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Original Research Paper

Determining an Assurance Non-Archimedean Epsilon in FDH Models with Variable Returns to Scale

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Abstract. Selecting an appropriate value for the non-Archimedean epsilon in Free Disposal Hull (FDH) models with Variable Returns to Scale (VRS) is a persistent challenge. If not properly managed, it can lead to the infeasibility of the multiplier form and the unboundedness of the envelopment form. This paper employs the framework of duality theory in linear programming to provide an analytical solution with mathematical guarantees. Specifically, a method has been devised that determines a global assurance interval for the epsilon parameter. The upper bound of this interval is calculated using a simple formula based solely on basic arithmetic operations applied to the input data. This framework provides a dual guarantee: any epsilon value within this interval simultaneously ensures the feasibility of the multiplier model and the boundedness of the envelopment model. Validation of the method using numerical examples and a real-world case study demonstrates that the model based on this assured epsilon yields the same results as the base model with complete accuracy. By presenting a robust mathematical foundation and a practical, efficient tool, this research takes a significant step towards overcoming the challenge of epsilon selection in non-convex FDH environments.

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1 Introduction

Data envelopment analysis (DEA) is a scientific approach that evaluates the performance of decision-making units (DMUs) through efficiency calculations. This method was first proposed by Charnes et al. [1], which led to the introduction of the CCR model with constant returns to scale (CRS) technology. Banker et al. [2] developed the DEA technique by presenting the BCC model with variable returns to scale (VRS) technology. Following this, Färe and Graskopf [3] proposed the FG model with non-increasing returns to scale (NIRS) technology, and Seiford and Thrall [4] presented the ST model with non-decreasing returns to scale (NDRS) technology. These models are known as the basic DEA models.

This technique has continuously evolved and found extensive applications across various fields of management, economics, and engineering. One of the most persistent challenges in applying DEA models is selecting an appropriate value for the non-Archimedean epsilon parameter. This parameter prevents the assignment of zero weights to inputs and outputs, thereby avoiding the unintended exclusion of influential indicators from the evaluation. An inappropriate selection of epsilon can lead to two fundamental problems:

1. Infeasibility of the multiplier form, where no feasible solution exists for the dual problem.

2. Unboundedness of the envelopment form, where the objective function tends to infinity, making efficiency evaluation impossible.

This challenge becomes more complex in non-convex environments, particularly in Free Disposal Hull (FDH) models, which abandon the convexity assumption. In FDH models with Variable Returns to Scale (FDH-VRS models), the non-convex structure and the dependency of the solution on mixed-integer programming require specific computational and theoretical approaches to determine an epsilon value that guarantees both the feasibility of the multiplier form and the boundedness of the envelopment form. In this paper, for the first time, an analytical solution with mathematical guarantees is proposed to determine an assurance value of epsilon in the FDH-VRS setting, simultaneously addressing the aforementioned challenges.

The key achievements and advantages of this solution can be summarized along the following axes:

Elimination of Zero-Weight Flaw: Using a reliable epsilon value prevents the assignment of zero weights to input and output factors, ensuring that all indicators are meaningfully incorporated into the evaluation process. This leads to a more comprehensive and realistic assessment.

Providing a Foundational and Computationally Efficient Solution: The proposed method is based on a simple formula that calculates the assurance value of epsilon solely through basic arithmetic operations (addition and multiplication) on the input data. This approach completely eliminates the need to solve complex linear programming problems or execute iterative algorithms, significantly reducing the computational burden. Consequently, the model can be efficiently implemented even for large datasets.

Theoretically Grounded Epsilon: The calculated epsilon is not merely an empirical or speculative value; rather, it is founded on a solid theoretical basis and provides two simultaneous mathematical guarantees: 1) guaranteeing the feasibility of the multiplier form, and 2) guaranteeing the boundedness of the envelopment form. This dual guarantee establishes a robust foundation for all subsequent efficiency calculations.

High Generalizability and Applicability: By defining a universal confidence interval for epsilon, a single value is obtained that can be used to evaluate all DMUs within the dataset. This feature makes applying the model to large-scale, practical problems straightforward and feasible.

By integrating theoretical innovation with practical application, this paper takes a significant step toward enhancing the reliability, accuracy, and computational efficiency of non-convex FDH-VRS models.

1.1 Literature Review

FDH models were proposed to complement DEA models by Diprin et al. [5]. Whereas basic DEA models have a convexity condition, FDH models operate without such a condition. Therefore, FDH models are solved through mixed-integer programming. An important feature of FDH models is that the reference unit for each inefficient DMU includes only one of the observed units, which is more evident in many real-world problems. In recent years, many studies have been conducted on FDH models, including the following.

Tulkens [6] and Agrell and Tind [7] presented different computational methods for solving the FDH model and proposed equivalent linear programming formulations. Podynowski [8] was able to present a linear model for determining returns to scale (RTS) in the FDH model. Soleimani-Damane and Reshadi [9] proposed an algorithm to determine the RTS at the efficient points of FDH, which has a short solution time and requires only simple ratio calculations. Simar and Zelniuk [10] introduced the first stochastic version of the DEA/FDH model and showed that it improves the estimators. Sanei and Mamizadeh [11] presented a new application of the FDH model for supply chain evaluation. Abbasi et al. [12] evaluated congestion using the FDH model. Soleimani-Domane and Mostafaei [13] developed a method for computing anchor points in the FDH model. García-Romero et al. [14] provided an application of the output-driven FDH model. Khanjani Shiraz et al. [15] presented a nonparametric approach for both convex DEA and non-convex FDH technologies, under different RTS assumptions. Badin et al. [16] proposed a bootstrap-based approach to overcome certain FDH limitations. Tavakoli and Mostafaee [17] extended network DEA models to non-convex FDH technologies. Mirmozaffari et al. [18] introduced a new DEA-based model called SBM-FDH. Asadi et al. [19] proposed a multiple period inverse FDH model to evaluate changes in outputs over multiple periods. Esteve et al. [20] used machine learning techniques to determine super-efficiency of DMUs via the random forest and FDH. Dibachi and Izadikhah [21] extended the FDH model to random data with lognormal distribution. Papaioannou and Podinovski [22] introduced a multi-component FDH (MFDH) model to study multi-component technologies. Pourmahmoud et al. [23] extended the concept of inverse DEA to FDH technology. Mehdiloo et al. [24] extended the use of production balances to the FDH technology model and its constant, non-increasing, and non-decreasing scale return types in a unified framework. In basic DEA models, a zero weight means that the corresponding criterion is ignored in the performance evaluation process, which is undesirable. To address this issue, Charnes et al. [25] proposed setting an infinitesimal value, called epsilon, as the lower bound for the weights. Therefore, to prevent the input and output weights from becoming zero, a positive epsilon must be considered.

