(m+1,n+1)}B_{(m+1,n+1)} - B_{(m+2,n+2)}B_{(m,n)}\,. \end{aligned} \tag{13}$$

Now, to illustrate and motivate our results, we present an example.

Example 3.4. Using the definition of bidimensional *balancing* matrices $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$, we have:

$$\begin{aligned} |\mathcal{B}_{(0,0)}| &= \det \begin{bmatrix} 30+30i & -(1+i) \\ 1+i & 0 \end{bmatrix} \\ &= (30+30i)(0) - [-(1+i)(1+i)] = 2i\,. \end{aligned}$$

And, in a similar way, we have

- $|\mathcal{B}_{(1,0)}| = \det \begin{bmatrix} 175+174i & -(6+5i) \\ 6+5i & -1 \end{bmatrix} = -164-114i$;
- $|\mathcal{B}_{(0,1)}| = \det \begin{bmatrix} 174+175i & -(5+6i) \\ 5+6i & -i \end{bmatrix} = 164-114i$;
- $|\mathcal{B}_{(1,1)}| = \det \begin{bmatrix} 1015+1015i & -(30+30i) \\ 30+30i & -(1+i) \end{bmatrix} = -230i$.

Note that $|\mathcal{B}_{(0,0)}| = 2i \neq 0$ has a null real part. This also occurs with $|\mathcal{B}_{(1,1)}| = -230i \neq 0$; that is, the real part is identically zero. After some calculations, it's shows that $|\mathcal{B}_{(2,2)}| = -8110i \neq 0$ and $|\mathcal{B}_{(3,3)}| = 138600i \neq 0$. Hence, we have that the real component also vanishes.

These previously specified cases are special illustrations of our main result.

Theorem 3.5. *Let $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$ be the sequence of matrices of order 2 defined in (6). If $n = m$, then*

(a) $|\mathcal{B}_{(m,m)}|$ has a null real part,

(b) $|\mathcal{B}_{(m,m)}|$ is a strictly imaginary complex number.

Proof. (a) Consider

$$|\mathcal{B}_{(n,n)}| = \det \begin{bmatrix} B_{(n+2,n+2)} & -B_{(n+1,n+1)} \\ B_{(n+1,n+1)} & -B_{(n,n)} \end{bmatrix} = B^2_{(n+1,n+1)} - B_{(n+2,n+2)}B_{(n,n)} \,.$$

Using (4), we get

$$B_{(n+1,n+1)} = B_{n+1}(B_{n+1}-B_n)+(B_{n+1}-B_n)B_{n+1}i = B_{n+1}(B_{n+1}-B_n)(1+i)\,,$$

and so,

$$B^2_{(n+1,n+1)} = 2B^2_{n+1}(B_{n+1}-B_n)^2 i\,.$$

Also

$$\begin{aligned} B_{(n+2,n+2)} &= B_{n+2}(B_{n+2}-B_{n+1})(1+i) \\ B_{(n,n)} &= B_n(B_n - B_{n-1})(1+i)\ . \end{aligned}$$

Hence

$$\begin{aligned} B_{(n+2,n+2)}B_{(n,n)} &= B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1})(1+i)^2 \\ &= 2B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1})i\ . \end{aligned}$$

Therefore,

$$\begin{aligned} & B^2_{(n+1,n+1)} - B_{(n+2,n+2)}B_{(n,n)} \\ = \ & 2\Big[B^2_{n+1}(B_{n+1}-B_n)^2 - B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1})\Big]i, \end{aligned}$$

and we obtain the result.

(b) By item (a), it is enough to show that

$$B^2_{n+1}(B_{n+1}-B_n)^2 - B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1}) \neq 0\,,$$

or equivalently,

$$B^2_{n+1}(B_{n+1}-B_n)^2 \neq B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1})\,.$$

We claim that,

$$gcd\Big(B^2_{n+1}(B_{n+1}-B_n)^2,\ \ B_{n+2}B_n(B_{n+2}-B_{n+1})(B_n-B_{n-1})\Big) = 1\ ,$$

so, for $n \geq 1$ each of the two expressions is strictly greater than 1; hence their equality would contradict the co-primality. Indeed, by Corollary 2.2, we have

$$gcd(B_{n+2}, B_{n+1}) = gcd(B_{n+1}, B_n) = 1\,,$$

and by Corollary 2.4 we obtain $gcd(B_{n+1}^2,\ B_{n+2}B_n) = 1$, so there are no common terms for B_{n+1}^2 and $B_{n+2}B_n$. And, in a similar way, applying Lemma 2.6, we get

$$gcd(B_{n+2} - B_{n+1},\ B_{n+1} - B_n) = gcd(B_{n+1} - B_n,\ B_n - B_{n-1}) = 1\,,$$

which verifies the result. □

As the determinant of the matrices $\{\mathcal{B}_{(m,m)}\}_{m\geq 0}$ is a non-zero imaginary number, an immediate consequence of Theorem 3.5 is presented in the following result.