Determining an appropriate value for epsilon is a challenging issue in the DEA literature [25]. Ali and Seiford [26], Using an infinitesimal epsilon may lead to computational inaccuracies in efficiency evaluations. Podinovski and Bouzdine-Chameeva [27] proved that there exists a threshold such that if epsilon is smaller than this value, no computational error occurs. Conversely, using a large epsilon may cause the multiplier model to be infeasible, or equivalently, make the envelopment model unbounded [26, 27]. To overcome this, Mehrabian et al. [28] proposed a linear programming model defining an assurance interval for epsilon that guarantees the feasibility of the multiplier form and the boundedness of the envelopment form. Amin and Toloo [29] introduced a new algorithm for determining an assurance value for epsilon in DEA models, computed via simple addition and multiplication operations on the data. This method presents a polynomial-time algorithm for finding epsilon in convex DEA. The current paper, by offering a deterministic, non-iterative formula, is computationally more efficient and has extended the approach to the non-convex FDH structure for which the method of Amin and Toloo was not designed.

Alirezaei [30] proposed an axis partitioning algorithm for determining the overall assurance interval for the non-Archimedean epsilon in DEA models, which obtains this interval by solving fewer linear programming problems. This reference provides an efficient algorithm for reducing the number of LPs required to find the overall assurance interval in convex DEA. In the present paper, this number is reduced to zero for the FDH-VRS setting through the discovery of an algebraic formula. Toloo [31] explained the role of the non-Archimedean epsilon in identifying the best efficient DMU. He developed several new models to find a suitable value for epsilon and applied his approach to identify the best professional tennis players. This reference focuses on the application of epsilon for ranking and selecting the most efficient unit in convex DEA. The present paper takes a step back and establishes a theoretical, guaranteed foundation for the epsilon parameter itself within a different framework. The innovation lies in stabilizing the parameter itself, not in its application for a specific purpose. Toloo and Tavana [32] proposed a new method for determining the most efficient DMU with pure input or pure output data. This reference addresses a specific selection problem (identifying an efficient unit with pure data) in convex DEA. The main superiority of this paper lies in integrating three unique achievements that, in combination, distinguish it from references 30 to 32. First, a shift from convex DEA to non-convex FDH, second, Replacing optimization-based methods with a simple arithmetic formula, and third, providing a theoretical foundation that simultaneously overcomes two key problems (feasibility and boundedness). This paper is not a competitor but an essential and foundational complement to prior work.

Flokou et al. [35], using advanced methods such as DEA and FDH, evaluate the efficiency of public primary health centers in Greece after the economic crisis and before the COVID-19 pandemic. Their findings indicate that these centers are, on average, inefficient and have significant potential to increase services approximately doubling outputs with existing resources, particularly in rural and remote areas. Papaioannou and Podinovski [36] develop multi-component Free Disposal Hull (MFDH) models, which enable the combination of independent processes from different DMUs and manage uncertainty in shared resource allocation. Their practical application in UK universities demonstrates that these models (especially when considering allocation constraints) provide better discriminatory power in efficiency assessments compared to standard FDH. Joo [37], by comparing FDH methods with machine learning methods (EAT and RFEAT), showed that machine learning-based methods, particularly RFEAT, exhibit higher discriminatory power in evaluating the efficiency of OECD health systems and reduce the problem of trivializing inefficiency.

References [35, 36, 37] demonstrate what can be done with the FDH model (application, development, comparison). The present paper focuses on providing a theoretical and practical foundation to enhance the reliability and ease of use of the FDH-VRS model itself. This, in turn, can improve the accuracy and dependability of all future applications and developments (including those similar to references [35, 36, 37]).

A review of the literature reveals that no prior work has addressed the issue of the epsilon parameter used in non-convex models. Therefore, in this study, the researchers have focused on the epsilon parameter applied in FDH models. Furthermore, since the returns to scale of the population under study are unknown, models with variable returns to scale have been utilized in this research. It should be noted that the concepts discussed in this paper can also be extended to models with constant, non-increasing, and non-decreasing returns to scale.

In the present paper, by presenting an epsilon-based linear model equivalent to the FDH-VRS model and utilizing the method of Mehrabian et al. [28], we determine a reliable value for the non-Archimedean epsilon to ensure the feasibility of the multiplier form and, consequently, the boundedness of the envelopment form. Given the structure of the FDH model, this assurance value of epsilon is obtained solely through simple addition and multiplication operations on the data. The key distinction is that the method of Mehrabian et al. [28] for determining the confidence interval in convex DEA requires solving an LP problem. However, the innovation of the present paper lies in the complete elimination of the LP-solving stage and its replacement with a direct algebraic formula for the FDH setting. Section 2 provides a review of FDH-VRS models. This research is organized as follows: Section 3 presents the method for determining the assurance value of non-Archimedean epsilon. Section 4 provides numerical examples, and Section 5 offers an empirical example. Finally, Section 6 presents the conclusion.

2 FDH Model under Variable Returns to Scale (FDH-VRS)

Consider n decision-making units (DMUs), each of which consumes m inputs to generate s outputs. Let $X_j \in \mathbb{R}_+^m$ and $Y_j \in \mathbb{R}_+^s$ denote the input and output vectors of DMU_j $(j = 1, \ldots, n)$, respectively. The FDH-VRS model for evaluating DMU_k can be formulated as:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j X_j + S^- = \theta X_k, \\
& \sum_{j=1}^{n} \lambda_j Y_j - S^+ = Y_k, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \in \{0,1\}, \quad j = 1, \ldots, n, \\
& S^- \geq \mathbf{0}, \quad S^+ \geq \mathbf{0}, \quad \theta \text{ free in sign},
\end{aligned}
\tag{1}
$$

where θ is the corresponding relative efficiency score DMU_k , S^- is the input slack vector, and S^+ is the output slack vector. Similarly, the epsilon-based FDH-VRS model for evaluating DMU_k can alternatively be written as:

$$
\begin{aligned}
\min \quad & \theta - \varepsilon(\mathbf{1}_m S^- + \mathbf{1}_s S^+) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j X_j + S^- = \theta X_k, \\
& \sum_{j=1}^{n} \lambda_j Y_j - S^+ = Y_k, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \in \{0,1\}, \quad j = 1, \ldots, n, \\
& S^- \geq \mathbf{0}, \quad S^+ \geq \mathbf{0}, \quad \theta \text{ free in sign},
\end{aligned}
\tag{2}
$$

where $\mathbf{1}_k^T \in \mathbb{R}^k$ is a vector of ones, and $\mathbf{1}_m S^-$, $\mathbf{1}_s S^+$ are the sums of the input and output slacks, respectively. As can be observed, Models (1) and (2) are mixed-integer linear programming problems. Since the theoretical foundation of this paper relies on duality and linear programming theorems, we employ the method of Agrell and Tind [1] to linearize Models (1) and (2). Therefore, using this method, the linear equivalents of Models (1) and (2) which are Models (3) and (4) (the FDH-VRS envelopment forms), respectively are presented as follows:

FDH-VRS Model Envelopment Form:

$$
\begin{aligned}
E^*_{1k} = & \min \sum_{j=1}^{n} \theta_j \\
\text{s.t.} \quad & \theta_j X_k - \lambda_j X_j - S^-_j = \mathbf{0}, \quad j = 1, \ldots, n, \\
& \lambda_j (Y_j - Y_k) - S^+_j = \mathbf{0}, \quad j = 1, \ldots, n, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \; S^-_j \geq \mathbf{0}, \; S^+_j \geq \mathbf{0}, \quad j = 1, \ldots, n, \\
& \theta_j \text{ free in sign}, \quad j = 1, \ldots, n,
\end{aligned}
\tag{3}
$$

and

FDH-VRS Epsilon Model Envelopment Form:

$$
\begin{aligned}
E^*_{2k} = \min \quad & \sum_{j=1}^{n} \theta_j - \varepsilon \sum_{j=1}^{n} (\mathbf{1}_m S^-_j + \mathbf{1}_s S^+_j) \\
\text{s.t.} \quad & \\
& \theta_j X_k - \lambda_j X_j - S^-_j = \mathbf{0}, \quad j = 1, \ldots, n, \qquad (4{-}1) \\
& \lambda_j (Y_j - Y_k) - S^+_j = \mathbf{0}, \quad j = 1, \ldots, n, \qquad (4{-}2) \\
& \sum_{j=1}^{n} \lambda_j = 1, \qquad (4{-}3) \\
& \lambda_j \geq 0, \; S^-_j \geq \mathbf{0}, \; S^+_j \geq \mathbf{0}, \quad j = 1, \ldots, n, \\
& \theta_j \text{ free in sign}, \quad j = 1, \ldots, n,
\end{aligned}
\tag{4}
$$

where S^-_j and S^+_j are the input and output slack vectors corresponding to DMU_j, $(j = 1, \ldots, n)$ respectively, for evaluating DMU_k .

Lemma 2.1. *Models* (3) *and* (4) *are feasible.*

Proof. It can be seen that Models (3) and (4) have the same feasible region and the only difference is in their objective function. These models are feasible because the following vector satisfies all their constraints:

$\Big(\bar{\boldsymbol{\theta}} = (\bar{\theta}_1, \ldots, \bar{\theta}_n) = \mathbf{e}_k,\ \bar{\boldsymbol{\lambda}} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \mathbf{e}_k,\ \bar{S}_j^- = \mathbf{0}\ (j = 1, \ldots, n),$
$\bar{S}_j^+ = \mathbf{0}\ (j = 1, \ldots, n)\Big),$

where $\mathbf{e}_k \in \mathbb{R}^n$ is kth coordinate vector, This is a vector having zero components, except for a 1 in the kth position. $\square$

To accurately evaluate the efficiency of DMUs using Model (3), we follow the phases below: First, in Phase 1, we execute Model (3) for all DMUs, which requires solving n models. Then, in Phase 2, to verify the accuracy of the efficiency value for efficient units, we use the following model, known as the maximum slack model:

Maximum Slacks Model:

$$
\begin{aligned}
S_k^* = \max & \sum_{j=1}^{n} (\mathbf{1}_m S_j^- + \mathbf{1}_s S_j^+) \\
\text{s.t.} \quad & X_k - \lambda_j X_j - S_j^- = \mathbf{0}, \quad j = 1, \ldots, n, \\
& \lambda_j (Y_j - Y_k) - S_j^+ = \mathbf{0}, \quad j = 1, \ldots, n, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0,\ S_j^- \geq \mathbf{0},\ S_j^+ \geq \mathbf{0}, \quad j = 1, \ldots, n.
\end{aligned}
\tag{5}
$$

This model corresponds to DMU_k, which was identified as efficient in Phase 1 ($E_{1k}^* = 1$). Here, for each efficient unit, Model (5) is solved to verify the accuracy of the efficiency value. If $S_k^* = \mathbf{0}$, it is concluded that DMU_k lies on the efficiency frontier and is an efficient unit; otherwise, DMU_k does not lie on the efficiency frontier and is not efficient.