Corollary 3.6. *The matrices $\{\mathcal{B}_{(m,m)}\}_{m\geq 0}$ are invertible.*

As usual, we denote by $\overline{x} = a - bi$ the conjugate of the complex number $x = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. The following result reveals a form of "symmetry" arising when the indices m and n are interchanged in the determinant of the matrices $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$.

Proposition 3.7. *If $|\mathcal{B}_{(m,n)}| = x + yi$ for some integers x and y, then $|\mathcal{B}_{(n,m)}| = -\overline{|\mathcal{B}_{(m,n)}|}$, where $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$ is the sequence of bidimensional balancing matrices defined in (6) and i is the imaginary unit.*

Proof. Applying Corollary 2.8, we get

$$B_{(m+2,n+2)} = a_2 + b_2 i,\quad B_{(m+1,n+1)} = a_1 + b_1 i \text{ and } B_{(m,n)} = a_0 + b_0 i\,,$$

for $a_t + b_t i \in \mathbb{C}$, with $t \in \{0, 1, 2\}$. So

$$\begin{aligned} |\mathcal{B}_{(m,n)}| &= \begin{vmatrix} B_{(m+2,n+2)} & -B_{(m+1,n+1)} \\ B_{(m+1,n+1)} & -B_{(m,n)} \end{vmatrix} = \begin{vmatrix} a_2 + b_2 i & -(a_1 + b_1 i) \\ a_1 + b_1 i & -(a_0 + b_0 i) \end{vmatrix} \\ &= (a_1^2 - b_1^2 - a_0 a_2 + b_0 b_2) + (2a_1 b_1 - a_2 b_0 - a_0 b_2)i \\ &= x + yi, \end{aligned}$$

making $x = a_1^2 - b_1^2 - a_0 a_2 + b_0 b_2$ and $y = 2a_1 b_1 - a_2 b_0 - a_0 b_2$.

By Proposition 2.8, we have

$$\begin{aligned} |\mathcal{B}_{(n,m)}| &= \begin{vmatrix} B_{(n+2,m+2)} & -B_{(n+1,m+1)} \\ B_{(n+1,m+1)} & -B_{(n,m)} \end{vmatrix} = \begin{vmatrix} b_2 + a_2 i & -(b_1 + a_1 i) \\ b_1 + a_1 i & -(b_0 + a_0 i) \end{vmatrix} \\ &= (b_1^2 - a_1^2 + a_0 a_2 - b_0 b_2) + (2a_1 b_1 - a_2 b_0 - a_0 b_2)i \\ &= -(a_1^2 - b_1^2 - a_0 a_2 + b_0 b_2) + (2a_1 b_1 - a_2 b_0 - a_0 b_2)i \\ &= -x + yi = -(x - yi), \end{aligned}$$

and this concludes the proof. □

Note that, for all non-negative integers m and n, in the complex plane, the numbers $|\mathcal{B}_{(m,n)}|$ and $|\mathcal{B}_{(n,m)}|$ are symmetric with respect to the complex axis.

Example 3.8. Using Equation (13) and previously computed values in Table 1 we obtain,

$$|\mathcal{B}_{(2,0)}| = (35 + 29i)^2 - (1020 + 1014i) \cdot 6 = -5736 - 4054i\,,$$

and

$$|\mathcal{B}_{(2,1)}| = (175 + 174i)^2 - (5916 + 5915i)(6 + 5i) = -5572 - 4170i.$$

Now, continuing from Proposition 3.7 we obtain that

$$|\mathcal{B}_{(0,2)}| = 5736 - 4054i \text{ and } |\mathcal{B}_{(1,2)}| = 5572 - 4170i\,.$$

Remember that Theorem 3.5 demonstrates that every bidimensional matrix in the family $\{B_{(m,m)}\}_{m\geq 0}$ possesses an inverse. We close this section with an open question.