Since Models (3) and (4) have the same feasible region, their dual variables are similar. Therefore, to write the dual of these models, if we assume $V_j \geq \mathbf{0} \quad (j = 1, \ldots, n)$ that is the vector of dual variables corresponding to constraints (4-1), $U_j \geq \mathbf{0} \quad (j = 1, \ldots, n)$ is the vector of dual variables corresponding to constraints (4-2), and $z \in \mathbb{R}$ is the dual variable corresponding to constraint (4-3), then the dual of Models (3) and (4) are Models (6) and (7), respectively, as follows:

FDH-VRS Model Multiplier Form:

$$
\begin{aligned}
E_{1k}^{*} = & \max \; z \\
\text{s.t.} \;\; & V_j X_k = 1, \quad j = 1, \ldots, n, \\
& U_j(Y_j - Y_k) - V_j X_j + z \leq 0, \quad j = 1, \ldots, n, \\
& V_j \geq \mathbf{0}, \quad j = 1, \ldots, n, \\
& U_j \geq \mathbf{0}, \quad j = 1, \ldots, n, \\
& z \text{ free in sign,}
\end{aligned}
\tag{6}
$$

and

FDH-VRS Epsilon Model Multiplier Form:

$$
\begin{aligned}
E_{2k}^{*} = & \max \; z \\
\text{s.t.} \;\; & V_j X_k = 1, \quad j = 1, \ldots, n, \\
& U_j(Y_j - Y_k) - V_j X_j + z \leq 0, \quad j = 1, \ldots, n, \\
& V_j \geq \varepsilon \mathbf{1}_m, \quad j = 1, \ldots, n, \\
& U_j \geq \varepsilon \mathbf{1}_s, \quad j = 1, \ldots, n, \\
& z \text{ free in sign.}
\end{aligned}
\tag{7}
$$

Here, vectors V_j and U_j are called the input and output weight vectors of the jth unit in evaluating DMU_k, respectively.

Lemma 2.2. *Models (3) and (6) have a finite optimal solution.*

Proof. Model (6) is feasible, since the following vector satisfies all constraints:

$$\left(\bar{U}_j = \mathbf{0} \quad (j = 1, \ldots, n), \bar{V}_j = \tfrac{1}{x_{\ell k}} \mathbf{e}_\ell \quad (j = 1, \ldots, n), \bar{z} = 0\right),$$

where $x_{\ell k} = \max\limits_{i=1,\ldots,m} x_{ik}$. Since Models (3) and (6) are feasible, then by the fundamental duality theorem, Model (6) has a finite optimal solution. $\square$

Despite the feasibility of Model (6), the feasibility of Model (7) depends on the appropriate choice of value for ε. Because by choosing its value inappropriately, Model (7) may become infeasible and therefore its dual, Model (4), may become unbounded.

Definition 2.3. *Any $\varepsilon > 0$ for which Model (7) becomes feasible and Model (4) becomes bounded is called an assurance value of non-Archimedean epsilon for the FDH-VRS epsilon model for evaluating DMU_k.*

According to Definition 1, determining an assurance value of non-Archimedean epsilon is necessary for the feasibility of Model (7) and the boundedness of Model (4).

3 Determining an Assurance Value of Non-Archimedean Epsilon for Model (7)

We are looking for values of ε that guarantee the feasibility of Model (7), which implies the boundedness of Model (4). Moreover, we are looking for the largest ε that guarantees this feasibility. Thus, we have to solve the following linear program model:

$$
\begin{array}{lr}
\varepsilon_k^* = \max\ \varepsilon_k & \\
\text{s.t.} & \\
\quad V_j X_k = 1, \quad j = 1, \ldots, n, & (8-1) \\
\quad U_j(Y_j - Y_k) - V_j X_j + z \leq 0, \quad j = 1, \ldots, n, & (8-2) \\
\quad \varepsilon_k \mathbf{1}_m - V_j \leq \mathbf{0}, \quad j = 1, \ldots, n, & (8-3) \\
\quad \varepsilon_k \mathbf{1}_s - U_j \leq \mathbf{0}, \quad j = 1, \ldots, n, & (8-4) \\
\quad z \text{ free in sign.} &
\end{array}
\tag{8}
$$

Theorem 3.1. *Model (8) has a finite optimal solution.*

Proof. Model (8) is feasible, since the following vector satisfies all constraints: $\left(\bar{\varepsilon}_k = 0, \bar{U}_j = \mathbf{0}(j = 1, \ldots, n), \bar{V}_j = \frac{1}{x_{\ell k}} \mathbf{e}_\ell(j = 1, \ldots, n), \bar{z} = 0\right)$, where $x_{\ell k} = \max\limits_{i=1,\ldots,m} x_{ik}$. For $j = k$, from the set of constraints (8-3), we have:

$$\varepsilon_k \mathbf{1}_m X_k - V_k X_k \leq 0 \rightarrow \varepsilon_k \mathbf{1}_m X_k \leq 1 \rightarrow \varepsilon_k \leq \frac{1}{\mathbf{1}_m X_k}.$$

Therefore, the objective function of Model (8) is bounded from above and so, this model has a finite optimal solution. □

Since Model (8) has a finite optimal solution, then according to the optimality condition theorem, its dual also has an optimal solution. Now, to find the optimal value of Model (8), we first write its dual. If we assume that $\theta_j \in \mathbb{R}$ $(j = 1, \dots, n)$ are the dual variables corresponding to constraints (8-1), $\lambda_j \geq 0$ $(j = 1, \dots, n)$ are the dual variables corresponding to constraints (8-2), $\mathbf{w}_j \in \mathbb{R}^m_+$ $(j = 1, \dots, n)$ are the vectors of dual variables (8-3), and $\mathbf{z}_j \in \mathbb{R}^s_+$ $(j = 1, \dots, n)$ are the vectors of dual variables (8-4), then the dual of Model (8) will be in the form of Model (9), which is given below:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \theta_j \\
\text{s.t.} \quad & \\
& \sum_{j=1}^{n} \mathbf{1}_m \mathbf{w}_j + \sum_{j=1}^{n} \mathbf{1}_s \mathbf{z}_j = 1, && (9-1) \\
& \theta_j X_k - \lambda_j X_j - \mathbf{w}_j = \mathbf{0}, \quad j = 1, \dots, n, && (9-2) \\
& \lambda_j (Y_j - Y_k) - \mathbf{z}_j = \mathbf{0}, \quad j = 1, \dots, n, && (9-3) \\
& \sum_{j=1}^{n} \lambda_j = 0 && (9-4) \\
& \theta_j \in \mathbb{R},\ \lambda_j \geq 0,\ \mathbf{w}_j \geq \mathbf{0},\ \mathbf{z}_j \geq \mathbf{0},\ j = 1, \dots, n.
\end{aligned}
\tag{9}
$$

Theorem 3.2. *The optimal value of Model (8) is as follows:*

$$
\varepsilon_k^* = \frac{1}{\mathbf{1}_m X_k} \tag{10}
$$

Proof. According to Theorem 3.1, Model (8) has a finite optimal solution, and therefore, by the duality theorems, Model (9) also has a finite optimal solution. Given this, in the optimal solution of Model (9), from constraint (9-4) and considering the non-negativity of the λ_j's, it follows that $\lambda_j = 0 \quad (j = 1, \dots, n)$. Subsequently, from constraints (9-3), we obtain $\mathbf{z}_j = \mathbf{0} \quad (j = 1, \dots, n)$. Then, from constraints (9-2), we have $\mathbf{w}_j = \theta_j X_k \quad (j = 1, \dots, n)$ and then $\mathbf{1}_m \mathbf{w}_j = \theta_j \mathbf{1}_m X_k \quad (j = 1, \dots, n)$.

Finally, from constraint (9-1) we obtain:

$$\sum_{j=1}^{n} \mathbf{1}_m \mathbf{w}_j = 1 \rightarrow \sum_{j=1}^{n} \theta_j \mathbf{1}_m X_k = 1 \rightarrow \mathbf{1}_m X_k \sum_{j=1}^{n} \theta_j = 1.$$

Therefore, the optimal value of the objective function of Model (9) is $\sum_{j=1}^{n} \theta_j = \frac{1}{\mathbf{1}_m X_k}$, which, by the Fundamental Duality Theorem, equals the optimal value of the objective function of Model (8). Hence, the optimal value of the objective function of Model (8) is $\varepsilon_k^* = \frac{1}{\mathbf{1}_m X_k}$. $\square$

Therefore, for any choice $\varepsilon \in (0, \varepsilon_k^*]$, Model (7) is feasible and Model (4)) is bounded.

Lemma 3.3. *For $\varepsilon > \varepsilon_k^*$, Model (7) becomes infeasible.*

Proof. By contradiction, suppose Model (7) is feasible. From the set of constraints $V_j X_k = 1, \quad j = 1, \ldots, n$, we have $V_k X_k = 1$ for $j = k$. Also from the set of constraints $V_j \geq \varepsilon \mathbf{1}_m, \quad j = 1, \ldots, n$, we have $\varepsilon \mathbf{1}_m \leq V_k$. Then $\varepsilon \mathbf{1}_m X_k \leq V_k X_k$, and therefore $\varepsilon \leq \frac{1}{\mathbf{1}_m X_k} = \varepsilon_k^*$, which is a contradiction. $\square$

According to Lemma 3.3, we can give the following definition:

Definition 3.4. *The interval $(0, \varepsilon_k^*]$ is called assurance interval for Models (4) and (7) for evaluation DMU_k.*

From definition 3.4, it follows that any value of $\varepsilon \in (0, \varepsilon_k^*]$ is an assurance value in FDH-VRS epsilon models for evaluation DMU_k.

Definition 3.5. *The interval $(0, \varepsilon^*]$, which is the intersection of all assurance intervals for the feasibility of Model (7) and the bounded of Model (4) for the evaluation of all $DMUs$, is called the overall assurance interval for epsilon in FDH-VRS epsilon models, where $\varepsilon^* = \min\{\varepsilon_1^*, \ldots, \varepsilon_n^*\}$.*

Lemma 3.6. *The optimal value of ε^* is as follows:*

$$\varepsilon^* = \frac{1}{\max\limits_{k=1,\ldots,n} \mathbf{1}_m X_k}. \tag{11}$$

Proof. From Eq. (10) and Definition 3.5, we have:

$$\varepsilon^* = \min\{\varepsilon_1^*, \ldots, \varepsilon_n^*\} = \min_{k=1,\ldots,n} \frac{1}{\mathbf{1}_m X_k} = \frac{1}{\max\limits_{k=1,\ldots,n} \mathbf{1}_m X_k}.$$

□

Theorem 3.7. *Interval* $(0, \varepsilon^*]$, *is the largest overall assurance interval.*

Proof. Assume $(0, \varepsilon']$ is the largest overall assurance interval for the non-Archimedean ε for the FDH-VRS epsilon models. Since $(0, \varepsilon^*]$ is an overall assurance interval for ε, then $(0, \varepsilon^*] \subseteq (0, \varepsilon']$, ie $\varepsilon^* \leq \varepsilon'$. If $\varepsilon^* < \varepsilon'$ then there is a $\delta > 0$ such that $\varepsilon' = \varepsilon^* + \delta$. On the other hand, suppose that $\varepsilon^* = \min\{\varepsilon_1^*, \ldots, \varepsilon_n^*\} = \varepsilon_t^*$ $(t \in \{1, \ldots, n\})$. Therefore, there is a feasible solution such as: $\left(\varepsilon_t^* + \delta, \bar{U}_j(j = 1, \ldots, n), \bar{V}_j(j = 1, \ldots, n), \bar{z}\right)$, for Model (8) to evaluate DMU_t. Since ε_t^* is the optimal value of Model (8) to evaluate DMU_t, then $\varepsilon_t^* \geq \varepsilon_t^* + \delta$ and this is a contradiction. Therefore $\varepsilon' = \varepsilon^*$. □

From Definition 3.5 and Theorem 3.7, it follows that each $\varepsilon \in (0, \varepsilon^*]$ is an assurance value of epsilon for evaluating all DMUs in the FDH-VRS epsilon models.

4 Numerical Example

Example 4.1. To further understand the difference between FDH-VRS model and FDH-VRS epsilon models, consider two DMUs with two inputs and one output, the information about which is taken from the article by Molla-Alizadeh-Zavardehi et al. [33] and is given in Table 1:

Table 1: Data for Example 4.1.

DMUs	A	B
Input1	2	2
Input2	3	5
Output	4	4

Using Model (6), the efficiency scores of DMUs A and B are evaluated. The optimal solutions are reported in Table 2.

Table 2: Results of Model (6) in Evaluating DMUs of Example 4.1.