Conjecture 3.9. *Consider the sequence $\{\mathcal{B}_{(m,n)}\}_{m,n\geq 0}$ of bidimensional balancing matrices. For every pair of non-negative integers m and n, then* $\det \mathcal{B}_{(m,n)} \neq 0$.

Note that the Conjecture 3.9 is motivated by Theorem 3.5 and Examples 3.4 and 3.8.

4 Conclusion

In this paper, we explored a systematic approach to matrices with balancing bidimensional sequences as components. The results made a significant contribution to the domain of mathematics, providing new insights into matrix sequences, offering a new way for further research, and laying a foundation for practical applications. In future work, the other properties will be examined, highlighting the potential of bidimensional balancing matrices to improve the development of applications involving matrices, for instance, we will try to know if there is a direct relation between $\mathcal{B}_{(m,n)}$ and the one-dimensional Q_B matrix.

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References

[1] A. Behera and G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.*, vol. 37, no. 2, pp. 98–5 (1999).

[2] P. Catarino, H. Campos, and P. Vasco, On some identities for balancing and cobalancing numbers, *Ann. Math. Inform.*, vol. 45, pp. 6–24 (2015).

[3] J. Chimpanzo, P. Catarino, P. Vasco, and A. Borges, Bidimensional extensions of balancing and Lucas-balancing numbers, *J. Discrete Math. Sci. Cryptogr.*, vol. 27, no. 1, pp. 95-65 (2024).

[4] J. Chimpanzo, M. V. Otero-Espinar, A. Borges, P. Vasco, and P. Catarino, Bidimensional extensions of cobalancing and Lucas-cobalancing numbers, *Annales Mathematicae Silesianae*, vol. 38, no. 2, pp. 241–262 (2023).

[5] E. A. Costa and P. Catarino, On matrices with bidimensional Fibonacci numbers, *Annales Mathematicae Silesianae*, vol. 40, no. 1, pp. 59-76 (2026). 10.2478/amsil-2025-0014

[6] R. Finkelstein, The house problem, *The American Mathematical Monthly*, vol. 72, no. 10, pp. 1082–1088 (1965).

[7] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*. Oxford university press, 1979.

[8] K. Liptai, Lucas-balancing Numbers, *Acta Mathematica Universitatis Ostraviensis*, vol. 14, pp. 43–47 (2006).

[9] I. Niven, H. S. Zuckerman and H. L. Montegomery, *An introduction to the theory of numbers*, John Wiley and Sons, 1991.

[10] K. Lipati, F. Luca, A. Pinter, and L. Szalay, Generalized balancing numbers, *Indag. Mathem.*, vol. 20, no. 1, pp. 87–100 (2009).

[11] P. Olajos, Properties of balancing, cobalancing and generalized cobalancing numbers, *Ann. Math. Inform.*, vol. 37, pp. 125-–138 (2010).

[12] G. K. Panda, Sequence balancing and cobalancing numbers, *Fibonacci Quart.*, vol. 45, no. 3, pp. 265–271 (2007).

[13] G. K. Panda, Some fascinating properties of balancing numbers, In: *Proc. Eleventh Internat. Conference on Fibonacci Numbers and Their Applications*, vol. 194, pp. 185–189 (2009).

[14] G. K. Panda and P. k. Ray, Some Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers, *Bul. of Inst. of Math. Acad. Sinica*, vol. 6, no. 1, pp. 41–72 (2006).

[15] P. K. Ray, *Balancing and Cobalancing Numbers*, Ph.D. thesis, Department of Mathematics, National Institute of Technology, Rourkela, India, 2009. Available online at http://ethesis.nitrkl.ac.in/2750/1/Ph.D._Thesis_of_P.K._Ray.pdf

[16] P. K. Ray, Certain matrices associated with balancing and Lucas-balancing numbers, *Matematika*, vol. 28, no. 1, pp. 15–22 (2012).

[17] N. J. Sloane *et al.*, The On-Line Encyclopedia of Integer Sequences, 2026. Available online at https://oeis.org

[18] M. Uysal and E. Özkan, Balancing and Lucas-Balancing hybrid numbers and some identities, *Journal of Information and Optimization Sciences*, vol. 45, no. 5, pp. 293–1304 (2024).

[19] S. M. Zahid, Some new properties of Lucas-balancing and Lucas-cobalancing number, *Matematika*, vol. 33, no. 2, pp. 207–214 (2017).

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