DMUs \ Optimal values	V_{1A}^*	V_{2A}^*	V_{1B}^*	V_{2B}^*	z^*
A	0	0.7143	0	0.7143	1
B	0.5	0	0	0.2	1

In Table 2, both DMUs A and B are efficient and their efficiency value is equal to unity under Model (6). But as can be seen from Table 1, DMU A dominates DMU B, because DMU A produces the same output, which is 4, with fewer inputs. Therefore, DMU B is inefficient and DMU A is efficient. But in the evaluation of these DMUs, both DMUs were evaluated as efficient by Model (6). See the column corresponding to z^* in Table 2. This problem arises from considering zero optimal weight values for the multipliers of inputs and outputs. To solve this problem, we use the epsilon Model (7). In this model, we first calculate ε^* from Eq. (11). So, we have:

$$\max\{\mathbf{1}X_A, \mathbf{1}X_B\} = \max\{2+3, 2+5\} = 7 \rightarrow \varepsilon^* = \tfrac{1}{7}.$$

Now, with this epsilon value, we proceed to evaluate DMUs A and B. The optimal values obtained are listed in Table 3.

Table 3: Results of Model (7) in Evaluating DMUs of Example 4.1, for $\varepsilon = \varepsilon^*$.

DMUs \ Optimal values	V_{1A}^*	V_{2A}^*	V_{1B}^*	V_{2B}^*	z^*
A	0.2857	0.1429	0.2857	0.1429	1
B	0.1429	0.1429	0.1429	0.1429	0.7143

As can be seen from Table 3, using the epsilon-based Model (7), DMUs A and B have been evaluated. As expected, DMUs A is assessed as efficient and unit B as inefficient. The weak duality theorem helps to intuitively understand this example of why choosing an inappropriate epsilon is problematic. We consider the overall assurance interval $(0, \varepsilon^*]$ for epsilon, where $\varepsilon^* = \frac{1}{7}$. For the choice of $\varepsilon = \frac{1}{10} < \varepsilon^*$, we have the results of Table 4 and for the choice of $\varepsilon = \frac{1}{6} > \varepsilon^*$, we have the results of Table 5 by applying Model (7) to the data of numerical Example 4.1.

Table 4: Results of Model (7) in Evaluating the Units of Example 4.1, for $\varepsilon = \frac{1}{10} < \varepsilon^*$

Optimal values / DMUs	V_{1A}^*	V_{2A}^*	V_{1B}^*	V_{2B}^*	z^*
A	0.3500	0.1000	0.1000	0.2667	1
B	0.2500	0.1000	0.2500	0.1000	0.8

Table 5: Results of Model (7) in Evaluating the Units of Example 4.1, for $\varepsilon = \frac{1}{6} > \varepsilon^*$

Optimal values / DMUs	V_{1A}^*	V_{2A}^*	V_{1B}^*	V_{2B}^*	z^*
A	0.2500	0.1667	0.2500	0.1667	1
B	-	-	-	-	infeasible

Analysis of Tables 4 and 5, clearly shows that the choice of the value of epsilon relative to the critical value of ε^* has a decisive effect on the feasibility of the results of Model (7). By choosing ε smaller than $\varepsilon^*(\varepsilon = \frac{1}{10} < \varepsilon^*)$, the model constraints become flexible and the model provides a reasonable optimal solution for both units A and B. However, by choosing ε larger than $\varepsilon^*(\varepsilon = \frac{1}{6} > \varepsilon^*)$, the model constraints become very strict, such that unit B becomes infeasible and only unit A with efficiency 1 can be evaluated. These results emphasize the importance of choosing an appropriate ε within overall assurance interval, because choosing one outside this interval can lead to the feasibility of the model and, consequently, meaningless efficiency analysis.

Example 4.2. Consider the six DMUs with one input and one output as shown in Table 6

Table 6: Data for Example 4.2.

DMUs	A	B	C	D	E	F
Input	3	5	8	10	6	9
Output	2	4	7	8	1	5

Figure 1 shows the FDH-VRS model frontier. As can be seen, A, B, C, and D are efficient DMUs, and E and F are inefficient DMUs.

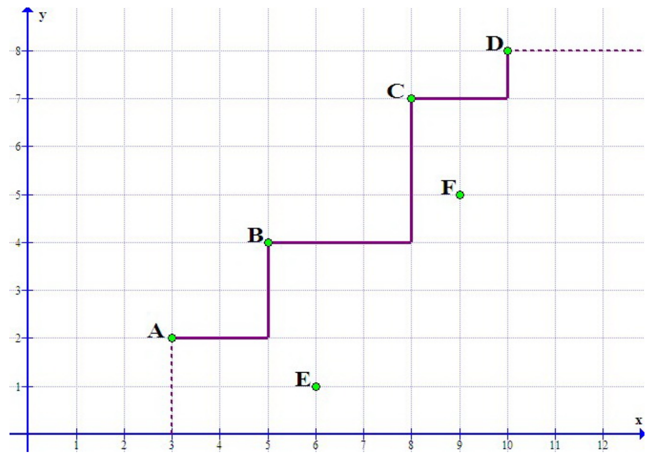

Figure 1: FDH-VRS efficiency frontier for Example 4.2.

Evaluating the DMUs of Example 4.2, using Model (7), results in Table 7. In this example, the epsilon calculated in Model (7) is obtained from Eq. (11) as follows:

$$\max\{\mathbf{1}X_A, \mathbf{1}X_B, \mathbf{1}X_C, \mathbf{1}X_D, \mathbf{1}X_E, \mathbf{1}X_F\} = \max\{3, 5, 8, 10, , 6, 9\} = 10 \rightarrow \varepsilon^* = \frac{1}{10}$$

Therefore, overall assurance interval of epsilon for Example 4.2 is obtained as $(0, \frac{1}{10}]$.

Table 7: Results of Model (7) in evaluating DMUs of Example 4.2.

Optimal values / DMUs	$V_1^*(j = A, \ldots, F)$	U_A^*	U_B^*	U_C^*	U_D^*	U_E^*	U_F^*	z^*
A	0.3333	0.1185	0.1000	0.1000	0.1000	0.1000	0.1000	1
B	0.2000	0.2000	0.1000	0.1000	0.1000	0.1000	0.1000	1
C	0.1250	0.1250	0.1250	0.1000	0.1000	0.1000	0.1000	1
D	0.1000	0.1167	0.1250	0.2000	0.1000	0.1000	0.1000	1
E	0.1667	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.4000
F	0.1111	0.1185	0.1333	0.1000	0.1000	0.1000	0.1000	0.6889

The efficiency values of the DMUs are shown in Table 7, in the column corresponding to z^*. As mentioned earlier, in this table, A, B, C and D are efficient DMUs (efficiency values of 1) and E and F are inefficient DMUs (efficiency values less than 1).

5 An Experimental Example

In this section, we apply a dataset of 25 supermarkets in Finland (Table 6) taken from the paper by Korhonen and Siriänen [34] to illustrate our models. The inputs and outputs of this dataset are as follows:

Inputs:

I1: Man-Hours: The total working hours of personnel over a specified period (in person-hours). This variable represents the labor input.

I2: Size: The total sales floor area (in square meters). This variable represents the physical capital input and the store's capacity.

Outputs:

O1: Sales: The total revenue generated from the sale of goods (in local currency units). This variable represents sales performance.

O2: Profit: The net operating profit (in local currency units). This variable represents economic efficiency and cost management performance.

Supermarkets, whose data are shown in Table 8, are a real-world example of DMUs that are often analyzed in operations and supply chain management. This selection of a set of supermarkets shows that the proposed model can be applied in real environments.

Table 8: Data for 25 supermarkets in Finland.

DMU	Inputs		Outputs	
	I1	**I2**	**O1**	**O2**
B01	4.99	79.1	115.3	1.71
B02	3.30	60.1	75.2	1.81
B03	8.12	126.7	225.5	10.39
B04	6.70	153.9	185.6	10.42
B05	4.47	65.7	84.5	2.36
B06	4.08	76.8	103.3	4.35
B07	2.53	50.2	78.8	0.16
B08	2.47	44.8	59.3	1.30
B09	2.32	48.1	65.7	1.49
B10	4.91	89.7	163.2	6.26
B11	2.24	56.9	78.7	2.80
B12	5.42	112.6	142.6	2.75
B13	6.28	106.9	127.8	2.70
B14	3.14	54.9	62.4	1.42
B15	4.43	48.8	55.2	1.38
B16	3.98	59.2	95.9	0.74
B17	5.32	74.5	121.6	3.06
B18	3.69	94.6	107.0	2.98
B19	3.00	47.0	65.4	0.62
B20	3.87	54.6	71.0	0.01
B21	3.31	90.1	81.2	5.12
B22	4.25	95.2	128.3	3.89
B23	3.79	80.1	135.0	4.73
B24	2.99	68.7	98.9	1.86
B25	3.10	62.3	66.7	7.41

Table 8, shows the results of evaluating supermarkets with Models (3) and (4). These results were obtained using the Python programming language, version 3.8.

Table 9: Results of Models (3) and (4) on the experimental example data, for $\varepsilon^* = 6.23 \times 10^{-3}$.

DMU_j	E^*_{1j}	Reference Unit of DMU_j in model (3)	S^*_j	E^*_{2j}	Reference Unit of DMU_j in model (4)
B01	1	B01	0	1	B01
B02	0.9468	B11	5.3743	0.9468	B11
B03	1	B03	0	1	B03
B04	1	B04	0	1	B04
B05	1	B05	0	1	B05
B06	1	B06	0	1	B06
B07	1	B07	0	1	B07
B08	1	B08	0	1	B08
B09	1	B09	0	1	B09
B10	1	B10	0	1	B10
B11	1	B11	0	1	B11
B12	0.9059	B10	36.4147	0.9059	B10
B13	0.7493	B23	10.1456	0.7493	B23
B14	0.8761	B09	3.8011	0.8761	B09
B15	0.9857	B09	12.6565	0.9857	B09
B16	1	B16	0	1	B16
B17	1	B17	0	1	B17
B18	1	B18	0	1	B18
B19	1	B19	0	1	B19
B20	0.9194	B07	8.9781	0.9194	B07
B21	1	B21	0	1	B21
B22	0.8918	B23	12.3360	0.8918	B23
B23	1	B23	0	1	B23
B24	1	B24	0	1	B24
B25	1	B25	0	1	B25

5.1 Experimental Example Findings

- **Introduction to Table 9 columns.**

  The first column (DMU_j) is the names of the supermarkets. The second column (E^*_{1j}) is the efficiency obtained from Model (3) for DMUs under evaluation. The third column is the reference unit of the supermarkets, which are the $\lambda^*_j (j = 1, \ldots, 25)$ in the optimal solution obtained from Model (3). The fourth column is the sum of the optimal slacks in Model (5). The fifth column (E^*_{2j}) is the efficiency obtained from Model (4) for DMUs under evaluation, and the sixth column is the reference unit of the supermarkets, which is obtained from Model (4).

- **On the methodological nature of FDH.**

  The FDH model is a nonparametric model that removes the convexity assumption from classical DEA models. This feature allows efficient DMUs to be compared with only one real DMU (and not a combination of them), as is evident in the third and fifth columns of Table 9.

- **Methodological optimization and validation.**

  Before discussing the results, it is necessary to mention the steps of the method used to conduct this study. To find the results in Table 9, first, in Phase 1, Model (3) was run on the data in Table 8, which requires solving 25 models (the number of supermarkets). The results of this implementation are shown in the second column of Table 9, in which 18 supermarkets, including supermarkets B01, B03 to B11, B16 to B19, B21, and B23 to B25, were identified as efficient DMUs, and the rest of them (7 supermarkets) were identified as inefficient. Then, in Phase 2, to ensure the accuracy of efficient supermarkets, to verify the efficiency and calculate slacks, Model (5) was solved for each efficient DMU. Here, for each efficient supermarket, a Model (5) was solved to determine the accuracy of the efficient DMUs. Therefore, traditionally, in Phases 1 and 2, a total of 43 (=25+18) linear programming models were solved to verify the correctness of the efficient DMUs. The results of running Model (5) for all supermarkets are listed in the fourth column of Table 9. These results show that DMUs identified by the efficient Model (3) were correctly identified. In Phase 3 of the implementation method, the value of $\varepsilon^*(= 6.23 \times 10^{-3})$ was first calculated from Eq. (11). Then, by placing this value in Model (4) (combination of Phases 1 and 2) and solving it, with only 25 units, the results of the fifth and sixth columns were obtained. The most important point here is that the results from Model (4) were completely consistent with the results from Model (3). This consistency proves two conclusions.

1. The obtained assurance epsilon lies exactly within the global assurance interval and prevents artificial variables from entering the optimal basis, while keeping the dual model feasible.
2. The efficiency gained from epsilon-based Model (4) reduces the computational volume by almost half without losing accuracy, which is a great advantage for large data sets.

- **Returns to scale in FDH.**

  The FDH model inherently operates on the assumption of VRS. This means that each DMU is only compared to units of its own size or units operating at a similar scale.

- **Interpretation of slack values.**

  The value of S_k^*, or the sum of slack values in Model (5), indicates the amount of surplus in inputs or the amount of shortage in outputs of each DMU. This value tells us exactly how much an inefficient DMU must reduce its resources or increase its production to reach the efficiency frontier. This provides a quantitative, measurable goal. In this study, the highest are for B12 with a value of 36.4148. This very high number indicates a significant lack of optimal allocation of resources. This supermarket requires a major structural change in the use of inputs to reach its reference unit level (B10). S_k^* is zero for efficient DMUs, which makes perfect sense, since these DMUs are right on the frontier.

- **Importance of an assurance epsilon selection.**

  The theoretical material focused on finding an assurance value for the non-Archimedean epsilon. This value should be small enough to mathematically prevent artificial variables from entering the optimal basis, but large enough to keep the model feasible. The epsilon calculated by Eq. (11) causes the values of E_{1j}^* and E_{2j}^* to be the same for all DMUs, and the complete agreement of the results of Model (3) and Model (4) indicates that this calculated epsilon value ($\varepsilon^* = 6.23 \times 10^{-3}$) is exactly within the overall assurance interval of epsilon. If this value of epsilon were chosen to be either too large or too small, not only would Model (7) not be feasible, but the results of Model (4) would also differ from Model (3). This dual consistency confirms the absolute correctness of the calculations and the choice of parameters.

- **Analysis of specific DMUs.**

  Supermarket B13, with an efficiency of 0.7493, has the lowest efficiency among all supermarkets, with its reference unit being Supermarket B23 (a perfectly efficient unit). This result says that B13 uses 36% ($\approx$ 100%-73.43%) more resources to produce the same amount of output as B23. The value of S^*_{B13}= 10.1456 also confirms the exact amount of correction needed in the inputs. DMUs such as B13 with very low efficiency and B12 with the highest total slack should be given top priority for reviewing processes and reallocating resources.

- **Best practices identification.**

  Certain efficient DMUs such as B09 and B23 have emerged as references for several inefficient DMUs. B09 is the reference for DMUs B14 and B15, and B23 is the reference for DMUs B13 and B22.These benchmarks indicate that these efficient supermarkets are the best practices in the data set and that other units should bring their processes closer to these benchmarks. Also, managers should focus on these supermarkets (B09 and B23) as best practices.

- **Model validation.**

  The perfect alignment between the results of the primary and dual models (E^*_{1j} and E^*_{2j}), along with the consistency between the results of Model (3) and Model (4), demonstrates that the analytical method possesses complete mathematical stability and computational accuracy. Consequently, its results can be relied upon with full confidence. As a result, this analysis, through an optimal and validated method, provides decision-makers not only with a performance report but also with a precise road-map for improving efficiency.

Table 10: Results of Model (4) on the experimental example data, for ε.

DMU_j	$E^*_{2j}(\varepsilon = 6.23 \times 10^{-7} < \varepsilon^*)$	$E^*_{2j}(\varepsilon = 8 \times 10^{-3} > \varepsilon^*)$
B01	1	0.6601
B02	0.9468	0.7528
B03	1	infeasible
B04	1	infeasible
B05	1	0.6950
B06	1	0.6872
B07	1	1
B08	1	1
B09	1	1
B10	1	0.9492
B11	1	1
B12	0.9059	0.4379
B13	0.7493	0.3577
B14	0.8761	0.7870
B15	0.9857	0.8870
B16	1	0.9237
B17	1	0.8034
B18	1	0.5415
B19	1	1
B20	0.9194	0.8437
B21	1	0.4666
B22	0.8918	0.6395
B23	1	0.9451
B24	1	0.8999
B25	1	1

Table 10 shows the efficiency values E^*_{2j} resulting from running Model (4) on the experimental example data for two choices ε. In the first choice, $\varepsilon = 6.23 \times 10^{-7} < \varepsilon^*$ is within overall assurance interval. With this choice, as can be seen in the first column of Table 8, all units can be evaluated. In the second choice, $\varepsilon = 8 \times 10^{-3} > \varepsilon^*$ is outside the overall assurance interval. With this choice, as can be seen in the second column, the efficiency values have decreased significantly and even the evaluation of the two units B03 and B04 has become infeasible. These results clearly confirm the importance of choosing the appropriate value of ε in making the model feasible and the calculated efficiency values more realistic.

6 Conclusion

This research addresses one of the long-standing and practical challenges in Data Envelopment Analysis (DEA), particularly in Free Disposal Hull models with Variable Returns to Scale (FDH-VRS). The main problem was determining the optimal value of the non-Archimedean epsilon; an incorrect choice of this value can lead to two serious issues: first, the infeasibility of the multiplier form, and second, the unboundedness of the envelopment form. These two problems completely undermine the validity and reliability of performance evaluation results. Using an assurance value of epsilon not only prevents the assignment of zero weights to inputs and outputs but also ensures that all data are incorporated into the evaluation process, resulting in a more comprehensive and realistic assessment.

The key finding of this research is the derivation of a simple formula ε^* for calculating an assurance value of epsilon. This approach has a methodological advantage over previous methods (such as Mehrabian et al. [28] and Alirezaee [30] for convex DEA) that require solving linear programming problems or iterative algorithms: the complete elimination of the optimization step and its replacement with a simple algebraic calculation. The model presented in this paper has several advantages over existing methods. First, it provides simultaneous guarantees for the feasibility of the multiplicative model and the boundedness of the covering model based on the strong duality theorem. Second, it does not require specialized LP solvers and is easy to code. Overall, this paper has taken a step towards increasing the reliability and ease of use of non-convex FDH-VRS models in real applications by providing a simple, efficient, and theoretically sound solution.

For future research, the following are suggested to extend this framework:

- Generalizing the method to other returns to scale assumptions (constant, non-increasing, and non-decreasing returns to scale) in the FDH environment.

- Integrating this method with more advanced analyses such as network FDH (NFDH) models or FDH with undesirable factors.

- Investigating the performance of the method on very large datasets and integrating it with machine learning platforms.

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